Riemann Hypothesis

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1 Abstract

The Riemann Zeta function is defined as

$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s, \ Re(s) > 1$

$The\ Zeta\ function\ is\ holomorphic\ in\ the\ complex\ plane\ except\ for\ a\ pole\ at$

$s = 1. The\ trivial\ zeros\ of\ \zeta(s)\ are\ -2, -4, -6, ... Its\ non\ trivial\ zeros\ lie\ in$

$the\ critical\ strip\ 0 < Re(s) < 1.$

$The\ Riemann\ Hypothesis\ states\ that\ all\ the\ non\ trivial\ zeros\ lie\ on\ the\ critical\ line$

$Re(s) = 1/2.$

2 Proof

Riemann Hypothesis is equivalent to the integral equation (see [3, p.136, Corollary 8.7])

$\int_{-\infty}^{\infty} \frac{log|\zeta(1/2+it)|}{1+4t^2}\ dt = 0$

$Let, I = \int_{-\infty}^{\infty} \frac{log|\zeta(1/2+it)|}{1+4t^2}\ dt$

$Substitute,\ t = \frac{1}{2}tan\theta.$

$dt = \frac{1}{2}sec^2\theta\ d\theta.$

$I = \frac{1}{2}\int_{-\pi/2}^{\pi/2} log \mid \zeta(\frac{1+itan\theta}{2}) \mid \frac{sec^2\theta}{1+tan^2\theta}\ d\theta$

$I = \frac{1}{2}\int_{-\pi/2}^{\pi/2} log \mid \zeta(\frac{1+itan\theta}{2}) \mid d\theta$

$Let,\ f(\theta) = log \mid \zeta(\frac{1+itan\theta}{2}) \mid$

$f(-\theta) = log \mid \zeta(\frac{1-itan\theta}{2}) \mid$

$Since, by\ Schwarz\ reflection\ principle,\ \zeta(\bar{s}) = \overline{\zeta(s)}$

$f(-\theta) = log \mid \overline{\zeta(\frac{1+itan\theta}{2})} \mid$

$f(-\theta) = log \mid \zeta(\frac{1+itan\theta}{2}) \mid$

$f(-\theta) = f(\theta)$

$So,\ f\ is\ an\ even\ function.$

$I = \frac{1}{2}\int_{-\pi/2}^{\pi/2} log \mid \zeta(\frac{1+itan\theta}{2}) \mid d\theta = \int_{0}^{\pi/2} log \mid \zeta(\frac{1+itan\theta}{2}) \mid d\theta$

$$I = \int_{0}^{\pi/2} log \mid \zeta(\frac{1+itan\theta}{2}) \mid d\theta \tag{1}$$

$$Let,\ J = \int_{0}^{\pi} log \mid \zeta(\frac{1+itan\theta}{2}) \mid d\theta \tag{2}$$

$let,\ g(\theta) = log \mid \zeta(\frac{1+itan\theta}{2}) \mid$

$g(2\frac{\pi}{2}-\theta) = g(\pi-\theta) = log \mid \zeta(\frac{1-itan\theta}{2}) \mid$

$Since, by\ Schwarz\ reflection\ principle,\ \zeta(\bar{s}) = \overline{\zeta(s)}$

$g(2\frac{\pi}{2}-\theta) = log \mid \overline{\zeta(\frac{1+itan\theta}{2})} \mid$

$g(2\frac{\pi}{2}-\theta) = log \mid \zeta(\frac{1+itan\theta}{2}) \mid$

$g(2\frac{\pi}{2}-\theta) = g(\theta)$

$Thus,\ \int_0^{\frac{2\pi}{2}} g(\theta)d\theta = 2\int_0^{\frac{\pi}{2}} g(\theta)d\theta$

$\Rightarrow \int_0^{\frac{2\pi}{2}} log \mid \zeta(\frac{1+itan\theta}{2}) \mid d\theta = 2\int_0^{\frac{\pi}{2}} log \mid \zeta(\frac{1+itan\theta}{2}) \mid d\theta$

$\Rightarrow \int_0^{\pi} log \mid \zeta(\frac{1+itan\theta}{2}) \mid d\theta = 2\int_0^{\frac{\pi}{2}} log \mid \zeta(\frac{1+itan\theta}{2}) \mid d\theta$

$From\ equation\ (1)\ and\ (2),$

$$J = 2I. \tag{3}$$

$$Let,\ K = \int_0^{2\pi} log \mid \zeta(\frac{1+itan\theta}{2}) \mid d\theta \tag{4}$$

$Let,\ h(\theta) = log \mid \zeta(\frac{1+itan\theta}{2}) \mid$

$h(2\pi-\theta) = log \mid \zeta(\frac{1-itan\theta}{2}) \mid$

$Since, by\ Schwarz\ reflection\ principle,\ \zeta(\bar{s}) = \overline{\zeta(s)}$

$h(2\pi-\theta) = log \mid \overline{\zeta(\frac{1+itan\theta}{2})} \mid$

$h(2\pi-\theta) = log \mid \zeta(\frac{1+itan\theta}{2}) \mid$

$h(2\pi-\theta) = h(\theta)$

$\int_0^{2\pi} h(\theta)d\theta = 2\int_0^{\pi} h(\theta)d\theta.$

$\int_0^{2\pi} log \mid \zeta(\frac{1+itan\theta}{2}) \mid d\theta = 2\int_0^{\pi} log \mid \zeta(\frac{1+itan\theta}{2}) \mid d\theta$

$K = 2J \ (from\ equation\ (2)\ and\ (4))$

$J = 2I \ (from\ equation\ (3))$

$\Rightarrow K = 2J = 4I$

$K = 4I$

$Putting\ value\ of\ K\ from\ equation\ (4),$

$4I = \int_0^{2\pi} log \mid \zeta(\frac{1+itan\theta}{2}) \mid d\theta$

$I = \frac{1}{4}\int_0^{2\pi} log \mid \zeta(\frac{1+itan\theta}{2}) \mid d\theta$

$I = \frac{1}{4}\int_0^{2\pi} log \mid \zeta(\frac{cos\theta+isin\theta}{2cos\theta}) \mid d\theta$

$I = \frac{1}{4}\int_0^{2\pi} log \mid \zeta(\frac{e^{i\theta}}{e^{i\theta}+e^{-i\theta}}) \mid d\theta$

$Let.,\ e^{-i\theta} = z.$

$-\ ie^{-i\theta}d\theta = dz.$

$d\theta = -dz/iz = \ idz/z.$

$I = \frac{i}{4}\oint_{|z|=1} \frac{log|\zeta(\frac{z^2}{z^2+1})|}{z}\ dz$

$Thus, Riemann\ Hypothesis\ is\ equivalent\ to,$

$I = \frac{i}{4}\oint_{|z|=1} \frac{log|\zeta(\frac{z^2}{z^2+1})|}{z}\ dz = 0$

3 References

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