How truly random Is The Brownian Motion ? The Hidden Connection with Riemann ZetaFunction & Riemann Hypothesis! Is Randomness truly random Or the existence of Duality ?

Author :

Pankaj Mani,CQF,FRM

(manipankaj9@gmail.com)

New Delhi,India

Random Numbers and processes are used to model random processes to model financial assets which typically involves **Brownian** motion/Wiener process. In this article I'll explore how random Brownian motion truly is and is it really compatible and capable to model randomness in financial variables ? The logic behind all these goes to the existence of Prime Numbers and their dynamics in pure mathematics from where they originate . Computer generated Random Numbers being used are based on certain algebraic equations made out of prime numbers that are also feed into random processes.

The legendary John von Neumann said : Anyone who attempts to generate random numbers by deterministic means is, of course, living in a state of sin.

Brownian motion claims to model random processes in real-world . Here, I will try to cross validate this claim by looking at the fundamental origination of Brownian motion. I will try to focus on this taking clue from one of the most celebrated mathematical functions called Riemann Zeta function. Is Brownian motion hidden inside Riemann Zeta functions? Riemann Zeta function was found in attempt to find out the hidden pattern in the occurrence of prime numbers which appear random in the first place. But are they really random ? The non-trivial zeros of the Riemann Zeta function upon analytic continuation over entire complex plane has deep resemblance with the prime number occurrence. The hypothesis linked to Riemann Zeta function popularly called as Riemann Hypothesis is still considered probably as the most important unsolved problem in mathematics .We will see how Brownian motion is linked to Riemann Zeta function and try to explore how random it's truly ?

Let's have deeper look at the structure of Riemann Zeta function which is special and the most basic case of Dirichlet L functions

Riemann Zeta Function & Riemann Hypothesis

The formula below is the Riemann Zeta function defined for s >1 where s is the complex variable.

$$\zeta_s = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ ...$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}.$$

More specifically, if we define the completed zeta function

ˆζ(s) by ˆζ(s) := π^ (− s /2) Γ(s/ 2)ζ(s),

Theorem :The completed zeta function ˆζ has an analytic continuation to the entire complex plane except for simple poles at s = 0, 1. Furthermore, this function ˆζ satisfies the functional equation

ˆζ(s) = ˆζ(1 − s).

The Riemann zeta function ζ(*s*) is a function whose argument *s* may be any complex number other than 1, and whose values are also complex. It has zeros at the negative even integers; that is, ζ(*s*) = 0 when *s* is one of −2, −4, −6, ... These are called its *trivial zeros*. However, the negative even integers are not the only values for which the zeta function is zero. The other ones are called *non-trivial zeros*. The Riemann hypothesis is concerned with the locations of these non-trivial zeros, and states that:

The real part of every non-trivial zero of the Riemann zeta function is 1/2.

Thus, if the hypothesis is correct, all the non-trivial zeros lie on the critical line consisting of the complex numbers 1/2 + *it*, where *t* is a real number and *i* is the imaginary unit.

The formula below is the reciprocal of Riemann Zeta function which contains

Mobius function in the numerator

$$\frac{1}{\zeta(s)} = 1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{1}{6^s} - \frac{1}{7^s} + \frac{0}{8^s} + \frac{0}{9^s} + \frac{1}{10^s} \ldots\ldots\ldots\ldots\ldots$$

The above reciprocal of the Riemann Zeta function contains Mobius function sequence in the numerator.

The sequence of Numerator above

+1 ,-1,-1,0,-1,+1,-1,0,0,+1……….

This is similar to Random walk process where the probability of occurrence of +1 & -1 is around 6/pi*pi =0.608 close to 0.5 as is the case with Brownian motion . This result was first due to Von Sternbach in 1896 as in Prime Numbers by David Well.

This points to a very deep structural connection between Riemann Zeta function and the Brownian motion i.e. Wiener Process/Random walk motion.

Now I give a very deep intriguing connection between Brownian motion and Riemann Zeta function and Riemann Hypothesis.

Consider a Pinned Brownian Motion on R . This is a Standard Brownian Motion on the line Bt (R)

t>=0; B(0) =B(1)=0

Now let

$Z = \max_{0\le t\le 1} B_t - \min_{0\le t\le 1} B_t$

be the length of the range of B_t.

Then the expectation of Z^s is known to be

$E[Z^s] = \xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$.

here it can be seen the Riemann Zeta function hidden connection with Brownian motion!

In another Brownian system, there exists following equivalent of the Riemann hypothesis .

Theorem (Balazard, Saias, Yor).

Consider two-dimensional Brownian motion in the plane, starting at (0, 0).

Let (1 /2 , W) be the first point of contact with the line

$X = 1/2$.

Then the Riemann hypothesis is equivalent to

$E[\log|\zeta(W)|] = 0$.

Now given the very much possibility that Riemann Hypothesis is True, the above results clearly show mathematically that Brownian motion behavior ,would also have hidden deterministic pattern as well like Riemann Zeta function (as Riemann Hypothesis)!

Riemann Hypothesis Truthfulness indicates that though the non-trivial zeros linked to prime number distribution appear random locally but they have a very deterministic well defined trajectory on the straight critical line R(s)=1/2. Mathematical structures in Nature arrange themselves quite deterministically in global sense based on some symmetrical aspects. As Brownian motion is linked to Riemann Zeta function, given this globally deterministic behavior of the later, mathematically it does mean that Brownian motion behavior is also globally well determined and patterned rather being truly random process(though appearing random locally) !

But then Stock prices supposed to have a random component in its process should not be compatible with Brownian motion, which is likely fundamentally Not Deterministic in Global Sense ? It raises fundamental questions on the compatibility of Brownian motion while modeling the supposedly random financial variables .

How this deep connection between Brownian motion and Riemann Zeta function impacts the practical trading world in Quantitative Finance ,let's look at a very important connection between the two in context of Options Pricing. The formula derived by Broadie, Kou, Glasserman for finding the discrete adjustment using continuous price distribution of Knock-in/out Options, which is quite popular among traders in banks and institutional players that involves Riemann Zeta function ..

Formula : Let V (H) be the price of a continuous barrier option, and Vm(H) be the price of an otherwise identical barrier option with m monitoring points.

Then for any of the eight discrete monitored regular barrier options, we have the approximation

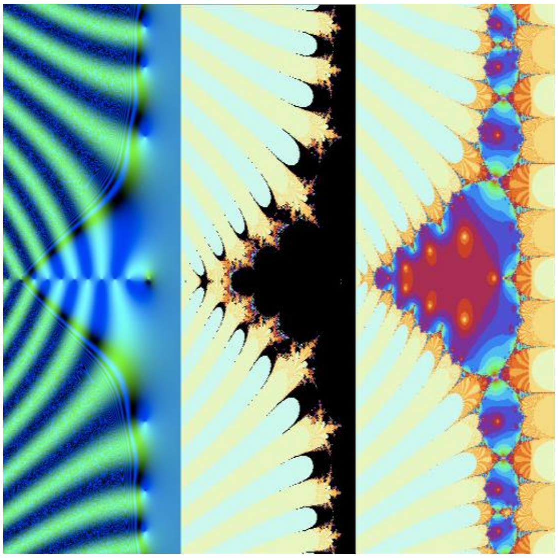
$V_m(H) = V(He^{\pm\beta\sigma\sqrt{T/m}}) + o(1/\sqrt{m})$,

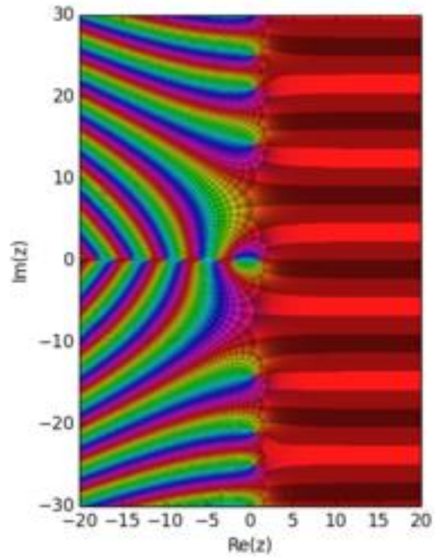
with + for an up option and − for a down option, where the constant $\beta = -(\zeta(1/2)/\sqrt{2\pi}) \approx 0.5826$, ζ the Riemann Zeta function. The constant β was calculated in Chernoff (1965).

How do these deep patterns inside so called random numbers and processes guide us in finding the right impartial mathematical models to price the financial assets ? Is Brownian motion really random fundamentally to model random processes of asset prices and other variables in quantitative finance ? Before we could make some conclusive statement on this, let's analyse Riemann Zeta Function & Riemann Hypothesis .

Riemann Hypothesis is the holy grail of pure mathematics often considered as the most important problem in mathematics the reason being there are so many other results based on this. The riddle involving prime numbers are used in day to day life in digitial security, cryptography etc.. Recently I have proposed my idea based on Game Theory(the great result by legend John Nash) & Symmetry results to show that Riemann Hypothesis is True for Riemann Zeta Function ,which is partly acknowledged as interesting (and currently under discussion to be followed deeply!) by a leading expert and international authority (who has himself recently resolved centuries old one of the most fundamental problem in number theory since Euler and found hidden patterns inside certain fundamental numbers). In any case, what I have discovered in my approach that in the game of arranging Zeros in the complex plane, they are globally governed by some symmetrical and equilibrium laws in quite deterministic manner like Nature and hence the global trajectory confined to the critical line for non-trivial zeros. This further reveals some deep aspects for the foundation and Nature of mathematics as a whole which is beyond the context of this paper. Empirically, Riemann Hypothesis has also been verified as True for Riemann Zeta function for billions of its non-trivial zeros. **The Truthfulness of Riemann Hypothesis establishes towards the globally deterministic pattern hidden inside the random appearing prime numbers distribution**

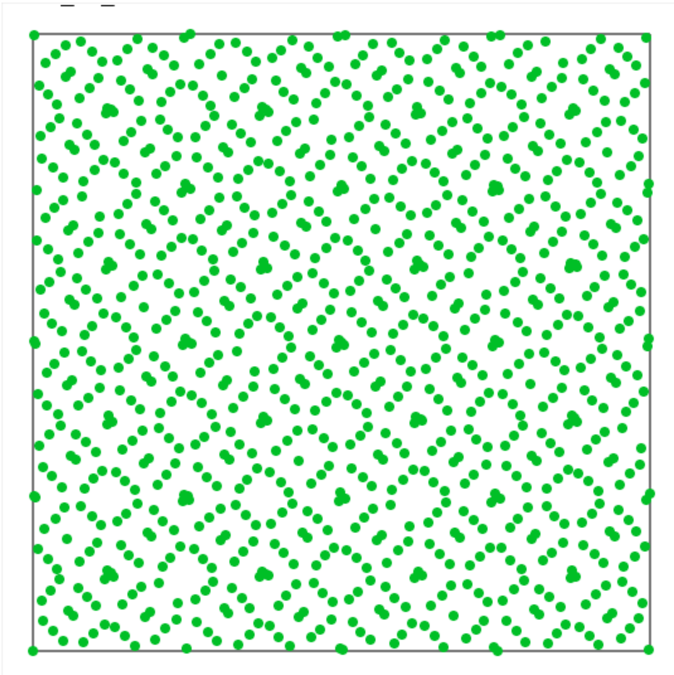
It certainly reveals the Fundamental Nature of Mathematics as a whole and the beauty hidden within random appearing numbers. Now, let's visualize the beautiful global deterministic structure inside Riemann Zeta Function graphically .





Similar beautiful patterns appear even within the random and quasi random numbers empirically and graphically which are used to simulate financial prices say in Monte Carlo Simulations. So, these intrinsic aspects of those random numbers must be taken into account while taking decision before applyting it on any particular financial product.

Beautiful Geometrical Pattern inside th Sobol Sequence below:



As we have seen that **Randomness appearing the mathematical structure of Numbers locally but their behavior is quite deterministic globally**. **There is duality existing out here as in Nature.** It can possibly also point towards the fact that even in Nature, Randomness might appear locally but quite Determinsitic globally . The beautiful geometrical patterns can also be empirically found within the random, quasi-random number sequences, which are treated as random in the first place. The Prime Numbers are used to computationally generate random numbers using algorithms , which are also used to simulate Brownian Motion to model stochastic trajectory of asset paths. Where do these beautiful deterministic geometrical structure hidden within so-called random processes, numbers come from if they are truly random in nature ? How the non-trivial zeros linked to prime number distribution could behave so deterministically (given the possibility that Riemann Hypothesis is True) . Recently in 2016, two number theorists from Stanford University were surprised to find pattern inside the Random-appearing Prime Numbers(5&6). Recently,a leading international expert of Number Theory has discovered fractal pattern inside Partition Numbers!! These findings strongly support that random appearing numbers have hidden pattern inside to discover !(7)

There is hidden synchronicity in every random event . So, is so-called randomness itself might not be random in true sense ! My view is that even market or broadly Nature might appear Random locally in space-time but globally they could be quite deterministic to varying degree .

Stock market is also run by human behavior but then do human behave truly random ? There exists some sort of Duality in Nature and even human in context of Randomness. It might be human ignorance as of now or incompetence to decipher those patterns out of randomness!

But the reason behind all these philosophical discussion in above few lines is to examine the compatibility and effectiveness of our mathematical tools e.g. Brownian Motion/Wiener Process etc to model Physical Randomness in Nature ! Before we could model Nature's randomness which includes Financial Market Randomness , we need to verify if the structure of mathematical tools is itself capable of modeling such randomness ? If not, the entire philosophy of modeling randomness in quantitative finance would be foundationally objectionally and incompatible.

"So, I think the use of Brownian Motion as a mathematical tool to model True Randomness in financial variables is fundamentally and foundationally incompatible and objectionable as I have shown above mathematically in Riemann Zeta function & Riemann Hypothesis globally deterministic Connection !"

The Big Question(?) to explore for the future of quantitative finance:

Given the much possibility that Brownian Motion/Wiener process is not really a random process fundamentally as shown mathematically, the financial modeling based on Brownian motion/Wiener Process could be incompatible while modeling Randomness in Quantitative finance ? This fundamental biasness in the Brownian Motion could also induce unfairness in the quantitative finance models say while pricing, modeling etc. What impact will it have on the way randomness in financial assets/variables are modeled in future ? Hence its application in modeling the random behavior of asset prices or other financial variables should be questioned fundamentally . It is likely to impact fundamentally the pricing of various financial products as mentioned above in its option pricing applications. It should not be unjustified to state that Brownian motion is not truly random process foundationally. These foundational aspects need to be delved and looked at seriously ! I leave these questions open-ended for readers to delve upon and debate in order to reach a conclusive stage for the progress of quantitative finance.

References & Further readings :

1. http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.434.9647&rep=rep1&type=pdf
2. Prime Numbers by David Wells (John Wiley & Sons, 2005)
3. A Continuity Correction for Discrete Barrier Options, Mark Broadie, Paul Glasserman (Columbia Univerity), Steven Kou (University of Michigan), Mathematical Finance, Vol 7, No. 4 (October 1997)
4. https://www.claymath.org/millennium-problems/riemann-hypothesis
5. https://arxiv.org/abs/1603.03720
6. https://www.newscientist.com/article/2080613-mathematicians-shocked-to-find-pattern-in-random-prime-numbers/
7. https://aimath.org/news/partition/