Sedeonic Generalization of London Equations

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Abstract

We discuss the generalization of phenomenological equations for electromagnetic field in superconductor based on algebra of space-time sedeons. It is shown that the combined system of London and Maxwell equations can be reformulated as a single sedeonic wave equation for the field with nonzero mass of quantum, in which additional conditions are imposed on the scalar and vector potentials, relating them to the deviation of charge density and currents in the superconducting phase. Also we considered inhomogeneous equations including external sources in the form of charges and currents of the normal phase. In particular, a screening of the Coulomb interaction of external charges in a superconducting media is discussed.

1 Introduction

The phenomenological relativistic-invariant theory of the electromagnetic field in a superconductor was first proposed by the London brothers in 1935 [1]. In particular, one of the main predictions that followed from the London equations is the effect of the penetration of electric and magnetic fields into the superconductor to a certain depth, depending on material parameters. The London model is widely used for qualitative analysis of the effects associated with the interaction of magnetic field with superconductors [2,3]. The effects of the interaction between superconductor and electric field cause discussion [4-7] and continue to be intensively discussed in the literature [8-10].

In recent time, there is the great progress in the reformulation of the electromagnetic field equations based on the different algebras of hypercomplex numbers. In simplest case the use of four-component quaternions consisting of scalar and vector parts allow adequate describing the four-vector concept of electromagnetic potentials and field sources in special relativity [11-14]. In order to distinguish the spatial-temporal properties of electric and magnetic fields the eight-component octonions [15-18] and octons [19-21] are used. However, a consistent relativistic approach implies equally the space and time symmetries that require using the sixteen-component space-time algebras. Recently we proposed the algebra of sixteen-component sedeons, which takes into account the full space-time symmetry of physical values and realizes the scalar-vector representation of Poincare group [22]. In particular, we considered the equations for massive and massless fields based on sedeonic potentials and space-time operators [23-25]. In the present paper we apply the sedeonic algebra to the generalization of London model of electromagnetic field in superconductor.

2 Preliminary remarks

The equilibrium and electroneutrality conditions for a superconductor can be written as follows:

$$\begin{aligned} &\int\limits_V (\vec{j}_i + \vec{j}_e) dV = 0, \\ &\int\limits_V (\rho_i + \rho_e) dV = 0, \end{aligned} \tag{1}$$

where ρ_i and ρ_e are volume densities of charges, while $\vec{j}_i$ and $\vec{j}_e$ are the volume densities of currents for ions and electrons respectively. Since ions are fixed at the sites of the crystal lattice, with a good approximation we have

$$\begin{aligned} &\vec{j}_i = 0, \\ &\rho_i = const, \end{aligned} \tag{2}$$

while the electron gas is a mobile system that responds to external fields. The feature of a superconductor below critical temperature is the absence of dissipation in the electronic subsystem and, as a consequence, the existence of persisting Meissner currents that compensate the external magnetic field.

The fundamental position of the London's theory is the relationship between the volume densities of charge and current of electrons in the superconducting phase and the scalar φ and vector $\vec{A}$ potentials of the electromagnetic field [1]. Let us denote ρ_s the deviation of volume density of electrons in superconducting phase from equilibrium density ρ_{e0}

$$\begin{aligned} &\rho_s = \rho_e - \rho_{e0}, \\ &\rho_s << \rho_{e0}, \end{aligned} \tag{3}$$

and $\vec{j}_s$ as volume density of supercurrent. Then the London relations for potentials in modern notation are written as follows:

$$\begin{aligned} &4\pi\lambda^2 \rho_s = -\varphi, \\ &\frac{4\pi}{c}\lambda^2 \vec{j}_s = -\vec{A}. \end{aligned} \tag{4}$$

Here λ is the parameter of London theory, c is the speed of light in a vacuum. From these relationships, using the definition of electric and magnetic field strengths

$$\begin{aligned} &\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\varphi, \\ &\vec{B} = \vec{\nabla} \times \vec{A}. \end{aligned} \tag{5}$$

one can obtain the following London relations

$$\begin{aligned} &\vec{E} = \frac{4\pi}{c^2}\lambda^2 \frac{\partial \vec{j}}{\partial t} + 4\pi\lambda^2 \vec{\nabla}\rho_s, \\ &\vec{B} = -\frac{4\pi}{c}\lambda^2 \vec{\nabla} \times \vec{j}_s. \end{aligned} \tag{6}$$

On the other hand, the fields in the superconductor satisfy the Maxwell equations:

$$\begin{aligned} &\vec{\nabla} \cdot \vec{E} = 4\pi\rho_s, \\ &\vec{\nabla} \cdot \vec{B} = 0, \\ &\vec{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \\ &\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}_s. \end{aligned} \tag{7}$$

Substitution of (**??**) into (**??**) leads to the following London wave equations [1]:

$$\begin{aligned}
\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle + \frac{1}{\lambda^2}\right)\vec{E} = 0,\\
\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle + \frac{1}{\lambda^2}\right)\vec{B} = 0,\\
\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle + \frac{1}{\lambda^2}\right)\vec{j}_s = 0,\\
\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle + \frac{1}{\lambda^2}\right)\rho_s = 0.
\end{aligned} \tag{8}$$

In form, these equations coincide with the equations for a field with nonzero mass of quantum [23], which is expressed in terms of the London parameter as follows:

$$m_0 = \frac{\hbar}{c\lambda}, \tag{9}$$

where $\hbar$ is the Planck constant.

3 Sedeonic form of London equations

For compact writing the equations we use the algebra of sixteen component sedeons [22,23], which are the tensor product of two four component algebras of Macfarlane quaternions [26]. In this formalism any vector is presented in the basis of unit vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ as

$$\vec{A} = A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3, \tag{10}$$

with the following rules of multiplication and commutation

$$\mathbf{a}_n\mathbf{a}_m = \delta_{nm} + i\varepsilon_{nmk}\mathbf{a}_k, \tag{11}$$

where δ_{nm} is Kronecker delta, ε_{nmk} is Levi-Civita symbol $(n, m, k \in \{1, 2, 3\}$) and i is the imaginary unit ($i^2 = -1$). The main advantage of Macfarlane algebra is the existence of Clifford product of vectors. For example, the product of two vectors $\vec{A}$ and $\vec{B}$ is

$$\vec{A}\vec{B} = \vec{A}\cdot\vec{B} + i\,\vec{A}\times\vec{B}. \tag{12}$$

that allows to write the equations in the very compact form. To compactify the operator parts of equations we use additional Macfarlane units $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ with the same rules of multiplication

$$\mathbf{e}_n\mathbf{e}_m = \delta_{nm} + i\varepsilon_{nmk}\mathbf{e}_k, \tag{13}$$

The basis $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ is associated with spatial rotation of vector values, while basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ connected with space-time inversion [22]. The unit vectors $\mathbf{a}_n$, commute with units $\mathbf{e}_m$

$$\mathbf{a}_n\mathbf{e}_m = \mathbf{e}_m\mathbf{a}_n. \tag{14}$$

Let us denote for convenient

$$\begin{aligned}
\rho = 4\pi\lambda^2\rho_s,\\
\vec{J} = \frac{4\pi}{c}\lambda^2\vec{j}_s.
\end{aligned} \tag{15}$$

Then the London equations (**??**) can be represented as the following sedeonic relation:

$$\left(i\mathbf{e}_1\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_2\vec{\nabla}\right)\left(i\mathbf{e}_1\varphi + i\mathbf{e}_1\rho + \mathbf{e}_2\vec{A} + \mathbf{e}_2\vec{J}\right) = 0. \tag{16}$$

Indeed, using the Lorentz gauge and the definitions of $\vec{E}$ and $\vec{B}$ fields (**??**), we obtain from (**??**) the London equations in the following form:

$$\begin{aligned}
&\vec{E} - \vec{\nabla}\rho - \frac{\partial \vec{J}}{\partial t} = 0, \\
&\vec{B} + \vec{\nabla} \times \vec{J} = 0, \\
&\frac{1}{c}\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot J = 0.
\end{aligned} \tag{17}$$

The third equation in this system is the continuity equation. Returning to the original notation, it has the following form:

$$\frac{\partial \rho_s}{\partial t} + \vec{\nabla} \cdot j_s = 0. \tag{18}$$

4 Sedeonic wave equation for electromagnetic field in superconductor

The electromagnetic field in a superconductor can be described by the following generalized sedeonic wave equation:

$$\begin{aligned}
&\left(i\mathbf{e}_1 \frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_2 \vec{\nabla} - i\mathbf{e}_3 m \right) \left(i\mathbf{e}_1 \frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_2 \vec{\nabla} - i\mathbf{e}_3 m \right) \left(i\mathbf{e}_1 \varphi + \mathbf{e}_2 \vec{A} \right) \\
&\qquad = -4\pi i \mathbf{e}_1 \rho_n - \mathbf{e}_2 \vec{j}_n,
\end{aligned} \tag{19}$$

where

$$m = \frac{m_0 c}{\hbar} = \frac{1}{\lambda}. \tag{20}$$

Here m_0 is the mass of quantum of London field (**??**, ρ_n is the volume density of external (normal) charges, $\vec{j}_n$ is the volume density of external (normal) current.

Multiplying the operators on the left side of (**??**) and separating the values of different types, we obtain a system of inhomogeneous wave equations

$$\begin{aligned}
&\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle + \frac{1}{\lambda^2} \right) \varphi = 4\pi\rho_n, \\
&\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle + \frac{1}{\lambda^2} \right) \vec{A} = \frac{4\pi}{c}\vec{j}_n.
\end{aligned} \tag{21}$$

On the other hand, the sequential action of the operators in (**??**) leads us to the combined system of Maxwell and London equations. Indeed, let us introduce the scalar and vector field strengths [23] according to the relations

$$\begin{aligned}
&g = -m\varphi, \\
&\vec{G} = -m\vec{A}, \\
&\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\varphi, \\
&\vec{B} = \vec{\nabla} \times \vec{A}.
\end{aligned} \tag{22}$$

Then the action of the first operator in equation (**??**) is reduced to

$$\left(i\mathbf{e}_1 \frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_2 \vec{\nabla} - i\mathbf{e}_3 m \right) \left(i\mathbf{e}_1 \varphi + \mathbf{e}_2 \vec{A} \right) = -i\mathbf{e}_2 g + \mathbf{e}_1 \vec{G} + \mathbf{e}_3 \vec{E} - i\vec{B} \tag{23}$$

and the equation (**??**) takes the form

$$\left(i\mathbf{e}_1 \frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_2 \vec{\nabla} - i\mathbf{e}_3 m \right) \left(-i\mathbf{e}_2 g + \mathbf{e}_1 \vec{G} + \mathbf{e}_3 \vec{E} - i\vec{B} \right) = -4\pi i \mathbf{e}_1 \rho_n - \mathbf{e}_2 \vec{j}_n, \tag{24}$$

Separating the values of different types, we obtain a system which combines inhomogeneous Maxwell-Proca equations and London equations

$$
\begin{aligned}
&\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla}\times\mathbf{E},\\
&\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \vec{\nabla}\times\vec{B} - m\vec{G} - \frac{4\pi}{c}\mathbf{j}_n,\\
&\vec{\nabla}\cdot\vec{E} = mg + 4\pi\rho_n,\\
&\vec{\nabla}\cdot\vec{B} = 0,\\
&\frac{1}{c}\frac{\partial \vec{G}}{\partial t} = m\vec{E} - \vec{\nabla}g,\\
&\vec{\nabla}\times\mathbf{G} = -m\vec{B},\\
&\frac{1}{c}\frac{\partial \mathbf{g}}{\partial t} = -\vec{\nabla}\cdot\vec{G}.
\end{aligned}
\tag{25}
$$

In the absence of external charges and currents, this system coincides with equations (**??**) and (**??**). On the other hand, applying the operator

$$
\left(i\mathbf{e}_1\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_2\vec{\nabla} - i\mathbf{e}_3 m\right) \tag{26}
$$

to the equation (**??**) and separating quantities of different types, we obtain the following system of inhomogeneous wave equations:

$$
\begin{aligned}
&\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle + m^2\right)\vec{B} = -\frac{4\pi}{c}\vec{\nabla}\times\vec{j}_n,\\
&\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle + m^2\right)\vec{E} = 4\pi\vec{\nabla}\rho_n + \frac{4\pi}{c^2}\frac{\partial \vec{j}_n}{\partial t},\\
&\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle + m^2\right)\vec{G} = -\frac{4\pi}{c}m\vec{j}_n,\\
&\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle + m^2\right)g = -4\pi m\rho_n.
\end{aligned}
\tag{27}
$$

5 Relations between energy and momentum of London field

From the system (**??**) we obtain the following relation between the energy and field momentum:

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\left(g^2 + \vec{G}^2 + \vec{E}^2 + \vec{B}^2\right) + \vec{E}\cdot[\vec{\nabla}\times\vec{B}] - \vec{B}\cdot[\vec{\nabla}\times\vec{E}],\\
&+g(\vec{\nabla}\cdot\vec{G}) + (\vec{G}\cdot\vec{\nabla}g) + \frac{4\pi}{c}\vec{E}\cdot\vec{j}_n = 0.
\end{aligned}
\tag{28}
$$

Introducing the following notations:

$$
\begin{aligned}
w &= \frac{1}{8\pi}\left(g^2 + \vec{G}^2 + \vec{E}^2 + \vec{B}^2\right),\\
\vec{P} &= \frac{c}{4\pi}\left(g\vec{G} + \vec{E}\times\vec{B}\right),
\end{aligned}
\tag{29}
$$

the expression (**??**) is represented as

$$
\frac{\partial w}{\partial t} + \vec{\nabla}\cdot\vec{P} + \vec{E}\cdot\vec{j}_n = 0, \tag{30}
$$

where w is volume density of energy and $\vec{P}$ is volume density of energy flux. The term $\vec{E}\cdot\vec{j}_n$ describes the Joule dissipation. The expression(**??**) is the analog of Poynting theorem for London field [1].

6 Interaction of external point charges in superconductor

Let us consider the point electrical charge q (for example interstitial impurity ion) in a superconductor. Corresponding volume charge density is

$$\rho_n = q\delta(\vec{r}), \tag{31}$$

where $\delta(\vec{r})$ is Dirac delta function. The electric field of this charge is screened due to the redistribution of charges of the superconducting phase. The screening is described by a stationary wave equation (**??**)

$$\left(\triangle - \frac{1}{\lambda^2}\right)\varphi(\vec{r}) = -q\delta(\vec{r}). \tag{32}$$

In spherical coordinates this equation has the following solution:

$$\varphi = \frac{q}{r} exp\left(-\frac{r}{\lambda}\right), \tag{33}$$

where r is the distance from point charge ($r = |\vec{r}|$). This solution describes the disturbance of volume density of charges in superconducting phase. The scalar field in a superconductor (see (**??**)) is

$$g = -\frac{1}{\lambda}\frac{q}{r} exp\left(-\frac{r}{\lambda}\right). \tag{34}$$

Accordingly, the electric field is equal to:

$$\vec{E} = \left(\frac{1}{r} + \frac{1}{\lambda}\right)\frac{q}{r} exp\left(-\frac{r}{\lambda}\right). \tag{35}$$

Let us consider the interaction of two point electrical charges q_1 and q_2 in an infinite superconductor. Following (**??**) the interaction energy of two point charges is

$$W_{12} = \frac{1}{4\pi}\int\limits_V (g_1 g_2 + \vec{E}_1 \cdot \vec{E}_2) dV, \tag{36}$$

Integrating this expression we get

$$W_{12} = \frac{q_1 q_2}{R} exp\left(-\frac{R}{\lambda}\right), \tag{37}$$

where R is the distance between point charges.

7 Conclusion

The sedeonic space-time algebra provides the Lorentz invariance of the equations and can be effectively applied for the high-symmetric description of classical and quantum fields. In the frames of London's model we have proposed nonhomogeneous sedeonic wave equation for electromagnetic field in superconductor. This equation includes the combined system of Maxwell-Proca and London equations and describes the field with nonzero mass of quantum. It leads to the specific screened interaction between electrical charges in superconducting media caused by overlapping of vector electric fields and scalar fields associated with nonhomogenety of charge density of superconducting phase. In general, considered approach can be applied for qualitative analysis of the electromagnetic effects in type II superconductors.

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