Is it a New Criterion for Riemann Hypothesis or True Proof?

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Abstract

There are tens of self-proclaimed proofs for Riemann Hypothesis and only 2 or 4 disproofs of it in arXiv. I am adding to the Status Quo my very short and clear evidence, which uses the peer-reviewed achievement of Dr. Solé and Dr. Zhu, which they made just 4 years ago in a serious mathematical journal INTEGERS.

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I. PRIOR RESEARCH RESULT

Because the paper of Dr. Zhu [1] is not published in a peer-review journal (for 4 years) and is very complicated, it could contain a fatal mistake. Thus, I do not start with the final result called "The probability of Riemann's hypothesis being true is equal to 1" but rather with the starting information of the papers [1, 2] (one of the papers is peer-reviewed), where is proven (cf. Theorem 2), that "limit inferior"

$$\lim_{n\to\infty} \inf d(n) \geq 0\,, \tag{1}$$

where $d(n) = D(n)/n$, and $D(n) = \mathrm{e}^{\gamma}\, n \ln\ln n - \sigma(n)$. Hereby the Riemann Hypothesis holds true, if $\lim\limits_{n\to\infty} \inf D(n) \geq 0$.

The main problem of the available Riemann Hypothesis proofs is the fatal mistakes somewhere in the text. If text is complicated enough, the mistake is practically impossible to find. The final result of Ref. [1] comes from too many theorems: 1,2,3 in the Ref. [2], so the risk of having a mistake is very high. But now I will demonstrate, that it is enough to hope for the validity of the Theorem 2 in the Ref. [2], i.e. I can prove the Riemann Hypothesis even without his theorems one and three. Recall, that the Riemann Hypothesis has been shown to hold unconditionally for n up to $N = \exp(\exp(26))$ as written in Refs. [2, 3]. Thus, it is enough to check the Riemann Hypothesis for the $n \gg 1$ area. Therefore, we do not need Theorem 3, because it is a trivial fact Dr. Zhu is proving, that if $D(n) \geq 0$ for $n > N \gg 1$ the Riemann Hypothesis is correct. And we do not need Theorem 1, because Theorem 2 already says, that holds the Eq.(1).

II. MY PROOF

Today the unchecked area of Riemann Hypothesis has extremely large values of $n > \exp(\exp(26))$ (including the unlimitely large n). From the Eq.(1) I conclude that for large $n \gg 1$ the

$$\frac{D(n)}{n} \equiv \mathrm{e}^{\gamma} \ln\ln n - \frac{\sigma(n)}{n} \geq -\beta(n)\,, \tag{2}$$

where the $\beta(n) = 0$, if $n \to \infty$. On the other hand the Riemann Hypothesis is true, if for every $n > 1$ holds [4]

$$\frac{\sigma(n)}{n} < \frac{H_n + \exp(H_n)\ln(H_n)}{n}\,, \tag{3}$$

where the harmonic number is

$$H_n = \gamma + \ln(n) + K(n)\,, \tag{4}$$

where $K(n) > 0$ and $K(n) = 0$ if $n \to \infty$. Let us insert the H_n from Eq.(4) into Eq.(3), we get

$$\frac{\sigma(n)}{n} < \mathrm{e}^\gamma \ln(\gamma + \ln(n)) + R(n)\,, \tag{5}$$

where $R(n) > 0$. From Eqs. (2), (5) follows, that Riemann Hypothesis is true, if for large n holds

$$\beta(n) + \mathrm{e}^\gamma \ln\ln n < \mathrm{e}^\gamma \ln(\gamma + \ln(n)) + R(n)\,. \tag{6}$$

Inequality (6) is satisfied, if

$$0 \le \beta(n) \le \beta_0(n)\,, \tag{7}$$

but is violated if $\beta(n) > \beta_0(n)$. Let us find the violation $\beta_0(n)$. From Eq.(6)

$$\beta_0(n) = \mathrm{e}^\gamma \ln(\gamma + \ln(n)) - \mathrm{e}^\gamma \ln\ln n + R(n) = \mathrm{e}^\gamma \ln([\gamma/\ln(n)] + 1) + R(n)\,. \tag{8}$$

Quote from the end of the paper [2]: "For instance, one cannot rule out the case that $D(n)$ behaves like $-\sqrt{n}$ when $n \to \infty$, which would not contradict the fact that $\liminf_{n\to\infty} d(n) = 0$." This points to my function $\beta(n) = (C\sqrt{n})/n = C/\sqrt{n}$, where $C \ge 0$, e.g. $C = 1$. The following holds true $C/\sqrt{n} < \beta_0(n)$ [5] then the Riemann Hypothesis is true. And because the Riemann Hypothesis is shown now true (for case if $D(n)$ acts like $-\sqrt{n}$), then to avoid contradiction with Robin's inequality we must assign $C = 0$.

Moreover, the $\beta(n) = C/n^x \ge 0$ results in the $C = 0$ by the same analysis, for all fixed powers $x > 0$, $x \ne 0$. That means, that if there exists the Taylor Series for $\beta(n)$ for large argument, then the Riemann Hypothesis is proven. But because the $\beta(n)$ is a monotonic slowly decreasing function, it is well justified, that it has non-zero derivative (then formally the n are taken to be continuous) somewhere in the first Taylor terms. If the $\beta(n)$ would be exponential, thus, non-analytically rapidly approaches zero, the Eq.(7) is still satisfied: $\exp(-n) \ll \sqrt{n}$.

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[1] Zhu Y., The probability of Riemann's hypothesis being true is equal to 1, arXiv:1609.07555v2 [math.GM] (2016, 2018)

[2] Solé P. and Zhu Y., An Asymptotic Robin Inequality. INTEGERS, Nr.A81, 16 (2016), http://math.colgate.edu/∼integers/q81/q81.pdf

[3] Briggs K., Abundant numbers and the Riemann hypothesis, Experiment. Math. 15 (2), 251–256 (2006).

[4] Lagarias J. C., An elementary problem equivalent to the Riemann hypothesis, The American Mathematical Monthly 109 (6), 534–543 (2002); Sandifer C. E., How Euler Did It, MAA Spectrum, Mathematical Association of America, p. 206 (2007)

[5] while demonstration one formally writes in Eq.(4) the $K(n) \equiv 0$ for all n, checks the resulting inequality, and then restores back the $K(n) > 0$