Quantum Field Theory with Fourth-order Differential Equations for Scalar Fields

Rui-Cheng LI *

School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

Abstract

We introduce a new class of higgs type complex-valued scalar fields U with Feynman propagator $-i/p^4$ and consider the matching to the traditional gauge fields with propagator $-ig^{\mu\nu}/p^2$ in the viewpoint of effective potentials at tree level. With some particular postulations on the convergence and the causality, there are a wealth of potential forms generated by the fields U, such as: by using U to construct a QED or a QCD theory,we can get an effective potential including a Coulomb potential, a linear potential and a logarithmic potential, which could be a source for the confinement effect; and, by using U to construct a gravitation theory, we can get an effective potential including a linear potential to serve for the dark energy effect, and the Newton's gravity accompanied by a relativistic effect correction to serve the dark matter effect. Moreover, for some limit cases, we can get some suppositions, such as: a nonlinear Klein-Gordon equation could generate in the low energy limit; the gauge symmetry and a linear QED could superficially generate in the weak field case; a fermion mass spectrum with generation structure and a seesaw mechanism for gauge symmetry and flavor symmetry could generate due to the multi-vacuum structure for a sine-Gordon type vector field induced by U.

Keywords linear potential, confinement, dark energy, dark matter

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*rui-chengli@163.com

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1 Introduction

As a very successful theory, the gauge field theory with the gauge invariance principle could be used to solve a huge part of questions for people. Certainly, there are some challenges to the gauge theory: one class about the extension for methods of application, such as the ones for non-perturbative problems; another class about the extension for new phenomenons, such as the ones for new particles or dark matter/dark energy effects; with an inevitable old topic about the unification and renormalization.

It's just the linear potential from the non-perturbative results in lattice gauge theory [1] that motivated us to consider a fourth order differential equations (D.E.). And, mathematically, a most straightforward way on the extensions for new particles could be related to the higher order D.E., generally with a trouble on dealing with the redundant unphysical/noncausual degrees of freedom (d.o.f) and an omittance/ignorance on the non-perturbative and unification problems. So, it would be significant to modify the higher order D.E. framework to cover the three sectors mentioned above, even with some man-made postulations or constraints. That is just what we have done in this paper.

In this paper, we have taken some postulations to construct our model within the 4th-order D.E. framework, mainly for the convergence (renormalization) and a reasonble performance on matching conditions of the model. For simplicity, we have concentrated our studies on the pro forma feasibility of the model in the view of effective potentials at tree level.

The remainder of this paper is organized as follows: Sect. 2 is for the Lagrangian construction for a linear potential; Sect. 3 is for the kinetics and the propagators from the Lagrangian; Sect. 4 is for the effective potentials generated from the Lagrangian, especially for the linear potential, the Coulomb potential and the gravitational potential; Sect. 5 is for some interesting suppositions uniquely occurring in our theory for some limit cases; Sect. 6 is for interpreting the causality in our theory; and Sect. 7, the final section, is for our conclusions.

2 Lagrangian for linear potential

2.1 Framework: effective potentials at tree-level

We can get the classic non-relativistic (NR) potential forms from the amplitudes of the tree-level 2→ 2 scattering process for a perturbative theory, within the Born-approximation framework, for instance, we can take [2]

$$(\text{vertex})_1 \ \otimes \ (\text{inner-line propagator}) \ \otimes (\text{vertex})_2 \Leftrightarrow \mathcal{V} \tag{1}$$

where the l.h.s is a part of the amplitude for a tree-level Feynman diagram, and the r.h.s is the classic potential. So, conversely, we can build theories for potentials with a definite form through the tree-level-correspondence, provided that the theories are perturbatively computable. For example, if there were neither momentums nor coordinates in the Feynman rules of vertices, we would extract different potentials with different inner-line propagators, such as:

$$\begin{aligned} \text{linear potential} \quad &\leftrightarrow \quad \frac{1}{p^4}\,, \\ \text{Coulomb potential} \quad &\leftrightarrow \quad \frac{1}{p^2}\,, \\ \text{short-distance potential} \quad &\leftrightarrow \quad \frac{1}{p^\alpha}\,, \text{with } \infty < \alpha < 2. \end{aligned} \tag{2}$$

2.2 Lagrangian

Firstly, we take a complex-valued scalar field U, a Dirac field ψ (and $\bar{\psi}$) as the physical field degree of freedom(d.o.f) [1], which have the transformation law under a $U(1)$ **global** group element V as

$$U \to VUV^{-1} = U\,,\ \psi \to V\psi\,, \bar{\psi} \to \bar{\psi}V^{-1}\,. \tag{3}$$

Secondly, in the method mentioned in Section 2.1, for a theory with a propagator form $\sim \frac{1}{p^4}$ for U, we write the Lagrangian of $\{U, \psi, \bar{\psi}\}$ as

$$\mathcal{L} = \mathcal{L}_U + \mathcal{L}_\psi + \mathcal{L}_I, \tag{4}$$

where the term

$$\begin{aligned}\mathcal{L}_U \;&=\; Tr\left\{-\partial^\mu\partial^\nu U^\dagger\partial_\mu\partial_\nu U - \Lambda_U^4[(U+U^\dagger)+i(U-U^\dagger)]\right.\\ &\qquad \left.+m_U^4 U^\dagger U - \lambda_U\Lambda_U^4 U^\dagger U U^\dagger U\right\}\end{aligned} \tag{5}$$

is purely of the complex-valued scalar field U, the term

$$\mathcal{L}_\psi = \bar{\psi}(i\partial\!\!\!/ - m_\psi)\psi \tag{6}$$

is purely of the matter field ψ, and the term

$$\begin{aligned}\mathcal{L}_I \;=\;& -\alpha\Lambda Q_\alpha\,\bar{\psi}\left\{[(U+U^\dagger)+i(U-U^\dagger)]\right\}\psi\\ & -\beta Q_\beta\,\bar{\psi}\left\{\sigma_{\mu\nu}\partial^\nu[(U+U^\dagger)+i(U-U^\dagger)]\right\}\gamma^\mu\psi\\ & -\xi\frac{1}{M}Q_\xi\bar{\psi}\left\{\sigma_{\mu\nu}\partial^\mu[(U+U^\dagger)+i(U-U^\dagger)]\right\}(i\overleftrightarrow{\partial}^\nu\psi)\\ & -\rho\frac{1}{M}Q_\rho\,\bar{\psi}\left\{\partial^\mu\partial^\nu[(U+U^\dagger)+i(U-U^\dagger)]\right\}\gamma_\mu\gamma_\nu\psi\\ & -\text{...(higher order 3-field terms)}\\ & -\kappa Q_\kappa\,\bar{\psi}\left\{\frac{\Lambda^2}{M}[U^\dagger U]\psi + \Lambda[U^\dagger i\overleftrightarrow{\partial^\mu}U]\gamma^\mu\psi + \frac{1}{M}\sigma_{\mu\nu}\partial^\mu[U^\dagger i\overleftrightarrow{\partial^\nu}U]\psi\right\}\\ & -\text{...(higher order multi-field terms)}\end{aligned} \tag{7}$$

is the invariant interaction term of ψ coupled to U under the transformations in (3) and the Lorentz transformation. The application of $\sigma^{\mu\nu}$ in the β term is to ensure a real-valued effective coupling in the Feynman rule language, by recalling the reduction of $-i\sigma_{\nu\mu}q^\mu \to q_\nu$; and $\gamma_\mu\gamma_\nu$ in the ρ term is to match the symmetric indices $\mu\nu$ in $\partial^\mu\partial^\nu U$.

Thirdly, we give some **postulations** as the illustrations of the variables in the Lagrangian of (4) as below.

1. Λ_U is a constant of the dimension of mass, m_U is the mass of field U, and λ_U is a dimensionless constant; m_ψ is the mass of field ψ.

2. Each of the coefficients $\{\alpha, \beta, \xi, \rho, \kappa, ...\}$ takes a real number value for the sake of hermiticity; and the notation "..." in (7) denotes terms for multi-field and higher-dimension operators .

3. For the parameters Λ and M, referring to Wilson's scheme for renormalization, for the interaction Lagrangian terms we can propose the postulations as:
(i) each U (not ∂U) is tied with one infrared (I.R.) energy scale Λ;

[1] Discussions for the vector field U^μ, the tensor field $U^{\mu\nu}$, and the massive $\{U, U^\mu\}$ have also been finished by the author, see Ref. [3].

(ii) all the terms with higher-dimensional ($D > 4$) are suppressed by a ultraviolet (U.V.) energy scale M.

For example, if we plan to construct a QED, a QCD or a gravitation theory with the U field, then the variable Λ and M might be respectively set as

$$\Lambda = \mu_{IR} \simeq \{0, \Lambda_{QCD}, 0\}, M = \mu_{UV} \simeq \{\mu_{EW}, \mu_{GUT}, \mu_{Plank}\}, \tag{8}$$

where μ_{IR} is the I.R. boundary energy scales, i.e., { value $\simeq 0$, the QCD scale $\Lambda_{QCD} \simeq 200\, MeV$, value $\simeq 0$}, and M is the U.V. boundary for the theory, i.e., {the electroweak (EW) scale $\mu_{EW} \sim 246\, GeV$, the grand unification theory (GUT) scale μ_{GUT}, the Plank scale μ_{Plank}}, for a QED,a QCD and a gravitation theory, respectively.

4. The variables $Q_{\{\alpha,\beta,\xi,\rho,\kappa,...\}}$ can be seemed as a kind of reconstructed charges (RC), and they are defined as

$$Q_{\{\alpha,\beta,\xi,...\}} \equiv Y \mathcal{Q}_{\{\alpha,\beta,\xi,...\}},\ Y = \pm 1, \tag{9}$$

where Y is the generator of the global $U(1)$ group with eigenvalues ± 1, and $\mathcal{Q}_\alpha$ is a generator of some other global group (such as the electromagnetic $U(1)$ group) corresponding to the current $\mathcal{J}_\alpha$, with the definitions

$$\mathcal{Q}_\alpha \equiv 1,\, Y; \qquad \mathcal{J}_\alpha \equiv \bar{\psi}\psi; \tag{10}$$

$$\mathcal{Q}_\beta \equiv T_{\mathcal{Q}}; \qquad \mathcal{J}_\beta \equiv \bar{\psi}\gamma^\mu\psi; \tag{11}$$

$$\mathcal{Q}_\xi \equiv Y; \qquad \mathcal{J}_\xi \equiv \bar{\psi} i \overleftrightarrow{\not\partial} \psi,\ ..., \tag{12}$$

where $\mathcal{Q}_\alpha = 1$ for neutral U (e.g. U for mediating a QED theory), $\mathcal{Q}_\alpha = Y$ for charged U (e.g. U for mediating a QCD theory); $T_{\mathcal{Q}} \equiv T_{QED}, T^a_{QCD}, ...$, is either the generator of the QED $U(1)$ group for constructing a QED theory with U, or one of the the generator of QCD $SU(3)$ group for constructing a QCD theory with U, etc.

Furthermore, if we define a kind of effective media field as

$$\begin{aligned}
(\mathcal{A}_I)_\alpha &\equiv -\alpha\Lambda\mathcal{Q}_\alpha \cdot [(U + U^\dagger) + i(U - U^\dagger)], && (13)\\
[(\mathcal{A}_I)_\beta]_\mu &\equiv -\beta\mathcal{Q}_\alpha \cdot \sigma_{\mu\nu}\partial^\nu[(U + U^\dagger) + i(U - U^\dagger)], && (14)\\
[(\mathcal{A}_I)_\rho]_{\mu\nu} &\equiv -\rho\frac{1}{M}\mathcal{Q}_\alpha \cdot \partial_\mu\partial_\nu[(U + U^\dagger) + i(U - U^\dagger)],\ .... && (15)
\end{aligned}$$

then the interaction Lagrangian terms in (7) can be remembered as

$$\mathcal{L}_I \equiv \mathcal{L}^{RC} = (\mathcal{A}_I) \cdot \mathcal{J} \cdot Y. \tag{16}$$

5. How to determine the value of Y and $\mathcal{Q}_\alpha$? Here we define: if the momentum of U flows "in" to the $\bar{\psi}U\psi$ vertex, then the charge at this vertex is $Y = +1$, motivated by an imagination that the effective mass of ψ would become bigger by "eating" a nonzero vacuum expectation value $\langle U \rangle$; on the contrary, if the momentum of U flows "out" of the $\bar{\psi}U\psi$ vertex, then the charge at this vertex is $Y = -1$. Similarly for the $\mathcal{Q}_\alpha$, $\mathcal{Q}_\beta$ and $\mathcal{Q}_\xi$, e.g.:

(i) for $\mathcal{Q}_\alpha$: in the case of a charged U for a QCD theory, in every physically allowed process, if the $\mathcal{Q}_\alpha$ charge of U flows "in" to the $\bar{\psi}U\psi$ vertex, then the $\mathcal{Q}_\alpha$ charge variation for the "current" $\mathcal{J}_\alpha \equiv \bar{\psi}^i\psi^j$ (with i, j the color indices) at this vertex is $\mathcal{Q}_\alpha = +1$, the same as the value of Y; on the contrary, if the $\mathcal{Q}_\alpha$ charge of U flows "out" of the $\bar{\psi}U\psi$ vertex, then the $\mathcal{Q}_\alpha$ charge variation for the "current" $\mathcal{J}_\alpha \equiv \bar{\psi}^i\psi^j$ at this vertex is $\mathcal{Q}_\alpha = -1$, the same as the value of Y; in the case of a neutral U for a QED theory, the $\mathcal{Q}_\alpha$ charge variation for the "current" $\mathcal{J}_\alpha \equiv \bar{\psi}\psi$ at both vertices are defined to be always 1;

(ii) for $\mathcal{Q}_\beta$: even in the case of a neutral U for a QED theory, the $\mathcal{Q}_\beta$ charge variation for the "electromagnetic current" $\mathcal{J}_\beta \equiv \bar{\psi}\gamma^\mu\psi$ is not 1, but to be the QED "charge" $T_{QED} \equiv \mathcal{Q}^{QED}$;

(iii) for $\mathcal{Q}_\xi$: even in the case of a neutral U for a gravitation theory, the $\mathcal{Q}_\xi$ charge variation for the "momentum current" $\mathcal{J}_\xi \equiv \bar{\psi} i \overleftrightarrow{\not\partial} \psi$ is not 1, but to be Y; etc.

6. To ensure the renormalizability, we need an extra **postulation**: all divergences can be removed by introducing cutoff for the amplitudes or the phase-space parameters.

2.3 On the $(\partial\partial U)^2$ term for kinetics term

The traditional kinetic term

$$(\partial U)^2 \text{ and } U^\dagger \partial\partial U$$

could not appear in our model, since it would give a term $\partial\partial U$ in the E.O.M so a propagator form $\sim 1/(p^4 - p^2)$. However, due to the singularity (pole) structure, we can't get the same results for the two propagators, $\sim 1/(p^4)$ and $\sim 1/(p^4 - p^2)$. Besides, if the E.O.M is not the form $\hat{p}^4 U = m^4 U$, that might break a generalized "charge" symmetry. So, we could only take $\partial\partial U$ to construct the kinetic term rather than ∂U.

For convenience, we would call the model for U defined with the $(\partial\partial U)^2$ term for kinetics term as a "**P4 type**", and the traditional model for U defined with the $(\partial U)^2$ term for kinetics term as a "**P2 type**". Besides, our P4 type theory in the high-order D.E. framework is different from the ones actually being a P2 type one [4].

It might be helpful for us to more easily understand the double partial term $(\partial\partial U)^2$ for the kinetics term, if we understand our U field as a classic **continuum medium field**. For the detail, for the continuum medium field ϕ we have the continuity equation

$$\partial_\mu \partial_\nu T^{\mu\nu} = 0, \tag{17}$$

with the energy-momentum tensor defined as

$$\begin{aligned} T^{\mu\nu} &= (\rho + p) u^\mu u^\nu + p g^{\mu\nu} \\ &= \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_\alpha)} \partial^\mu \phi_\alpha - g^{\mu\nu} \mathcal{L} \\ &= (\partial_\mu \phi^\dagger \partial_\nu \phi + \partial_\nu \phi^\dagger \partial_\mu \phi) - g^{\mu\nu}(\partial_\alpha \phi^\dagger \partial^\alpha \phi - m^2 \phi^\dagger \phi) \\ &= \phi^\dagger \left[\overleftarrow{\partial}_\mu \partial_\nu \phi + \overleftarrow{\partial}_\nu \partial_\mu - g^{\mu\nu}(\overleftarrow{\partial}_\alpha \partial^\alpha - m^2) \right] \phi \\ &= \phi^\dagger \left[(i\partial_\mu i\partial_\nu \phi + i\partial_\nu i\partial_\mu) - g^{\mu\nu}(i\partial_\alpha i\partial^\alpha - m^2) \right] \phi. \end{aligned} \tag{18}$$

Formally, to fully describe a field ϕ, one might need the $\partial\partial \cdot \partial\partial$ operator acting on the field.

Moreover, we can write the E.O.M in another form,

$$\hat{p}^4 U(x) = [\hat{p}^2 \Phi(x)]^2 = [\hat{p}^2 \widetilde{\Phi}(x)] \cdot [\hat{p}^2 \Phi(x)], \tag{19}$$

with the correspondence for $\widetilde{\Phi}$ to Φ here is just like a generalized version of the case that the anti-particles $\bar{\psi}$ associated with the particles ψ, which also arised from the treatment that the Dirac equation was formally from the square root of the Klein-Gordon equation. Besides, we can see, if the E.O.M is not the form $\hat{p}^4 U = m^4 U$, then that might break a generalized "charge" symmetry between Φ and $\widetilde{\Phi}$. We can denote that as

$$\Phi(x) \sim \langle \bar{\psi}\psi \rangle \Rightarrow \quad \text{K-G eq.} \ = [\text{ Dirac eq. }]^2 \ , \tag{20}$$

$$U(x) \sim \langle \widetilde{\Phi}\Phi \rangle \Rightarrow \quad \text{U-eq.} \ = [\text{ K-G eq. }]^2 \ . \tag{21}$$

Then we can have the new E.O.M

$$\hat{p}^2 \Phi = m_U^2 \Phi \Rightarrow \Phi = c_1 e^{ip\cdot x} + c_2 e^{-ip\cdot x} \tag{22}$$

for the ordinary physical d.o.f, and

$$\hat{p}^2 \widetilde{\Phi} = -m_U^2 \widetilde{\Phi}, \quad \text{(tachyon/higgs)} \tag{23}$$

$$\not\Leftrightarrow \quad -\hat{p}^2 \widetilde{\Phi} = m_U^2 \widetilde{\Phi}, \quad \text{(phantom)} \tag{24}$$

for the so-called unphysical d.o.f: the tachyons in (23), with an imaginary number valued mass [5]; and the phantoms in (24), with a negative kinetic energy [6], respetively. The sign of the action corresponding to the E.O.Ms in (23) and (24) are different, which is not negligible [7].

Although there exist acausal solutions for differential equations with orders higher than 2, [2] we can just omit them by treating them as non-physical (or, frozen) d.o.f, or, treat them as effects of hidden new degrees of freedom (existent but can't be directly measured for some reasons, such as being confined or spreading to the higher dimensions) beyond the standard model(SM) in particle physics; the latter one case is just what we want to propose, as to be discussed in Section 2.4. We will revisit this topic in Section 6.

2.4 U is a kind of higgs-type field!

The self-interaction potential of field U is

$$V(U) \equiv -m_U^4 U^\dagger U + \lambda_U \Lambda_U^4 U^\dagger U U^\dagger U, \tag{25}$$

so, according to the minus sign in the mass term, U is a kind of higgs-type field. And, for convenience, in all this article for allowed cases we set

$$\langle U \rangle = 1. \tag{26}$$

But we should remind ourselves that $\langle U \rangle$ could be very large even when the energy scale is very low, that means, U with large $\langle U \rangle$ is a strong field!

For a higgs field U with a potential form in (25) plotted as the line-"b" in Fig.-1-(1), besides of the angular component U_θ as the conventional field (the Goldstone boson), there is also a radial-direction component $0 \le U_r \le +\infty$. Here, the most important point is, how to understand the U_r?

For a potential $V(U)$ of the form as the line-"a" in Fig.-1-(1), which is defined only for $0 \le |U| \le 1$ rather than for all the $|U| < \infty$ field configurations, we can not only treat the radial-direction component U_r as a stable (physical) fluctuation around the stable vacuum $|U| = 1$ (minimum of the potential $V(U)$), but also treat U_r as a $0 \le U_r \le 1$ oscillating around the point $|U| = 0$ maintained by the rebound from the potential barrier. Similarly, for a potential $V(U)$ of the form as the line-"b" in Fig.-1-(1), we can also understand the radial-direction component $0 \le U_r \le +\infty$ in two viewpoints: U_r is a stable (physical) field d.o.f U_r^{higgs} oscillating around the stable vacuum $|U| = 1$ (minimum of the potential $V(U)$), which could be seemed as the "traditional" P2 type excitation of "higgs particle"; or, U_r^{P4} is an unstable (unphysical) field d.o.f oscillating around the unstable vacuum $|U| = 0$ (local maximum of the potential $V(U)$), which would "decay/collapse" as what would happen in the more extreme two cases plotted as the line-"c" or line-"d" in Fig.-1-(1)).

However, we will just take the unstable (unphysical) U_r^{P4} d.o.f as the real component in our "untraditional" P4 type U field, with the purpose to design the U field to differ from the "traditional" P2 type field. Thus, from now on, we need not give too many query to the sign of the mass term in (5) any more. As discussed in Section 2.3, we can say: U is a kind of higgs-type field, and U does have a nonzero VEV, however, the U field with E.O.M. $\hat{p}^4 U = m^4 U$ is really designed to be neither a traditional higgs field with E.O.M. $\hat{p}^2 U = -m^2 U$ nor a phantom with E.O.M. $-\hat{p}^2 U = m^2 U$, see (23).

In a word, it should be emphasized that the choice for the sign of the mass term is very important and crucial for our following work.

[2] Note, this kind of acausality is different from the acausality discussed in Ref. [8], which only occurs in the classic mechanics case and can be removed in the framework of quantum mechanics through the uncertainty principle by treating all the observable variables as operators.

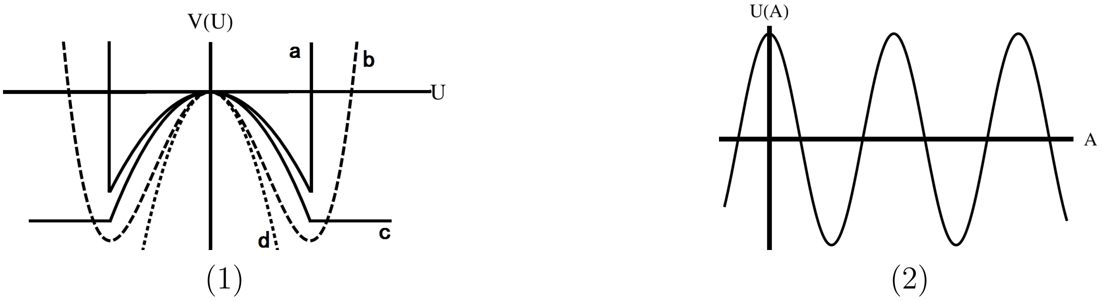

Figure 1: Self-interaction potentials for the field U and A.

3 The kinetics

3.1 The equation of motion of the U field

By the Euler-Lagrange equation [9]

$$\frac{\partial \mathcal{L}_U}{\partial U} - \partial_\mu \frac{\partial \mathcal{L}_U}{\partial(\partial_\mu U)} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}_U}{\partial(\partial_\mu \partial_\nu U)} = 0\,, \tag{27}$$

from (5) we can get the equation of motion(E.O.M) of free field U,

$$\begin{aligned} -\partial^\mu \partial^\nu \partial_\mu \partial_\nu U &= -m_U^4 U + \Lambda_U^4 &(28)\\ \Leftrightarrow \quad -\hat{p}^4 U &= -m_U^4 U + \Lambda_U^4\,,\ \hat{p}^\mu = i\partial^\mu\,, &(29)\end{aligned}$$

and the dynamical E.O.M for U, as

$$-\,\partial^4 U = -m_U^4 U + \Lambda_U^4 + \alpha Q \Lambda\, \bar{\psi}\psi + \ldots\,. \tag{30}$$

The appearance of term $(U + U^\dagger)$ in (5) must be in the combination with the term $U^\dagger U$, by the requirement for a stable vacuum, and, the role of term $(U + U^\dagger)$ is to provide a shift for the position of vacuum, as

$$\begin{aligned} V(U) &= Tr\left[\Lambda_U^4 (U + U^\dagger) - m_U^4 U^\dagger U\right] \\ &= -m_U^4 \left[\left(U - \frac{\Lambda_U^4}{m_U^4}\right)^\dagger \left(U - \frac{\Lambda_U^4}{m_U^4}\right) - (\frac{\Lambda_U^4}{m_U^4})^2\right]. \end{aligned} \tag{31}$$

3.2 The canonic commutator and propagator

Firstly, if we crudely copy the tradition of the quantization procedure for P2 type field theory, then, according to the **custom** on the choice of "±" sign in classic Poisson bracket

$$[p_i, x_j] = -i\delta_{ij}\,, \tag{32}$$

and its quantized version for scalar field

$$\left[\dot{U}_i(\boldsymbol{x}, t), U_i(\boldsymbol{y}, t)\right] = -i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})\,, \tag{33}$$

we just need assign the canonic commutators below to quantize our model:

$$\begin{aligned} \left[\partial^2 \dot{U}_i(\boldsymbol{x}, t), U_i(\boldsymbol{y}, t)\right] &= -i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) &(34)\\ \text{others} &= 0. &(35)\end{aligned}$$

Secondly, by inserting the "correlation function", i.e., one version of the definitions of propagator of U,

$$\begin{aligned} D_F(x-y) &\equiv \langle 0|TU(x)U(y)|0\rangle \\ &= \theta(x^0-y^0)\langle 0|U(x)U(y)|0\rangle + \theta(y^0-x^0)\langle 0|U(y)U(x)|0\rangle \end{aligned} \tag{36}$$

into the E.O.M, we can verify

$$\begin{aligned} & -(\partial^4 - m^4)_x D_F(x-y) \\ \equiv\ & (\partial^4 - m^4)_x \langle 0|TU(x)U(y)|0\rangle = +i\delta^{(4)}(x-y)\,. \end{aligned} \tag{37}$$

That means, $D_F(x-y)$ is really the "Green function", i.e., the other version of the definitions of propagator of U.

By setting $\Lambda_U = 0$, from (37) or its corresponding form in the momentum space

$$-(p^4 - m_U^4)\widetilde{D}_F(p) = \ i\,, \tag{38}$$

we can get the Feynman propagator in the momentum space, as

$$\widetilde{D}_F(p) = \frac{-i}{p^4 - m_U^4 + i\epsilon} = \frac{-i}{(p^2 + m_U^2 - i\epsilon)(p^2 - m_U^2 + i\epsilon)},\ (\Lambda_U = 0), \tag{39}$$

for $m_U \neq 0$, or

$$D_F(U) = \frac{-i}{p^4 + i\epsilon},\ (\Lambda_U = 0,\, m_U = 0). \tag{40}$$

So, the minus sign before the $\hat{p}^4$ operator in the E.O.M (29,30,37,38) is very crucial, which represents the sign of the mass term in Lagrangian, and, without this "-1" factor, everything will be different! After all, the U here isn't the traditional scalar field, as we said in Section 2.4.

The position and residue of a pole in the propagator is crucial for the calculation results of the amplitudes. For a general case, a contour integration in the p^0 complex plan would be equivalent to a complex integration $\int_{-\infty}^{+\infty} dp^0 + \int_{-i\infty}^{+i\infty} dp^0$, however, if we migrate the imaginary unit i in ip^0 into ix^0 for the product $ip\cdot x$ in $e^{-ip\cdot x}$ and treat ix^0 as the temperature T, then, in a zero-temperature field theory, we can omit effects of the two poles $\{iE_U + \epsilon, -iE_U - \epsilon\}$, with $E_U = \sqrt{\boldsymbol{p}_U^2 + m_U^2}$ the energy of U. Besides, for the $m_U = 0$ case, it's much more convenient for us since we can reduce the four simple-poles to just one quadruple-pole. For the $m_U \neq 0$ case, we have shown the results in another work [3].

4 Effective potentials

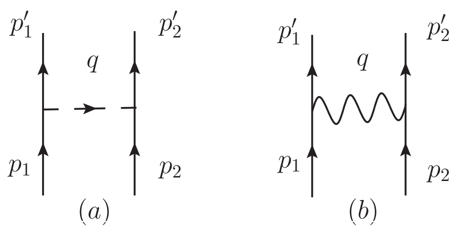

Figure 2: The Feynman diagrams for the leading order tree level processes, with (a) mediated by a U and (b) mediated by a photon A^μ.

At the beginning, we set the variables for the particles in the scattering processes shown in Fig. 2, as below:

$$p_1 = (m, \boldsymbol{p}_1), \qquad p_2 = (m, \boldsymbol{p}_2), \tag{41}$$

$$p_1' = (m, \boldsymbol{p}_1'), \qquad p_2' = (m, \boldsymbol{p}_2') \,. \tag{42}$$

In the non-relativistic approximation, $q^0 = 0$ (which is also called on-shell approximation), we have the relations for kinetics variables as

$$q = p_1 - p_1' \Rightarrow q^2 = (p_1 - p_1')^2 \xrightarrow[q^0=0]{\text{(NR limit)}} -|\boldsymbol{q}|^2 = -|\boldsymbol{p}_1 - \boldsymbol{p}_1'|^2, \tag{43}$$

and

$$\bar{u}^{s'}(p')u^s(p) = 2m\delta^{ss'}, \; \bar{u}^{s'}(p')\gamma^\mu u^s(p) \xrightarrow{\text{(NR limit)}} v^\mu 2m\delta^{ss'} \,. \tag{44}$$

4.1 Interaction I: coupled to intrinsic charges, Coulomb force?

Now, for the interaction term

$$\begin{aligned} \mathcal{L}_{\alpha\beta} &= -\alpha\Lambda Q_\alpha \bar{\psi} \left\{[(U+U^\dagger) + i(U-U^\dagger)]\right\} \psi \\ &\quad -\beta Q_\beta \bar{\psi} \left\{\sigma_{\mu\nu}\partial^\nu[(U+U^\dagger) + i(U-U^\dagger)]\right\} \gamma^\mu \psi, \end{aligned} \tag{45}$$

which was extracted from the total interaction Lagrangian (7), by defining the couplings

$$\alpha_{1,2} \equiv \alpha(Q_\alpha)_{1,2}, \; \beta_{1,2} = \beta(Q_\beta)_{1,2}, \tag{46}$$

and by using $i\sigma_{\mu\nu}q^\nu = -q_\mu$, $\gamma^\mu \to v^\mu$ from (44), we can write the corresponding amplitude for Fig. 2-(a), as[3]

$$\begin{aligned} i\mathcal{M}_a &= \bar{u}^{s'} i[-\alpha_1\Lambda - \beta_1\sigma_{\mu\nu}(iq^\nu)\gamma^\mu]u^s \cdot \frac{-i}{q^4} \\ &\quad \cdot \bar{u}^{r'} i[-\alpha_2\Lambda - \beta_2\sigma_{\alpha\beta}(-iq^\beta)\cdot\delta_{\mu\alpha}\gamma^\alpha]u^r \\ &= \bar{u}^{s'} i[-\alpha_1\Lambda + \beta_1 q_\mu\gamma^\mu]u^s \cdot \frac{-i}{q^4} \cdot \bar{u}^{r'} i[-\alpha_2\Lambda - \beta_2 q_\alpha \cdot \delta_{\mu\alpha}\gamma^\alpha]u^r \\ &= \bar{u}^{s'}[-i\alpha_1\Lambda]u^s \cdot \frac{-i}{q^4} \cdot \bar{u}^{r'}[-i\alpha_2\Lambda]u^r \\ &\quad +[\alpha_2\Lambda\beta_1(q\cdot v_1) - \alpha_1\Lambda\beta_2(q\cdot v_2)] \cdot \frac{-i}{q^4} \cdot \bar{u}^{s'}u^s\bar{u}^{r'}u^r \\ &\quad +\bar{u}^{s'}[i\beta_1\gamma^\mu]u^s \cdot \frac{-ig_{\mu\alpha}}{q^2} \cdot \bar{u}^{r'}[-i\beta_2\gamma^\alpha]u^r, \end{aligned} \tag{47}$$

where the indices in $\delta_{\mu\alpha}$ are not controlled by the Einstein summation convention, and we have taken the replacement

$$q_\mu q_\alpha \delta_{\mu\alpha} \to q^2 g^{\mu\alpha}. \tag{48}$$

The use of the $\delta_{\mu\alpha}$ in (47) could be understood by this reason: as there is only the single U field exchanged in a $2 \to 2$ scattering process, once the μ-component of the momentum q^μ of U is absorbed into one vertex in the Feynman diagram, there must be the same μ-component of the momentum q^μ is emitted out from the other vertex in the Feynman diagram! Thus, by treating $\partial^\mu U \sim \mathcal{A}^\mu$ as an P2 type effective media field as in (14), for which there is a propagator with the form of $\frac{-ig^{\mu\alpha}}{q^2}$, we can see, the last term in (47) is just of the form of an amplitude for a scattering process corresponding to the Coulomb potential in QED, as shown in Fig.2-(b).

[3]For simplicity, here we can only consider the contributions from U_1, and, for the contributions from U_2, the result just need a double.

In the NR limit of $q^0 = 0$, with the definitions

$$\boldsymbol{q} \cdot \boldsymbol{v}_1 \equiv \lambda_1 |\boldsymbol{q}|,\ \boldsymbol{q} \cdot \boldsymbol{v}_2 \equiv \lambda_2 |\boldsymbol{q}|, \tag{49}$$

and the approximation $\gamma^\mu \gamma^\alpha g_{\mu\alpha} \to \gamma^0 \gamma^0 g_{00} = 1$, we can continue to get

$$i\mathcal{M}_a \ = \ -i \left\{ -\frac{\alpha_1 \alpha_2 \Lambda^2}{|\boldsymbol{q}|^4} + \frac{(\alpha_2 \beta_1 \lambda_1 - \alpha_1 \beta_2 \lambda_2)\Lambda}{|\boldsymbol{q}|^3} - \frac{\beta_1 \beta_2}{|\boldsymbol{q}|^2} \right\} \cdot \bar{u}^{s'} u^s \cdot \bar{u}^{r'} u^r, \tag{50}$$

The amplitude $i\mathcal{M}$ should be compared with the Born approximation to the scattering amplitude in non-relativistic quantum mechanics, written in terms of the potential function $V(\boldsymbol{x})$: [2]

$$i\mathcal{M} \sim {}_{NR}\langle p' | iT | p \rangle_{NR} = -i\widetilde{V}(\boldsymbol{q})(2\pi)\delta(E_{\boldsymbol{p}'} - E_{\boldsymbol{p}}),\ \boldsymbol{q} = \boldsymbol{p} - \boldsymbol{p}', \tag{51}$$

with

$$\boldsymbol{p} = \eta_2 \boldsymbol{p}_1 - \eta_1 \boldsymbol{p}_2,\ \boldsymbol{p}' = \eta_2 \boldsymbol{p}'_1 - \eta_1 \boldsymbol{p}'_2,\ \eta_i = \frac{m_i}{m_1 + m_2},\ i = 1, 2. \tag{52}$$

By dealing with the kinetics factors as $2m\delta^{ss'} \to \delta^{ss'}$ and $(2\pi)\delta(E_{\boldsymbol{p}'} - E_{\boldsymbol{p}}) \to 1$, we can have

$$\widetilde{V}(\boldsymbol{q}) = -\frac{\alpha_1 \alpha_2 \Lambda^2}{|\boldsymbol{q}|^4} + \frac{(\alpha_2 \beta_1 \lambda_1 - \alpha_1 \beta_2 \lambda_2)\Lambda}{|\boldsymbol{q}|^3} - \frac{\beta_1 \beta_2}{|\boldsymbol{q}|^2}, \tag{53}$$

and the inverse Fourier transformation

$$V(\boldsymbol{x}) = \mathcal{F}^{-1}[\widetilde{V}(\boldsymbol{q})]\,. \tag{54}$$

Then, we can get the potential

$$V(r) = +\frac{\alpha_1 \alpha_2 \Lambda^2}{8\pi} r - \frac{(\alpha_2 \beta_1 \lambda_1 - \alpha_1 \beta_2 \lambda_2)\Lambda}{2\pi^2} (\log \frac{r}{r_0} + \gamma_E - 1) + \frac{-\beta_1 \beta_2}{4\pi r}, \tag{55}$$

with $r_0 = 1\, GeV^{-1}$ put by hand to balance the dimension, and γ_E the Euler constant. Moreover, by applying (9,10,11, 46) to get[4]

$$\begin{aligned}
& \text{(for QED:)} \\
\alpha_1 \alpha_2 \ &= \ \alpha^2 (Q_\alpha)_1 (Q_\alpha)_2 = \alpha^2 (Y \cdot 1)_1 (Y \cdot 1)_2 = -\alpha^2, && (56) \\
-\beta_1 \beta_2 \ &= \ -\beta^2 (Q_\beta)_1 (Q_\beta)_2 = -\beta^2 (Y \mathcal{Q}_\beta)_1 (Y \mathcal{Q}_\beta)_2 \\
&= \ -\beta^2 (Y_1 \mathcal{Q}_1^{QED})(Y_2 \mathcal{Q}_2^{QED}) = \beta^2 \mathcal{Q}_1^{QED} \mathcal{Q}_2^{QED}, && (57) \\
\alpha_1 \beta_2 \ &= \ \alpha\beta (Q_\alpha)_1 (Q_\beta)_2 = \alpha\beta (Y \cdot 1)_1 (Y Q^{QED})_2 = -\alpha\beta \mathcal{Q}_2^{QED}, && (58) \\
\alpha_2 \beta_1 \ &= \ \alpha\beta (Q_\alpha)_2 (Q_\beta)_1 = \alpha\beta (Y \cdot 1)_2 (Y Q^{QED})_1 = -\alpha\beta \mathcal{Q}_1^{QED}, && (59)
\end{aligned}$$

with $\mathcal{Q}^{QED}$ just the electric charge in QED, and

$$\begin{aligned}
& \text{(for QCD:)} \\
\alpha_1 \alpha_2 \ &= \ \alpha^2 (Q_\alpha)_1 (Q_\alpha)_2 = \alpha^2 (Y \cdot Y)_1 (Y \cdot Y)_2 = \alpha^2, && (60) \\
-\beta_1 \beta_2 \ &= \ -\beta^2 \mathcal{Q}_1^{QCD} \mathcal{Q}_2^{QCD}, && (61) \\
\alpha_1 \beta_2 \ &= \ \alpha\beta (Y \cdot Y)_1 (Y Q^{QCD})_2 = \alpha\beta (Y_2 Q_2^{QCD}) = \alpha\beta \mathcal{Q}_2^{QCD}, && (62) \\
\alpha_2 \beta_1 \ &= \ \alpha\beta (Y \cdot Y)_2 (Y Q^{QCD})_1 = \alpha\beta (Y_1 Q_1^{QCD}) = -\alpha\beta \mathcal{Q}_1^{QCD}, && (63)
\end{aligned}$$

[4] It is very important to set $Y_1 = -1$ and $Y_2 = +1$ to match the Feynman rules used in Eq. (47)!

with $\mathcal{Q}^{QCD}$ just the color charge in QCD, by combining with (46,56,57,58,59), the potential in (55) will become

$$\begin{aligned} V(r)_{QED} &= -\frac{\alpha^2\Lambda_{QED}^2}{8\pi}\cdot r + \frac{\alpha\beta\Lambda_{QED}(\mathcal{Q}_1^{QED}\lambda_1 - \mathcal{Q}_2^{QED}\lambda_2)}{2\pi^2}\cdot\log\frac{r}{r_0} \\ &\quad + \frac{\beta^2\mathcal{Q}_1^{QED}\mathcal{Q}_2^{QED}}{4\pi r}, \end{aligned} \tag{64}$$

$$\begin{aligned} V(r)_{QCD} &= \frac{\alpha^2\Lambda_{QCD}^2}{8\pi}\cdot r + \frac{\alpha\beta\Lambda_{QCD}(\mathcal{Q}_1^{QCD}\lambda_1 + \mathcal{Q}_2^{QCD}\lambda_2)}{2\pi^2}\cdot\log\frac{r}{r_0} \\ &\quad - \frac{\beta^2\mathcal{Q}_1^{QCD}\mathcal{Q}_2^{QCD}}{4\pi r}, \end{aligned} \tag{65}$$

By recalling that we have performed the derivations in the framework of a collision process in the center-of-mass frame, that is to say, $\boldsymbol{p}_1 = -\boldsymbol{p}_2$, by combining the on-shell approximation $|\boldsymbol{p}_1| = |\boldsymbol{p}_1'|$, we can indeed determine the relations

$$\begin{aligned} &\boldsymbol{q}\cdot(\boldsymbol{p}_1+\boldsymbol{p}_2) = (m_1\lambda_1 + m_2\lambda_2)|\boldsymbol{q}| = 0, \\ &\boldsymbol{q}\cdot(\boldsymbol{p}_1-\boldsymbol{p}_2) = (m_1\lambda_1 - m_2\lambda_2)|\boldsymbol{q}| > 0, \\ &(\boldsymbol{q}\cdot\boldsymbol{p}_1)(\boldsymbol{q}\cdot\boldsymbol{p}_2) = m_1 m_2\lambda_1\lambda_2|\boldsymbol{q}|^2 < 0. \end{aligned} \tag{66}$$

So, for the case of $m_1 = m_2$, we will have $\lambda_1 - \lambda_2 > 0$ and $\lambda_1 + \lambda_2 = 0$. As in (47), we can see again, the last term in (64) or (65) is coincidentally for the Coulomb interaction in QED or QCD, respectively!

Besides, there is a linear potential and a logarithmic potential in both (64) and (65). In (64), since the infrared energy scale boundary Λ_{QED} for the QED is about zero, the linear potential and the logarithmic potential could be negligible; however, in some cosmological experiments, the linear and the logarithmic term might give corrections to the electromagnetic observables such as the red-shift, e.g., an indications of a spatial variation of the electromagnetic fine structure constant [10]. In(65), since the infrared energy scale boundary Λ_{QCD} for the QCD is about $200\,MeV$, the linear potential could be significant to serve as the major part of the confinement in QCD, while the logarithmic potential could serve as a minor part of the confinement.

4.2 Interaction II: coupled to momentum, gravity?

Now we consider the interaction terms,

$$\begin{aligned} \mathcal{L}_I &= -\alpha\Lambda Q_\alpha\,\bar{\psi}\left\{[(U+U^\dagger) + i(U-U^\dagger)]\right\}\psi \\ &\quad -\xi\frac{1}{M}Q_\xi\bar{\psi}\left\{\sigma_{\mu\nu}\partial^\mu[(U+U^\dagger)+i(U-U^\dagger)]\right\}(i\overleftrightarrow{\partial}^\nu\psi), \end{aligned} \tag{67}$$

which was extracted from the total interaction Lagrangian (7). By defining the couplings as in (46)

$$\alpha_{1,2} \equiv \alpha(Q_\alpha)_{1,2},\ \xi_{1,2} \equiv \xi(Q_\xi)_{1,2} \tag{68}$$

and by using $i\sigma_{\mu\nu}q^\nu = -q_\mu$, $q_\nu q_\beta \delta_{\nu\beta} \to q^2 g_{\nu\beta}$ as in (47,48), we can write the corresponding amplitude for Fig. 2-(a), as

$$\begin{aligned} i\mathcal{M} &= \bar{u}^{s'} i \left\{ -\alpha_1\Lambda - \frac{\xi_1}{M}\sigma_{\mu\nu}\cdot(iq^\mu)\cdot[i\cdot i(p_1+p_1')^\nu] \right\} u^s \cdot \frac{-i}{q^4} \\ &\cdot \bar{u}^{r'} i \left\{ -\alpha_2\Lambda - \frac{\xi_2}{M}\sigma_{\alpha\beta}\cdot(-iq^\alpha)\cdot\delta_{\nu\beta}\cdot\left[i\cdot i(p_2+p_2')^\beta\right] \right\} u^r \\ &= \frac{-i}{q^4}\cdot\left\{ -\alpha_1\alpha_2\Lambda^2 + \frac{\Lambda}{M}q\cdot[\alpha_2\xi_1(p_1+p_1') - \alpha_1\xi_2(p_2+p_2')] \right. \\ &\left. + \frac{\xi_1\xi_2}{M^2}\cdot q_\nu q_\beta \delta_{\nu\beta}\cdot(p_1+p_1')^\nu(p_2+p_2')^\beta \right\}\cdot\bar{u}^{s'}u^s\cdot\bar{u}^{r'}u^r \\ &= -i\cdot\left\{ -\frac{\alpha_1\alpha_2\Lambda^2}{|\boldsymbol{q}|^4} - \frac{\Lambda}{M}\cdot\left[\frac{(2\alpha_2\xi_1\boldsymbol{q}\cdot\boldsymbol{p}_1 - 2\alpha_1\xi_2\boldsymbol{q}\cdot\boldsymbol{p}_2)}{|\boldsymbol{q}|^4}\right] \right. \\ &+ \frac{\Lambda}{M}\cdot\left[\frac{(\alpha_1\xi_2+\alpha_2\xi_1)}{|\boldsymbol{q}|^2}\right] - \frac{\xi_1\xi_2}{M^2}\cdot\frac{4p_1\cdot p_2}{|\boldsymbol{q}|^2} \\ &\left. + \frac{\xi_1\xi_2}{M^2}\cdot\frac{2(\boldsymbol{q}\cdot\boldsymbol{p}_1 - \boldsymbol{q}\cdot\boldsymbol{p}_2)}{|\boldsymbol{q}|^2} - \frac{\xi_1\xi_2}{M^2} \right\}\cdot 2m\delta^{ss'}2m\delta^{rr'},\ |\boldsymbol{q}|>0. \end{aligned} \tag{69}$$

In the non-relativistic limit, with $\boldsymbol{p}_{1,2} = m_{1,2}\boldsymbol{v}_{1,2}$, and the definitions

$$\boldsymbol{q}\cdot\boldsymbol{v}_1 \equiv \lambda_1|\boldsymbol{q}|,\ \boldsymbol{q}\cdot\boldsymbol{v}_2 \equiv \lambda_2|\boldsymbol{q}|, \tag{70}$$

we can get

$$\boldsymbol{q}\cdot\boldsymbol{p}_1 = m_1\lambda_1|\boldsymbol{q}|, \quad \boldsymbol{q}\cdot\boldsymbol{p}_2 = m_2\lambda_2|\boldsymbol{q}|. \tag{71}$$

Thus the non-relativistic effective potential in the momentum space will be

$$\begin{aligned} & \widetilde{V}(\boldsymbol{q}) = -\mathcal{M} \\ = & -\frac{\alpha_1\alpha_2\Lambda^2}{|\boldsymbol{q}|^4} - \frac{\Lambda}{M}\cdot\left[\frac{(2\alpha_2\xi_1 m_1\lambda_1 - 2\alpha_1\xi_2 m_2\lambda_2)}{|\boldsymbol{q}|^3}\right] \\ & + \frac{\Lambda}{M}\cdot\left[\frac{(\alpha_1\xi_2+\alpha_2\xi_1)}{|\boldsymbol{q}|^2}\right] - \frac{\xi_1\xi_2}{M^2}\cdot\frac{4p_1\cdot p_2}{|\boldsymbol{q}|^2} \\ & + \frac{\xi_1\xi_2}{M^2}\cdot\frac{2(m_1\lambda_1 - m_2\lambda_2)}{|\boldsymbol{q}|} - \frac{\xi_1\xi_2}{M^2},\ |\boldsymbol{q}|>0. \end{aligned} \tag{72}$$

The last term in (72), $-\frac{\xi_1\xi_2}{M^2}$, is effective to a Feynman rule of a vertex for a four-fermion contact term, so we will drop it in the non-relativistic limit due to the probability conservation law in the non-relativistic quantum mechanics framework.

Then, by performing the inverse Fourier transformation $V(\boldsymbol{x}) = \mathcal{F}^{-1}[\widetilde{V}(\boldsymbol{q})]$, we can get the potential in the coordinate space (with $|\boldsymbol{q}| > 0$ equivalent to a step function $\theta(|\boldsymbol{q}|)$, γ_E the Euler constant) as

$$\begin{aligned} V(r) &= \frac{\alpha_1\alpha_2\Lambda^2}{8\pi}r - \frac{\Lambda(2\alpha_2\xi_1 m_1\lambda_1 - 2\alpha_1\xi_2 m_2\lambda_2)}{M}\cdot\left[-\frac{1}{2\pi^2}(\log\frac{r}{r_0}+\gamma_E-1)\right] \\ &+ \frac{\Lambda(\alpha_1\xi_2+\alpha_2\xi_1)}{M}\cdot\frac{1}{4\pi r} - \frac{4\xi_1\xi_2 p_1\cdot p_2}{M^2}\cdot\frac{1}{4\pi r} \\ &+ \frac{2\xi_1\xi_2(m_1\lambda_1 - m_2\lambda_2)}{M^2}\cdot\frac{1}{4\pi^2 i r}\delta(r), \quad (r>0), \end{aligned} \tag{73}$$

with $r_0 = 1\,GeV^{-1}$ put by hand to balance the dimension. The last term of the $\delta(r)$ function in (73) is from the $\frac{1}{|\boldsymbol{q}|}$ term in (72), and it could also be dropped due to $r \neq 0$. At last, with the values

$Y_1 = -1$, $Y_2 = +1$, we have

$$\begin{aligned} \alpha_1 \xi_2 &= \alpha\xi (Q_\alpha)_1 (Q_\xi)_2 = \alpha\xi (Y \cdot 1)_1 (Y \cdot Y)_2 = -\alpha\xi, && (74) \\ \alpha_2 \xi_1 &= \alpha\xi (Q_\alpha)_2 (Q_\xi)_1 = \alpha\xi (Y \cdot 1)_2 (Y \cdot Y)_1 = \alpha\xi, && (75) \\ \xi_1 \xi_2 &= \xi^2 (Q_\xi)_1 (Q_\xi)_2 = \xi^2 (Y \cdot Y)_1 (Y \cdot Y)_2 = \xi^2, && (76) \end{aligned}$$

by combining with $\alpha_1\alpha_2 = -\alpha^2$ in (56) and $m_1\lambda_1 + m_2\lambda_2 = 0$ in (66), we can get the potential form

$$V(r) = -\frac{\alpha^2 \Lambda^2}{8\pi} r - \frac{4\xi^2 p_1 \cdot p_2}{M^2} \cdot \frac{1}{4\pi r}, \quad r > 0. \tag{77}$$

As expected, the linear potential also arises in (77) is the same as in (64), which could be corresponding to the dark energy effects. And the second term in (77) is happily to be the Newton's gravity form! Besides, a potential term with form of $-\frac{\boldsymbol{v}^2}{r}$ included in the factor

$$p_1 \cdot p_2 = p_1^0 p_2^0 - \boldsymbol{p}_1 \cdot \boldsymbol{p}_2 \simeq \frac{m_1}{\sqrt{1 - \boldsymbol{v}_1^2}} \cdot \frac{m_2}{\sqrt{1 - \boldsymbol{v}_2^2}} + m_1^2 |\boldsymbol{v}_1|^2 \tag{78}$$

with $\boldsymbol{p}_2 = -\boldsymbol{p}_1$ in the center-of-mass frame, can be treated as the one of the source of the dark matter effects [11], which is just of a relativistic effects! Moreover, there will be an extra relativistic corrections from the spinor basis $u^s(p)$, by instead m to p^0 in (44), as

$$\bar{u}^{s'}(p') u^s(p) = 2p^0 \delta^{ss'}, \; \bar{u}^{s'}(p') \gamma^\mu u^s(p) \xlongequal{\text{(NR limit)}} v^\mu 2m \delta^{ss'} . \tag{79}$$

Besides, we want to point out that, for a N-body system, potential terms in (77) will be additive and they will be enlarged only by the factor $(NQ_1) \cdot (NQ_2)$, rather than $(N\xi_1)(Nm_1) \cdot (N\xi_2)(Nm_2)$.

5 Induced theories in some limit cases

5.1 Effects of the nonzero $\langle U \rangle$

Now that U is a kind of higgs field, it should show its higgs-like property. According to the higgs mechanism, with the interaction term $\alpha\Lambda\bar{\psi}U\psi$ in (7), the fermions will get a mass correction

$$\triangle m \sim \alpha\Lambda \langle U \rangle. \tag{80}$$

If we set $\langle U \rangle_G = \frac{1}{L} \simeq 10^{-41} GeV$ as the gauge symmetry breaking energy scale of gravitation, with $L \simeq 10^{11} l.y.$ corresponding to the size of the universe, and $\langle U \rangle_{EW} \simeq 10^2 GeV$ as the gauge symmetry breaking energy scale of electroweak interaction, we will get a lucky coincidence for the ratio of the magnitudes of Newton's gravity force F_G and the Coulomb force F_C,

$$\frac{F_G}{F_C} = \left[G \left(\frac{m_e}{e} \right)^2 \frac{e^2}{r^2} \right] / \left[k \frac{e^2}{r^2} \right] \simeq 10^{-43} \to \frac{\langle U \rangle_G}{\langle U \rangle_{EW}} \tag{81}$$

where m_e is the mass of electron, $k \simeq 9 \times 10^9 (N \cdot m^2 \cdot C^{-2})$ is the Coulomb constant(in SI unit). If this is true, we might say, the smallness of gravitation constant G comes from its small VEV $\langle U \rangle_G$ (or the huge size of the universe).

Furthermore, if we set $\langle U \rangle_{TC}$ as the gauge symmetry breaking energy scale of the technicolor (TC) interaction [7], and the ratio

$$\frac{\langle U \rangle_{TC}}{\langle U \rangle_{EW}} \simeq \frac{g_s^2}{e^2} \simeq\simeq \frac{0.1^2}{0.01^2} = 100 \tag{82}$$

will give us a value of $\langle U \rangle_{TC} \simeq 10^4 \, GeV = 10 \, TeV$ for the typical technicolor energy scale.

5.2 Field U out a nutshell: generation of nonlinear Klein-Gordon equation

Here we need the self-interaction term of U, which could be written as

$$\mathcal{L}_I = -g_U \Lambda_U^2 U \partial_\mu U \partial^\mu U + m_U^4 U^2 \,. \tag{83}$$

For a pure U-field system, if its kinetic energy is very small, down to $p^2 \ll \Lambda_U \Lambda$ (or, in the sense of de Broglie wavelength, we can say, the system is "out of a nutshell"), then the kinetic energy term could be dropped, then we can get a E.O.M for U according to the Euler-Lagrangian equation, as

$$g_U \Lambda_U^2 (\partial U)^2 - 2 g_U \Lambda_U^2 U \partial^2 U = m_U^4 U \;\Rightarrow\; (\partial U)^2 - 2U\partial^2 U = \frac{m_U^4}{g_U \Lambda_U^2} U \,. \tag{84}$$

Apparently, that is a nonlinear 2nd-order D.E., so, we just call it "nonlinear Klein-Gordon equation". Particularly, for a special case, $\langle U\rangle \gg U - \langle U\rangle$ (i.e., the VEV large and the fluctuation small) and $\langle U\rangle \gg \partial U$ (i.e., the VEV large and the kinetic energy small), we can get the "linear" Klein-Gordon equation

$$-\partial^2 U = \frac{m_U^4}{2 g_U \langle U\rangle \Lambda_U^2} U \,, \tag{85}$$

and there should be the relation $2 g_U \langle U\rangle \Lambda_U^2 = m_U^2$. As said for (26), we should remind ourselves that $\langle U\rangle$ could be very large even when the energy scale is very low!

In a Lagrangian, there should be both the kinetic energy terms and the potential energy terms. However, there exists the freedom to choose which ones are the kinetic energy terms and which ones are the potential energy terms, that depends the choice of the d.o.f of the system. This is a kind of "kinetic-potential duality".

5.3 The constraint $U_1^2 + U_2^2 = \langle U\rangle^2$ in a weak field limit

5.3.1 U as a group element: the generation of gauge field A^μ

In the limit case of $U_1^2 + U_2^2 = \langle U\rangle^2$,

$$U = U_1 + iU_2 = \sigma(x) e^{-i\phi(x)} \to \langle U\rangle e^{-ig\phi(x)}, \tag{86}$$

that is to say, U becomes a group element, and the superficial **gauge symmetry** of the Lagrangian arises!

In (86), U_1 and U_2 are both P4 type field, and σ and ϕ are also both P4 type field; σ is purely unphysical field (i.e.,tachyon/instanton/phantom), while ϕ is physical field, as said in Sect. 2.4. Is the $\phi(x)$ really a detectable field? Mathematically to say, ϕ is a phase, and we can write

$$U \simeq \langle U\rangle e^{-ig\phi(x)} \to \langle U\rangle e^{-ig[\phi_0(x) + \epsilon n_\mu A^\mu(x) + \epsilon n_{\mu\nu} A^{\mu\nu}(x) + \ldots]} \,, \tag{87}$$

that means, the P4 type ϕ field can be generated by many different fields rather than only one field $\phi_0(x)$.

If only the $A^\mu(x)$ field is nonzero in (87), then, with

$$\bar\psi (U \partial\!\!\!/ U^\dagger) \psi \to e \bar\psi A\!\!\!/ \psi, \tag{88}$$

as a 4-particle-coupling term becoming to a 3-particle-coupling term, we get the gauge interaction term, with

$$\beta \cdot \langle U\rangle^2 = e \,. \tag{89}$$

Now, instead of the d.o.f. of $\phi(x)$, there exists a connection field (gauge filed) $A^\mu(x)$, induced by the Maurer-Cartan 1-form of $U(x)$ field. We name the constraint

$$U_1^2 + U_2^2 = \langle U\rangle^2, A^\mu(x) \neq 0, \phi_0(x) = A^{\mu\nu}(x) = ... = 0 \tag{90}$$

as "**Light Constraint**", in the reason that it survive only the field A^μ with the light speed after freezing the unphysical tachyon d.o.f. $\sigma(x)$ in (86) with speed over the light.

However, when both U_1 and U_2 are excited, the contribution of the massless U field includes an effect of a massless gauge field $A^\mu(x)$, see Fig. 2-(a). Now, as both the $\bar{\psi}\partial\!\!\!/ U^\dagger\psi$ term and the $\bar{\psi}A^\mu\psi$ term can generate the Coulomb potential, we would like to ask, is the gauge symmetry is necessary? We will return this question in Sect. 6.

5.3.2 Multi-vacuum structure for sine-Gordon type vector field A^μ

1. Multi-vacuum structure for A^μ

If we write [5]

$$U(x) = \exp[-ig\epsilon n^\mu A_\mu(x)] = \cos[g\epsilon n^\mu A_\mu(x)] - i\sin[g\epsilon n^\mu A_\mu(x)], \tag{91}$$

then the potential term

$$V(A) \sim U(A) + U^\dagger(A) = \cos[(g\epsilon)\cdot A], \tag{92}$$

would mean that the dynamics for the field A^μ is of a sine-Gordon type (or, a kind of generalized higgs type vector), see Fig. 1-(2). Thus, there might be many excitations for A^μ at different vacuums (or, VEVs), with heavy masses in the large g cases($g\epsilon \simeq 1$) and small masses in the small g cases.

2. Fermion mass spectrum with generation structure

Like the mass correction in (80) from U, with the term $\bar{\psi}A\!\!\!/\psi$, the fermions can get a mass correction from A^μ,

$$\Delta m \sim \alpha\Lambda\langle A\rangle \sim \alpha\Lambda\frac{(2n+1)\pi}{g\epsilon}\ ,\ n = 0, 1, 2, ... \tag{93}$$

where the number n might lead the fermion mass spectrum to a generation structure. Even for the same value of n, we can get the suppositions below:

a. if Δm is the mass differences between the current quarks and the constituent quarks, then, by setting

$$g \sim \frac{(2n+1)\alpha\Lambda}{\Delta m\cdot\epsilon} \xrightarrow{\mathcal{O}(\Lambda)\sim\mathcal{O}(\epsilon)} \frac{(2n+1)\alpha}{\Delta m} \sim 1\,, \tag{94}$$

with $\Delta m \sim 1GeV$ and $n = 0$, we have $\alpha \sim 1$.

b. if $g \sim 0.01$ for the E.W. interaction, then, $\Delta m \sim 100GeV$, corresponding to the possible heavy fermions.

3. A seesaw mechanism for gauge symmetry and flavor symmetry

See Fig. 1-(2), with (92), for a vacuum at $A = \langle A\rangle_i$, the potential could be written as

$$V(A \simeq A_i) \simeq -1 + (g\epsilon)^2(A - A_i) + \ldots, \tag{95}$$

which means the mass of the excitation $A' = A - \langle A\rangle_i$ is of order $\sim m = g\epsilon$. So, we can get the conclusions below:

(1) when $g \to 0$,

a. A'_μ is nearly massless, so the gauge symmetry is restored;

[5] As said for (26), we should remind ourselves that $\langle U\rangle$ could be very large even when the energy scale is very low!

b. the VEV $\langle A\rangle_i$ are of very different magnitudes, so, through (93), the fermion masses would be also of very different magnitudes, including very heavy fermions; this is a kind of flavor symmetry breaking for fermions;

(2) when $g \to \infty$,

a. A'_μ is massive, with the diagonal elements in its mass matrix being large, so the gauge symmetry is broken;

b. since the unphysical d.o.f (i.e.,tachyon/instanton/phantom) σ in (86) was excited now, the vacuum tunnelling (oscillating) effect would become strong, so the off-diagonal elements in the mass matrix of A'_i become large, too; or, in another viewpoint, now it's A'_μ that was frozen, and the tachyon was the real d.o.f for mediating interactions; we can treat the tachyon massless or nearly massless according to the absence of heavy bosons in a hadron;

c. the VEV $\langle A\rangle_i$ in the neighbour minimum are nearly equal, so, there would be a degenerate for the fermion mass, or, we can say, the flavor symmetry for fermions would be restored; besides, it's now allowed for very small fermion masses through (93), which might be an underlying reason for the feasibility of the "large N_c" or "large N_f" hypothesis for a real hadron, and for the possible neutrino oscillation.

So, maybe this is a new kind of dynamical symmetry breaking/restoring mechanism, with a see-saw for gauge symmetry and flavor symmetry.

5.3.3 Current to Field: from non-renormalizable to renormalizable

Besides the media field U, we can also treat the fermion matter field ψ as P4 type field. For convenience, we choose a scalar matter field ϕ and take the scalar QED as an example to illustrate our motivation.

If we treat the field ϕ as P4 type field, then the P2 type current of ϕ will become a P2 type field, as

$$\begin{aligned} J^\mu(x) = \phi^\dagger i\partial^\mu \phi(x) &\rightarrow \Phi^\dagger i\partial^\mu \Phi(x) \equiv \mathcal{A}^\mu(x), \\ (\text{P2 type field } \phi &\rightarrow \text{P4 type field } \Phi) \rightarrow (\text{Maurer-Cartan 1-from of } \Phi), \\ (\text{P2 type current } J^\mu) &\rightarrow (\text{P2 type field}). \end{aligned} \tag{96}$$

It is reasonable for (96), since the only difference between a current and a vector field is that: a field has a E.O.M, while a current hasn't; for other things, they could be treated as the same.

Thus, with the Light Constraint in (90), the old P2 type (nonrenormalizable) 3-particle interaction term will become a new 2-particle mixing term (which is also a kind of "kinetic-potential duality"), as

$$\mathcal{L}_I = eA_\mu \phi i \overleftrightarrow{\partial}^\mu \phi \rightarrow eA_\mu \Phi i \overleftrightarrow{\partial}^\mu \Phi = eA_\mu \mathcal{A}^\mu = \mathcal{L}_K. \tag{97}$$

Now the new d.o.f. will be A_μ and $\mathcal{A}^\mu$; and, the new 3-particle interaction term at leading-order will be the old P2 type high-order interaction term $\Phi\Phi A\cdot A$, which will be renormalizable. This method could be a useful reference to renormalization of the chiral perturbative theory and the gravitation theory.

5.3.4 A^μ back to excited U_1 and U_2: from non-perturbative to perturbative

Instead of the gauge field A^μ, the employment of the Wilson line $U(y,x)$ and Wilson loop $U_P(x,x)$, which are defined as [2]

$$U_P(x+\epsilon n, x) = 1 - ig\epsilon n^\mu A_\mu(x) + \mathcal{O}((g\epsilon)^2), \tag{98}$$

$$U_{P_{ij}}(x,x) = 1 - i\epsilon^2 g F_{ij} + \mathcal{O}(\epsilon^3), \tag{99}$$

ensured the availability of lattice gauge theory. It is just this subtle hint that inspired us to consider a field U, with a hidden correspondence of the Wilson loop U_P,

$$U_P \to U\,, \tag{100}$$

rather than the gauge field A as a possible effective d.o.f., with the Light Constraint in (90)

$$gA_\mu = U(x) i\partial_\mu U^\dagger(x), \text{ (U is a group element).} \tag{101}$$

Thus, as an inverse procedure, it is a useful try to solve the non-perturbative problem in stong QED by defining a P4 type complex scalar field $U = U_1 + iU_2$ with U_1 and U_2 are both excited.

6 The causality in theory with high-order differential equations

Let's go back to the causality problem mentioned in Section 2.3. As discussed in Ref. [8], in classic mechanics there exist acausal solutions for dynamical equations with derivative orders higher than 2, however, this kind of causality breaking can be removed in the framework of quantum mechanics through the uncertainty principle by treating all the observable variables as operators.

Here are the different expressions to causality in classic mechanics and quantum mechanics:

(a) in classic mechanics, the causality depends on the interval of the variables in the coordinate space, i.e., whether the interval is time-like or not;

(b) in the **Heisenberg picture** for quantum mechanics, the causality depends on the "interval" (defined by the commutator) of the variables (operators) in the algebra space, i.e., whether the "interval" is time-like or not;

(c) in the path integral framework, the causality depends on the time-order operator inserted for the Feynman propagators due to the retard potential boundary condition; etc.

However, in our P4 type field theory framework, we want to say, the causality arises with the vacuum symmetry breaking, i.e., the causality can only be well-defined after the VEV of a P4 type field is fixed. Some explanations are listed as below.

(1) In the limit of $V(U) = -m_U^4 U^\dagger U = +\lambda_U \Lambda_U^4 U^\dagger U U^\dagger U = 0$ in (25), the fields U_1 and U_2 (or, $\sigma(x)$ and $\phi(x)$) in (86) are both excited, and now, the higgs/tachyon/instanton/phatom effects are excited completely, which will be reflected in the detectable world. Now we might introduce a kind of symmetry between the inner region and the outer region of the light cone (i.e., between the time-like region and the space-like region), denoted as the "**Light Symmetry**".

(2) In the case of $0 < m_U, \lambda_U \ll 1$, only the physical d.o.f $e^{-ip\cdot x}$ survives as $t \to \infty$, and their residual effect can be detectable all the time; although the so-called nonphysical d.o.f $e^{-p\cdot x}$ in (23,24) are unstable, and their residual effect can be detectable until $t \to \infty$ (no matter the momentum p of field U is large or small; especially, they are fast-decay when p is large); besides, if we treat the solution $e^{-p\cdot x}$ as a stable solution in the imaginary time it or a semi-stable solution in the complex-value time $\tau + it$, then the time arrow (the causality) would become ill-defined.

(3) In the "Light constraint", $U_1^2 + U_2^2 = \langle U \rangle^2$, see (90) in Sect. 5.3.1, the gauge symmetry would arise automatically, then, only the physical speed value $c = 1$ for the light would become well-defined; as the "Light Symmetry" is broken, the light cone and the causality would become well-defined. On the other hand, if the "Light constraint" is not rigidly satisfied, then the light speed would fluctuate; generally, the "Light Symmetry" breaking process should be a dynamical process within finite time interval rather than a instantaneous one.

By combining the expressions to causality, maybe we can introduce a new terminology called "**vacuum picture**" to represent the"**vacuum tunnelling dynamics**", and we can express the causality in "vacuum picture" as that: the "interval" (defined by the inner product value $e^{i\theta}$) between the two vacuum states in the vacuum tunnelling (before the vacuum symmetry is spontaneously broken) should be unitary (the phase θ is real). For instance, the inner product between any two vacuum states in the "Light constraint" are unitary.

In a word, the "vacuum tunnelling dynamics" is the origin of the differences between our P4 type field theory and the P2 type theories.

7 Conclusion

We have introduced a new class of higgs type complex-valued scalar fields U ("P4 type") with a fourth-order differential equation as its equation of motion, motivated by the linear potential in the lattice gauge theory. The field U can generate a wealth of interaction forms with some postulations on the convergence being taken. After getting a propagator of the form of $-i/p^4$ from a $(\partial\partial U)^2$ term in the kinetics term in the canonic quantization framework, by computing the amplitudes of the tree-level $2 \to 2$ scattering processes mediated by the U field, we can get a wealth of classic non-relativistic effective potential form within the Born-approximation framework, such as: (1) by using U to construct a QED theory, we can get the Coulomb-type potential, with a negligible linear potential and logarithmic potential as correction; (2) by using U to construct a QCD theory, we can get the Coulomb-type potential, and a considerable linear potential to serve for the confinement, with a logarithmic potential as the next-leading order corrections; (3) by using U to construct a gravatition theory, we can get a linear potential to serve for the dark energy effect, and the Newton's gravity form accompanied by a relativistic effect correction of the form $-Gm^2\boldsymbol{v}^2/r$ to serve the dark matter effect.

Moreover, for some limit cases, we can get some interesting suppositions, such as: (1) in a low energy approximation of the dynamics of U, a nonlinear Klein-Gordon equation could be generated; (2) in a weak field limit with a constraint $U_1^2 + U_2^2 = \langle U \rangle^2$, U could become a group element, thus the gauge symmetry could superficially arise, with a linear QED to be generated by relating the field strength ∂U to the corresponding gauge field A^μ; (3) due to the multi-vacuum structure for a sine-Gordon type vector field A^μ induced from U, a fermion mass spectrum with generation structure and a seesaw mechanism for gauge symmetry and flavor symmetry could be generated, including heavy fermions; (4) by generalizing the P2 type field to a P4 type one, the corresponding P2 type current would become a P2 type field as a kind of "kinetic-potential duality", which provides a possible way for proposing a renormalizable gravitation theory; (5) besides, as an inverse procedure, with a correspondence of the Wilson loop to the field U, by treating U instead of the gauge field A^μ as the effective d.o.f, it is a useful try to solve the non-perturbative problem in QCD. So, the solution to the non-perturbative problem in QCD and a renormalizable gravitation theory might be practicable within our U field framework.

For the causality, in our P4 type field theory with high-order differential equations, we want to say, the causality arises with the vacuum symmetry breaking, i.e., the causality can only be well-defined after the VEV of a P4 type field U is fixed. In a word, the vacuum tunnelling dynamics is the origin of the differences between our P4 type field theory and the P2 type theories.

On the framework of model-building itself, if the results in our calculations are even partly right for the real physical processes, then it would be said that the P4 type theory is a more effective and more general theory, by contrast to the the P2 theory. According to the redefinition for the canonic d.o.f, e.g., from the coordinate to the wave function ($x \to \phi = e^{ip\cdot x}$) for the first quantization in quantum mechanics, and from the P2 type field to the P4 type field ($\phi \to U \sim e^{i\phi}$) in this paper, maybe we could ask, is there a principle about this redefinition of d.o.f (maybe we can call it "exponentialization")?

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References

[1] C.Y. Wong, "*Introduction to High-Energy Heavy-Ion Collisions*", World Scientific Publishing, (1994). Chapter 10-11; Page 207,214; Eq.(10.63),(11.13).

[2] M.E. Peskin, D.V. Schroeder, "*An Introduction to Quantum Field Theory*", (Boulder: Westview, 1995). Page: 18,30,43,51,121-126,186,193,294,310,482.
A.Zee, "*Quantum Field Theory in a Nutshell*", (Princeton University Press, 2010). Page: 518.

[3] R.-C. Li, Field Theory with Fourth-order Differential Equations,
`https://vixra.org/abs/1712.0487`

[4] J.-P. Hsu, "Yang-Mills Gravity in Flat Space-time, II. Gravitational Radiations and Lee-Yang Force for Accelerated Cosmic Expansion", Int.J.Mod.Phys.**A24**, 5217 (2009). arXiv:1005.3250v1 [gr-qc].

[5] G. Feinberg, "Possibility of Faster-Than-Light Particles", Physical Review.**159**, 1089 (1967).

[6] R. R. Caldwell, Phys.Lett.**B545**, 23 (2002). [arXiv:astro-ph/9908168].
Y.-F. Cai, E.N. Saridakis, M.R. Setare, et al., Phys.Rept.**493**, 1 (2010), arXiv:0909.2776v2 [hep-th].

[7] S. Weinberg, "*The Quantum Theory of Fields, Vol. I*", (Cambridge University Press, 1995). Page: 58; ditto, Vol. II , 1996, Pages: 326; 464-468.

[8] C. Itzykson, J.B. Zuber, "*Quantum Field Theory*", (New York: McGraw-Hill, 1980). Page: 42-44.

[9] T.P. Cheng, L.F Li, "*Gauge Theory of Elementary Particle Physics-Problems and Solutions*",(Oxford University Press, 2000). Page: 111.

[10] J.K. Webb, J. A. King, M. T. Murphy, et al., "Indications of a spatial variation of the fine structure constant", Phys. Rev. Lett., **107**, 191101 (2011). arXiv:1008.3907[astro-ph.CO].

[11] S. M. Faber, R. E. Jackson, "Velocity dispersions and mass-to-light ratios for elliptical galaxies", Astrophys.J. **204**, 668 (1976).