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The *abc* Conjecture: The Proof of $c < rad^2(abc)$

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Abstract

In this note, I present a very elementary proof of the conjecture $c < rad^2(abc)$ that constitutes the key to resolve the *abc* conjecture. The method concerns the comparison of the number of primes of c and $rad^2(abc)$ for large a, b, c using the prime counting function $\pi(x)$ giving the number of primes $\leq x$. Some numerical examples are given.

To my wife Wahida, my daughter Sinda and my son Mohamed Mazen

1. Introduction

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a). \prod_i a_i^{\alpha_i - 1} \tag{1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

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Conjecture 1.1. *Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists a constant $K(\epsilon)$ such that :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc), \quad K(\epsilon) \text{ depending only of } \epsilon. \tag{3}$$

The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine, in November 2018, about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. The difficulty to find a proof of the abc conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a+b$.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [1]. A conjecture was proposed that $c < rad^2(abc)$ [3]. It is the key to resolve the abc conjecture. In this note, I present for the case $c = a + 1$ an idea to obtain the proof of $c < rad^2(ac)$: I will compare the number of primes respectively $\leq c$ and $\leq rad^2(ac)$. The prime counting function noted by $\pi(x)$ is defined for x large as [4]:

$$\pi(x) = \int_2^x \frac{dt}{Logt} \tag{4}$$

We will study in details the case $c = a + 1$, for the second case $c = a + b$, the proof does not change without describing it.

The paper is organized as follows: in the second section, we present some preliminaries and formulas for counting the number of prime numbers less one integer. The details of the proof of the conjecture $c < rad^2(ac)$ are given in section three. In sections four and five, we present some numerical examples.

2. **Preliminaries and notations**

Let a, c positive integers relatively prime with $c = a+1$, $a \geq 2$. We note:

$$a = \mu_a . rad(a) = \mu_a . \prod_{i=1}^{i=N_a} a_i, \quad N_a \geq 2$$

$$\text{The number of primes} \leq a \text{ is } \pi(a) = I = N_a + d_a$$

$$c = \mu_c . rad(c) = \mu_c . \prod_{k=1}^{k=N_c} c_k, \quad N_c \geq 2$$

$$\text{The number of primes} \leq c \text{ is } \pi(c) = K = N_c + d_c$$

$$R = rad(ac) \Longrightarrow N_R = N_a + N_c$$

$$\text{The number of primes} \leq R \text{ is } \pi(R) = L = N_R + d_R$$

$$R^2 = rad^2(ac) \Longrightarrow N_{R^2} = N_a + N_c$$

$$\text{The number of primes} \leq R^2 \text{ is } \pi(R^2) = M = N_a + N_c + d_{R^2}$$

$$\Delta = \pi(R^2) - \pi(c) \tag{5}$$

In our study, we suppose that $c > R$ and a, c are large positive integers. The expression of Δ gives:

$$\Delta = \pi(R^2) - \pi(c) = M - K = (N_a + N_c + d_{R^2}) - (N_c + d_c) \Longrightarrow$$

$$\Delta = N_a + d_{R^2} - d_c = d_{R^2} + N_a - d_c \tag{6}$$

As $c > a$ and c, a are not prime integers, then $\pi(c) = \pi(a)$, we obtain:

$$\Delta = d_{R^2} + N_a - d_c = d_{R^2} + N_a - (\pi(c) - N_c) = d_{R^2} + N_c + N_a - \pi(c) \tag{7}$$

but $\pi(c) = \pi(a)$, the last equation can be written as:

$$\Delta = d_{R^2} + N_a - d_c = d_{R^2} + N_c + N_a - \pi(a) = d_{R^2} + N_c + N_a - Np_a - d_a$$

$$\Longrightarrow \boxed{\Delta = d_{R^2} + N_c - d_a = d_{R^2} + N_a - d_c} \tag{8}$$

Then we deduce an invariant:

$$\boxed{N_a - d_c = N_c - d_a} \tag{9}$$

As $c > R \Longrightarrow \pi(c) > \pi(R) \Longrightarrow N_c + d_c > N_a + N_c + d_R \Longrightarrow -d_R > N_a - d_c$. Then:

$$\boxed{N_a < d_c} \tag{10}$$

and the formulas (9) can written as:

$$\boxed{d_c - N_a = d_a - N_c > 0} \tag{11}$$

and we write Δ as :

$$\boxed{\Delta = d_{R^2} - (d_c - N_a) = d_{R^2} - (d_a - N_c)} \tag{12}$$

Let us take the example:

$$1 + 2.3^7 = 5^4.7 \Longrightarrow 1 + 4374 = 4375 \tag{13}$$

We find from $c = a + 1$:

$$\begin{gathered}
\pi(a) = \pi(4375) = 597, N_a = 2, d_a = 595 \Longrightarrow N_a \ll d_a \\
\pi(c) = \pi(4374) = 597, N_c = 2, d_c = 595 \Longrightarrow N_c \ll d_c \\
N_c \approx N_a \Longrightarrow d_c \approx d_a \qquad (14) \\
R = 2.3.5.7 = 210 \Longrightarrow \pi(210) = 46 \Longrightarrow d_R = 42 \Longrightarrow N_a, N_c \ll d_R \\
R^2 = (2.3.5.7)^2 = 210^2 = 44100 \Longrightarrow \pi(R^2) = \pi(44100) = 4412 > 597 \Longrightarrow \\
d_{R^2} = 4412 - 2 - 2 = 4408 \Longrightarrow d_a \ll d_{R^2}; d_c \ll d_{R^2}; d_R \ll d_{R^2} \Longrightarrow \\
\Delta = \pi(R^2) - \pi(c) = 4412 - 597 = 3815 > 0 \Longrightarrow c < R^2, \pi(c) \ll \pi(R^2) \\
(R = 210) < (c = 4375); (\mu_c = 5^3 = 125) > (rad(c) = 5.7 = 35) \\
\Longrightarrow (\mu_a = 3^6 = 729) > (rad(a) = 2.3 = 6)
\end{gathered}$$

And the conjecture $c < R^2$ is true. We give below the proof of $c < R^2$.

3. The Proof of $c < R^2$

Proof. : We will not use the formulas developed above but an analytic method. We will proceed by induction on n with $c_n = a_n + 1$, a_n, c_n not prime numbers but relatively comprime, so that $c_n > R_n$ where $R_n = rad(a_n c_n)$.

3.1. *Case* $k = 1$, $c_1 = a_1 + 1$. It gives $a_1 = 8, c_1 = 9 \Longrightarrow rad(a_1) = 2, rad(c_1) = 3 \Longrightarrow R_1 = rad(a_1 c_1) = 6 < c_1 \Longrightarrow R_1^2 = rad^2(a_1 c_1) = 36$ and $\pi(R_1^2) = \pi(36) = 11$ prime numbers={2,3,5,7,11,13,17,19,23,29,31}, $\pi(c_1) = \pi(9) = 4$ prime numbers={2,3,5,7}. Then we obtain $\Delta_1 = \pi(R_1^2) - \pi(c_1) = 11 - 4 = 7 > 0$ and the conjecture holds.

Assume that the conjecture $c < R^2$ has already been found to hold for k=2,3,...,n. Then we shall show that the conjecture also holds for $k = n + 1$ and hence by induction for all integers.

3.2. *Case* $k = n$, $c_n = a_n + 1$. We assume that a_n or c_n is not prime with $c_n > R_n$, and the conjecture holds for $k = n \Longrightarrow \pi(R_n^2) > \pi(c_n)$, with $\pi(c_n) \ll \pi(R_n^2)$. Then $c_n < R_n^2$. Now we consider the case $k = n + 1$.

3.3. *Case* $k = n + 1$. Let $a_{n+1} = c_n$, we obtain $c_{n+1} = a_{n+1} + 1$. We suppose that c_{n+1} is not a prime and $R_{n+1} = rad(c_{n+1})rad(c_n) < c_{n+1}$, if not, the conjecture $c < R^2$ holds. Then we take the first $c_n = c_n + r$ so that $c_n, c_{n+1} = c_n + 1$ verifying c_n or c_{n+1} not a prime and $c_{n+1} > R_{n+1}$. Let

$$\Delta_{n+1} = \pi(R_{n+1}^2) - \pi(c_{n+1}) \tag{15}$$

As a_n, c_n, c_{n+1} are not prime, then $\pi(c_{n+1}) = \pi(c_n)$ and we write equation (15) as:

$$\Delta_{n+1} = \pi(R_{n+1}^2) - \pi(R_n^2) + \pi(R_n^2) - \pi(c_n) \tag{16}$$

Using the case $k = n$, we know that $\pi(R_n^2) - \pi(c_n) > 0$, then:

- If $\pi(R_{n+1}^2) - \pi(R_n^2) > 0 \Longrightarrow \Delta_{n+1} > 0 \Longrightarrow c_{n+1} < R_{n+1}^2$. As $\pi(c_n) \ll \pi(R_n^2) \Longrightarrow \pi(c_n) \ll (\pi(R_n^2) + (\pi(R_{n+1}^2) - \pi(R_n^2))) \Longrightarrow \pi(c_n) \ll \pi(R_{n+1}^2)$. But $\pi(c_n) = \pi(c_{n+1}) \Longrightarrow \pi(c_{n+1}) \ll \pi(R_{n+1}^2)$. Then conjecture holds for the case $k = n + 1$.

- If $\pi(R_{n+1}^2) - \pi(R_n^2) < 0 \Longrightarrow R_{n+1}^2 < R_n^2 \Longrightarrow R_{n+1} < R_n$. So, we consider in the following that $R_{n+1} < R_n$. We will use an expression of the function $\pi(X)$ giving in [4] as:

Theorem 3.1. *There exists a constant $l > 0$ so that*:

$$\pi(X) = \int_2^X \frac{du}{Logu} + O(Xe^{-l(LogX)^{1/2}}) \tag{17}$$

It follows that, for $X > 4$:

$$\pi(X) = \frac{X}{LogX} + O\left(\frac{X}{Log^2 X}\right) \tag{18}$$

where $O(f)$ designs Landau O notation. We write the equation (17) as:

$$\pi(X) = \int_2^X \frac{du}{Logu} + \lambda(X), \quad \lambda(X) = O(Xe^{-l(LogX)^{1/2}}) \tag{19}$$

As a_n, c_n, c_{n+1} are not prime, it follows that $\pi(c_{n+1}) = \pi(c_n)$, it gives:

$$\Delta_{n+1} = \pi(R_{n+1}^2) - \pi(c_n) = \int_2^{R_{n+1}^2} \frac{du}{Logu} - \int_2^{c_n} \frac{du}{Logu} + \lambda(R_{n+1}^2) - \lambda(c_n) \tag{20}$$

- Case (i): we suppose that $R_{n+1}^2 > R_n$, we obtain:

$$\Delta_{n+1} = \pi(R_{n+1}^2) - \pi(c_n) = \int_{R_n}^{R_{n+1}^2} \frac{du}{Logu} - \int_{R_n}^{c_n} \frac{du}{Logu} + \lambda(R_{n+1}^2) - \lambda(c_n) \tag{21}$$

Using the mean value theorem, we obtain:

$$\int_{R_n}^{R_{n+1}^2} \frac{du}{Logu} = (R_{n+1}^2 - R_n).\frac{1}{Log\theta} \quad \theta \in]R_n, R_{n+1}^2[$$

Then we write that $1/Log\theta = (1 + \mu).1/LogR_{n+1}^2$ with $\mu > 1$. So we obtain:

$$\Delta_{n+1} = \frac{R_{n+1}^2}{LogR_{n+1}^2}\left(1 - \frac{R_n}{R_{n+1}^2}\right)(1 + \mu) - \int_{R_n}^{c_n} \frac{du}{Logu} + \lambda(R_{n+1}^2) - \lambda(c_n) \tag{22}$$

Using the same theorem for the second integral, we obtain :

$$\Delta_{n+1} > \frac{R_{n+1}^2}{LogR_{n+1}^2}\left(1-\frac{R_n}{R_{n+1}^2}\right)(1+\mu) - \frac{c_n}{Logc_n}\left(1-\frac{R_n}{c_n}\right)$$
$$+\lambda(R_{n+1}^2)-\lambda(c_n)$$

The last equation can written as:

$$\Delta_{n+1} > \frac{R_n^2}{LogR_n^2}.\frac{LogR_n^2}{LogR_{n+1}^2}.\frac{R_{n+1}^2}{R_n^2}\left(1-\frac{R_n}{R_{n+1}^2}\right)(1+\mu) - \frac{c_n}{Logc_n}\left(1-\frac{R_n}{c_n}\right)$$
$$+\lambda(R_{n+1}^2)-\lambda(c_n) \tag{23}$$

As $R_n > R_{n+1}$, we can write:

$$\frac{LogR_n^2}{LogR_{n+1}^2} > 1 \Longrightarrow \frac{LogR_n^2}{LogR_{n+1}^2} = 1+\epsilon, \quad , \epsilon > 0$$

$$\frac{R_{n+1}^2}{R_n^2}\left(1-\frac{R_n}{R_{n+1}^2}\right) = \frac{R_{n+1}^2}{R_n^2} - \frac{1}{R_n} > 0 \Longrightarrow$$

$$\frac{R_{n+1}^2}{R_n^2} - \frac{1}{R_n} - 1 = \frac{-(R_n^2 - R_{n+1}^2) - R_n}{R_n^2} < 0 \Longrightarrow 0 < \frac{R_{n+1}^2}{R_n^2} - \frac{1}{R_n} < 1$$

$$\Longrightarrow \frac{R_{n+1}^2}{R_n^2}\left(1-\frac{R_n}{R_{n+1}^2}\right) = 1-\epsilon', \quad \epsilon' > 0 \tag{24}$$

Then the equation (23) becomes:

$$\Delta_{n+1} > \frac{R_n^2}{LogR_n^2} - \frac{c_n}{Logc_n} + \frac{R_n}{Logc_n} + \frac{R_n^2}{LogR_n^2}(\mu+\epsilon-\epsilon')$$
$$+\lambda(R_{n+1}^2)-\lambda(c_n) \tag{25}$$

Using the equation (18), we obtain:

$$\Delta_{n+1} > \pi(R_n^2) - \pi(c_n) + \frac{R_n}{Logc_n} + \frac{R_n^2}{LogR_n^2}(\mu+\epsilon-\epsilon')$$
$$-O\left(\frac{R_n^2}{Log^2R_n^2}\right) + O\left(\frac{c_n}{Log^2c_n}\right) + \lambda(R_{n+1}^2)-\lambda(c_n) \tag{26}$$

As $\pi(R_n^2) - \pi(c_n) > 0$ and $\pi(c_n) \ll \pi(R_n^2)$ and from the equation above we can conclude, since c_n, R_n, R_{n+1} are large integers, that :

$$\Delta_{n+1} = \pi(R_{n+1}^2) - \pi(c_{n+1}) > 0 \Longrightarrow R_{n+1}^2 \geq c_{n+1} \Longrightarrow R_{n+1}^2 > c_{n+1} \tag{27}$$
$$and \quad \pi(c_{n+1}) \ll \pi(R_{n+1}^2) \tag{28}$$

Hence, the conjecture holds for $k = n+1$ in the case $R_{n+1}^2 > R_n$.

- Case (ii) :$R_{n+1}^2 < R_n$

Let A be the statement " If $c_{n+1} < R_{n+1}^2 \Longrightarrow R_n < R_{n+1}^2$". We have $R_{n+1}^2 > c_{n+1} > c_n > R_n$, then A is true. We consider its negation, we find: " If $R_n > R_{n+1}^2 \Longrightarrow c_{n+1} > R_{n+1}^2$. Then the case $R_{n+1}^2 < R_n$ is false.

Then the conjecture holds for $k = n + 1$.

In our proof, we have used the parameters $c_n, R_n, c_{n+1}, R_{n+1}$, then for the case $c = a + b$, the proof is unchanged. So we can announce the important theorem:

THEOREM 3.2. *Let a, b, c positive integers relatively prime with $c = a+b$, then*:

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{29}$$

This result, I think is the key to obtain a proof of the veracity of the *abc* conjecture. In the two following sections, we are going to verify some numerical examples.

4. Examples : Case $c = a + 1$

4.1. *Example* 1. The example is given by:

$$1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6 \tag{30}$$

$a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47\,045\,880 \Rightarrow \mu_a = (2 \times 3 \times 7)^2 = 1764$ and $rad(a) = 2 \times 3 \times 5 \times 7 \times 127$, in this example, $\mu_a < rad(a)$.
$c = 19^6 = 47\,045\,881 \Rightarrow rad(c) = 19$. Then $rad(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506\,730$.
We have $c > rad(ac)$ but $rad^2(ac) = 506\,730^2 = 256\,775\,292\,900 > c = 47\,045\,881 >$.

4.2. *Example* 2. We give here the example 2 from $https://nitaj.users.lmno.cnrs.fr$:

$$3^7 \times 7^5 \times 13^5 \times 17 \times 1831 + 1 = 2^{30} \times 5^2 \times 127 \times 353 \tag{31}$$

$a = 3^7 \times 7^5 \times 13^5 \times 17 \times 1831 = 424\,808\,316\,456\,140\,799 \Rightarrow rad(a) = 3 \times 7 \times 13 \times 17 \times 1831 = 8497671 \Longrightarrow \mu_a > rad(a)$,
$b = 1$, $rad(c) = 2 \times 5 \times 127 \times 353$ Then $rad(ac) = 849767 \times 448310 = 3\,809\,590\,886\,010 < c$. $rad^2(ac) = 14\,512\,982\,718\,770\,456\,813\,720\,100 > c$, then $c \leq 2rad^2(ac)$.

5. **Examples : Case** $c = a + b$

5.1. *Example* 1. We give here the example of Eric Reyssat [1], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \tag{32}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$.
$rad^2(abc) = 226\,261\,764 > c$.

5.2. *Example* 2. The example of Nitaj about the abc conjecture [1] is:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79 \tag{33}$$

$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311 \tag{34}$$

$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953 \tag{35}$$

$$rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110$$

$$rad^2(abc) = 831\,072\,936\,124\,776\,471\,158\,132\,100 >$$

$$c = 613\,474\,845\,886\,230\,468\,750$$

5.3. *Example* 3. It is of Ralf Bonse about the abc conjecture [3] :

$$2543^4.182587.2802983.85813163 + 2^{15}.3^{77}.11.173 = 5^{56}.245983 \tag{36}$$

$$a = 2543^4.182587.2802983.85813163$$

$$b = 2^{15}.3^{77}.11.173$$

$$c = 5^{56}.245983 = 3.41369987832962351603782735764498e + 44$$

$$rad(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163$$

$$rad(abc) = 1.5683959920004546031461002610848e + 33$$

$$rad^2(abc) = 2.4598659877230900595045886864951e + 66 > c$$

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