The Special Functions and the Proof of the Riemann's Hypothesis

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Abstract : By studying the Ⓢ function whose integer zeros are the prime numbers, and being inspired by the article [2], I give a new proof of the Riemann hypothesis.
Résumé : En étudiant la fonction Ⓢ dont les zéros entiers sont les nombres premiers, et en m'inspirant de l'article [2], je donne une nouvelle preuve de l'hypothèse de Riemann.

I- INTRODUCTION

The Riemann's hypothesis [2] conjectured that all nontrivial zeros of ζ are in the line $x = \frac{1}{2}$.
In this article, the study of the sghiar's function Ⓢ which I introduced and whose integer zeros are the prime numbers inspired me to use the function Gamma Γ. And miraculously a proof similar to that used in [2] allowed me to give a short and elegant proof of the Riemann Hypothesis.
In order not to recall everything, I suppose known - among others - the functions zeta ζ, Gamma Γ : $z \mapsto \int_0^{+\infty} t^{z-1}\, e^{-t}\, dt$ and their properties (See [3] and [4]).

II- THE PROOF OF THE RIEMANN HYPOTHESIS :

Theorem 1 (The Riemann hypothesis) *All non-trivial zeros of ζ are in the line $x = \frac{1}{2}$.*

Lemma 1

$$0 < Re(z) < 1 \Longrightarrow |\int_0^{+\infty} \frac{t^{z-1}}{e^t - 1} dt| \neq 0$$

Proof :
It suffices to prove that $Re(\int_0^{+\infty} \frac{t^{z-1}}{e^t-1} dt) \neq 0$ or $Im(\int_0^{+\infty} \frac{t^{z-1}}{e^t-1} dt) \neq 0$
Let $z = x+iy$, by change of variable, and by setting $t^{x-1} = e^u$, we deduce :

$$-Re(\int_0^{+\infty} \frac{t^{z-1}}{e^t - 1} dt) = \int_{-\infty}^{+\infty} \frac{e^u}{e^{e^{\frac{u}{x-1}}} - 1} cos(y\frac{u}{x-1}) \frac{1}{x-1} e^{\frac{u}{x-1}} du$$

Note :
As $\frac{e^u}{e^{e^{\frac{u}{x-1}}}-1} cos(y\frac{u}{x-1}) \frac{1}{x-1} e^{\frac{u}{x-1}}$ is zero for $u_k = (2k+1)\frac{\pi}{2}\frac{x-1}{y}$, $k \in \mathbb{Z}$ and oscillates increasing in amplitude because $g(u) = \frac{e^u}{e^{e^{\frac{u}{x-1}}}-1} \frac{1}{x-1} e^{\frac{u}{x-1}}$ is decreasing with u, we deduce that :
$\int_{u=(2k+1)\frac{\pi}{2}\frac{x-1}{y}}^{u=(2(k+\mathbf{2})+1)\frac{\pi}{2}\frac{x-1}{y}} \frac{e^u}{e^{e^{\frac{u}{x-1}}}-1} cos(y\frac{u}{x-1}) \frac{1}{x-1} e^{\frac{u}{x-1}} du$ is different from 0 and its sign does not depend on $k \in 2\mathbb{Z}$) (we have the same result if $k \in 2\mathbb{Z}+1$) :
Because : $\int_{u=(2k+1)\frac{\pi}{2}\frac{x-1}{y}}^{u=(2(k+\mathbf{2})+1)\frac{\pi}{2}\frac{x-1}{y}} \frac{e^u}{e^{e^{\frac{u}{x-1}}}-1} cos(y\frac{u}{x-1}) \frac{1}{x-1} e^{\frac{u}{x-1}} du = \int_{u_k}^{u_{k+2}} g(u) cos(y\frac{u}{x-1}) du = \int_{u_k}^{u_{k+1}} g(t) cos(y\frac{t}{x-1}) dt + \int_{u_{k+1}}^{u_{k+2}} g(u) cos(y\frac{u}{x-1}) du = \int_{u_{k+1}}^{u_{k+2}} cos(y\frac{u}{x-1})(g(u) - g(u-\tau)) du$ where $\tau = \frac{\pi}{\frac{y}{x-1}}$ (it is found by changing the variable $u = t + \tau$). and so the integral $\int_{u=(2k+1)\frac{\pi}{2}\frac{x-1}{y}}^{u=(2(k+\mathbf{2})+1)\frac{\pi}{2}\frac{x-1}{y}} \frac{e^u}{e^{e^{\frac{u}{x-1}}}-1} cos(y\frac{u}{x-1}) \frac{1}{x-1} e^{\frac{u}{x-1}} du$ is different from 0 and its sign does not depend on $k \in 2\mathbb{Z}$) (we have the same result if $k \in 2\mathbb{Z}+1$).

By using the note above :
Let $f(u) = \frac{e^u}{e^{e^{\frac{u}{x-1}}}-1} cos(y\frac{u}{x-1}) \frac{1}{x-1} e^{\frac{u}{x-1}}$, and $u_k = (2k+1)\frac{\pi}{2}\frac{x-1}{y}$, $k \in \mathbb{N}$

$$-Re(\int_0^{+\infty} \frac{t^{z-1}}{e^t - 1} dt) = \lim_{u_k \to +\infty} \int_{-\infty}^{u_k} f(u) du$$

If $\int_{-\infty}^{u_l} f(u) du \geq 0$:
So :
- Either $f'(u_l) \geq 0$ (f increasing in the vicinity of u_l)
In this case : $-Re(\int_0^{+\infty} \frac{t^{z-1}}{e^t-1} dt) = \int_{-\infty}^{u_l} f(u) du + \int_{u_l}^{u_{l+1}} f(u) du + \sum_{k \in 2\mathbb{N}} \int_{u_{k+l+1}}^{u_{(k+2)+l+1}} f(u) du \ngeq 0$
- Or either $f'(u_l) \leq 0$ (f decreasing in the vicinity of u_l)
In this case : $-Re(\int_0^{+\infty} \frac{t^{z-1}}{e^t-1} dt) = \int_{-\infty}^{u_l} f(u) du + \sum_{k \in 2\mathbb{N}} \int_{u_{k+l}}^{u_{(k+2)+l}} f(u) du \ngeq 0$
Similarly :
If $\int_{-\infty}^{u_l} f(u) du \leq 0$:
So :
- Either $f'(u_l) \geq 0$,

In this case : $-Re(\int_0^{+\infty} \frac{t^{z-1}}{e^t-1}dt) = \int_{-\infty}^{u_l} f(u)du + \sum_{k\in 2\mathbb{N}} \int_{u_{k+l}}^{u_{(k+2)+l}} f(u)du \nleq 0$

- Or either $f'(u_l) \leq 0$,

In this case : $-Re(\int_0^{+\infty} \frac{t^{z-1}}{e^t-1}dt) = \int_{-\infty}^{u_l} f(u)du + \int_{u_l}^{u_{l+1}} f(u)du + \sum_{k\in 2\mathbb{N}} \int_{u_{k+l+1}}^{u_{(k+2)+l+1}} f(u)du \nleq 0$

Proof of the theorem

We know ([3,4])that :

$$\zeta(z)\Gamma(z) = \int_0^{+\infty} \frac{t^{z-1}}{e^t-1}dt$$

As $\Gamma(z+1) = z\Gamma(z)$, then :

$$\zeta(z)(z-1)\Gamma(z-1) = \int_0^{+\infty} \frac{t^{z-1}}{e^t-1}dt$$

But the gamma function also checks the Legendre duplication formula [3] :

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\,\sqrt{\pi}\,\Gamma(2z).$$

So :

$$\Gamma(z-1)\,\Gamma\left(z-\frac{1}{2}\right) = 2^{3-2z}\,\sqrt{\pi}\,\Gamma(2z-2).$$

And we deduce :

$\zeta(z)(z-1)2^{3-2z}\sqrt{\pi}\Gamma(2z-2) = \Gamma\left(z-\frac{1}{2}\right)\int_0^{+\infty}\frac{t^{z-1}}{e^t-1}dt$

If $\zeta(s) = 0$ with s a non trivial zero of ζ, then, by symmetry of the zeros about the critical line $Re(z) = \frac{1}{2}$, we can assume that $s = \frac{1}{2} - \alpha + i\beta$ with $0 \leq \alpha \lneq \frac{1}{2}$ (because it is known that any non-trivial zero belongs to the critical strip : $\{s \in \mathbb{C} : 0 < Re(s) < 1\}$)

But from the Euler's reflection formula : $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \forall z \notin \mathbb{Z}$, we have $\Gamma\left(s-\frac{1}{2}\right) \neq 0$, so by tending z towards s and by using the **lemma 1**, we will have :$|\Gamma(2s-2)| = |\Gamma(-1-2\alpha+i2\beta)| = +\infty$, and consequently we deduce that : : $|\Gamma(-1-2\alpha)| = +\infty$

The study of *Gamma* -See Figure 1 - Shows that the only possible case is $-1-2\alpha = -1$, so $\alpha = 0$.

Theorem 2 *The sghiar's function and the prime numbers :*

Let $Ⓢ(z) = \zeta(-\frac{\Gamma(z)+1}{z/2})$.

if $z \in \mathbb{N}^*$ *then* $Ⓢ(z) = 0 \Longleftrightarrow z$ *is a prime number*

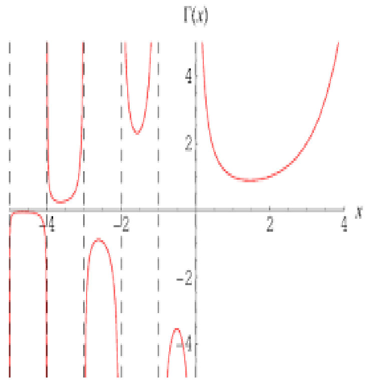

FIGURE 1 – Gamma function

Proof

It follows from Wilson's theorem [1] - which assures that p is a prime number if and only if $(p-1)! \equiv -1 \mod p$ - and the fact that the trivial zeros of ζ are $-2\mathbb{N}^*$.

III- Conclusion :

The Gamma function Γ and the Mertens function M are closely linked to the Riemann zeta function ζ.

What is curious is that by the same techniques the Mertens function allowed the proof of the Riemann hypothesis in [2], and the gamma function allowed also in this article a simple, short and elegant proof of the Riemann hypothesis.

IV- Acknowledgments :

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V- References

[1] Roshdi Rashed, Entre arithmétique et algèbre : Recherches sur l'histoire des mathématiques arabes, journal Paris, 1984,

[2] M. Sghiar. The Mertens function and the proof of the Riemann's hypothesis, International Journal of Engineering and Advanced Technology (IJEAT), ISNN :2249-8958, Volume- 7 Issue-2, December 2017

[3] https://en.wikipedia.org/wiki/Gamma_function.

[4] https://en.wikipedia.org/wiki/Riemann_zeta_function.