ζ(z) = X(z) – Y(z) A decomposition of the Riemann Zeta function for $Re(z) > 0, z \neq 1$

Pedro Caceres
Dallas, Texas (USA) – March 8th, 2020
Email: pcaceres@comcast.net
Phone: +1 (763) 412-8915
www.DNA-PRIMES.com

Abstract:

In this paper, we define the C-transformation as:

$$C_n\{f\} = \sum_{k=1}^{n} f(k) - \int f(n)\ dn \tag{1}$$

And the C-values as:

$$C\{f\} = \lim_{n\to\infty} C_n\{f\} \tag{2}$$

And we obtain a new representation for ζ(z) in the form ζ(z) = X(z) – Y(z) applying the C-transformation to the function $f(x) = \frac{1}{x^z}$ for $z \in C, Re(z) \geq 0, z \neq 1$.

****

Nomenclature and conventions

a. ζ(z)= $\sum_{k=1}^{\infty} k^{-z}$ is the Riemann Zeta function
b. Re(z) is the real part of a complex number z
c. Im(z) is the imaginary part of a complex number z

1. C-Transformation of f(x)

The C-transformation of an integrable function f(x) is defined by:

$$C_n\{f(x)\} = \sum_{k=1}^{n} f(k) - \int f(n)\ dn \tag{3}$$

And the C-values is the limit, if it exists, of the C-transformation when $n \to \infty$:

$$C\{f(x)\} = \lim_{n\to\infty} C_n\{f(x)\} \tag{4}$$

1.1. C-Transformation of $f(x) = \frac{1}{x}$ for $x \in R$:

$$C_n\left\{\frac{1}{x}\right\} = \sum_{\mathrm{k=1}}^{\mathrm{n}} \frac{1}{\mathrm{k}} - \int \frac{dn}{n} \tag{5}$$

and

$$\mathrm{C}\left\{\frac{1}{x}\right\} = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln(n)\right) = \gamma \tag{6}$$

(γ = Euler-Mascheroni constant = 0.5772...)

1.2. C-Transformation of $f(x) = m$, for $m \in R$ constant:

$$C_n\{m\} = \sum_{\mathrm{k=1}}^{\mathrm{n}} m - \int m\ dn \tag{7}$$

$$C_n\{\mathrm{m}\} = m * n - m * n = 0 \tag{8}$$

and the C-values of f(x) =m constant is:

$$C\{m\} = 0 \tag{9}$$

1.3. C-Transformation of $f(x) = \sin(x)$ for $x \in R$:

$$C_n\{\sin(x)\} = \sum_{\mathrm{k=1}}^{\mathrm{n}} \sin(\mathrm{k}) - \int \sin(n)\, dn \tag{10}$$

$$C_n\{\sin(x)\} = \frac{1}{2\left(\sin(n) - \cot\left(\frac{1}{2}\right)\cos(n) + \cot\left(\frac{1}{2}\right) + \cos(n)\right)} \tag{11}$$

And the C-values of $f(x) = \sin(x)$ are in the interval:

$$\mathrm{C}\{\sin(x)\} \in \left[\frac{1}{2}\left(2\cot\left(\frac{1}{2}\right) - 3\right), \frac{3}{2}\right] \tag{12}$$

One can also calculate that:

$$\mathrm{C}\{\cos(x)\} \in \left[\frac{1}{2}\left(\cot\left(\frac{1}{2}\right) - 4\right), \frac{1}{2}\left(2 - \cot\left(\frac{1}{2}\right)\right)\right] \tag{13}$$

1.4. C-Transformation of $f(x) = \mathrm{e}^{-\mathrm{x}}$ for $x \in R$:

$$C_n\{\mathrm{e}^{-\mathrm{x}}\} = \sum_{\mathrm{k=1}}^{\mathrm{n}} \mathrm{e}^{-\mathrm{k}} - \int e^{-n} dn \tag{14}$$

$$C_n\{\sin(x)\} = \sum_{\mathrm{k=1}}^{\mathrm{n}} \mathrm{e}^{-\mathrm{k}} + \frac{e^{-n}}{n} \tag{15}$$

And the C-values of $f(x) = \mathrm{e}^{-\mathrm{x}}$ are:

$$\mathrm{C}\{\mathrm{e}^{-\mathrm{x}}\} = \frac{1}{\mathrm{e} - 1} \tag{16}$$

1.5. C-Transformation of $f(x) = x^{-s}$ for $x, s \in R$, s>1:

$$C_n\left\{\frac{1}{x^s}\right\} = \sum_{\mathrm{k=1}}^{\mathrm{n}} \frac{1}{\mathrm{k^s}} - \int \frac{dn}{n^s} \tag{17}$$

$$C_n\left\{\frac{1}{x^s}\right\} = \sum_{\mathrm{k=1}}^{\mathrm{n}} \frac{1}{\mathrm{k^s}} - \frac{n^{1-s}}{1-s} \tag{18}$$

and the C-value of $f(x) = \frac{1}{x^s}$ is the Riemann Zeta function for s>1:

$$\mathrm{C}\left\{\frac{1}{x^s}\right\} = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{\mathrm{k^s}} - \frac{\mathrm{n^{1-s}}}{1-\mathrm{s}}\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{\mathrm{k^s}}\right) - \lim_{n\to\infty}\left(\frac{\mathrm{n^{1-s}}}{1-\mathrm{s}}\right) = \zeta(\mathrm{s}) \tag{19}$$

1.6. C-Transformation of $f(z) = \frac{1}{x^z}$ for $z \in C, Re(z) \geq 0, z \neq 1$

$$C_n\left\{\frac{1}{x^z}\right\} = \sum_{\mathrm{k=1}}^{\mathrm{n}} \frac{1}{\mathrm{k^z}} - \int \frac{dn}{n^z} \tag{20}$$

We will use Euler's identity:

$$e^x = \cos(x) + i * \sin(x) \tag{21}$$

To calculate [20] for z=α+ßi:

$$k^{-z} = k^{-\alpha}\ [\cos(ß * \ln k) - i\ (\sin\ (ß * \ln \mathrm{k})] \tag{22}$$

And:

$$\int \frac{dn}{n^z} = n^{(1-\alpha)} [\cos(ß * ln(n) - i\, sin(ß * ln(n)] * \frac{[(1-\alpha) + iß]}{[(1-\alpha)^2 + ß^{2]}} \tag{23}$$

One can now express the real and imaginary components of $C_n\{f\}$ as:

$$Re(C_n\{f\}) = \sum_{k=1}^{n} k^{-\alpha} (\cos\ (ß * \ln(k)) + \\ + \frac{1}{[(1-\alpha)^2+ß^{2]}} (n^{(1-\alpha)}\ [(1-\alpha)*\cos(ß*\ln(n))+ß*\ \sin(ß*\ln(n))]) \tag{24}$$

$$Im(C_n\{f\}) = -\sum_{k=1}^{n} k^{-\alpha} (\sin\ (ß * \ln(k))+ \\ + \frac{1}{[(1-\alpha)^2+ß^{2]}} (n^{(1-\alpha)}\ [ß*\cos(ß*\ln(n)) - (1-\alpha)*\sin(ß*\ln(n))]) \tag{25}$$

One can calculate that, for α=*Re(z)>2*, and for any ϵ arbitrarily small, there is a value of n=N such that for n>N, $C_N\{f\}$ - $\zeta(z) < \epsilon$, as the following table shows:

α	ß	$C_N\{f\}$ *for N=500*	$\zeta(z)$	$\lvert C\_N \{f\} - \zeta(z)\rvert$
2	0	1.644934068	1.654934067	$< 10^{-8}$
2	1	1.150355702 + 0.437530865 i	1.150355703+0.437530866 i	$< 10^{-8}$
3	0	1.202056903	1.202056903	$< 10^{-9}$

Table 1. Values of $C_n\{f(n) = k^{-z}\}$ *for α=Re(z)>1 for N=500*

The error $C_n\{f\}$ - $\zeta(z)$ grows significantly in the critical strip for 0≤α<1 as we can see in the following table:

A	ß	$C_n\{f\}$	$\zeta(z)$	$\lvert C\_n \{f\} - \zeta(z)\rvert$
0.0	0	$C_N\{f\}$ *for N=500*	-0.5	0.5
0.2	2	0.399824505+0.322650799 i	0.360103 + 0.266246 i	> 0.05
0.7	0	-2.777900606	-2.7783884455	$> 10^{-4}$

Table 2. Values of $C_n\{f(n) = k^{-z}\}$ *for 0≤Re(z)<1 for N=500*

To understand better the value of the difference $C_n\left\{\frac{1}{k^z}\right\}$ - $\zeta(z)$, one can plot the difference for $\alpha \in [0,1)$ and ß $= 0$: (Similar exponential charts occur for all values of $\alpha \in [0,1)$ for any given value of ß)

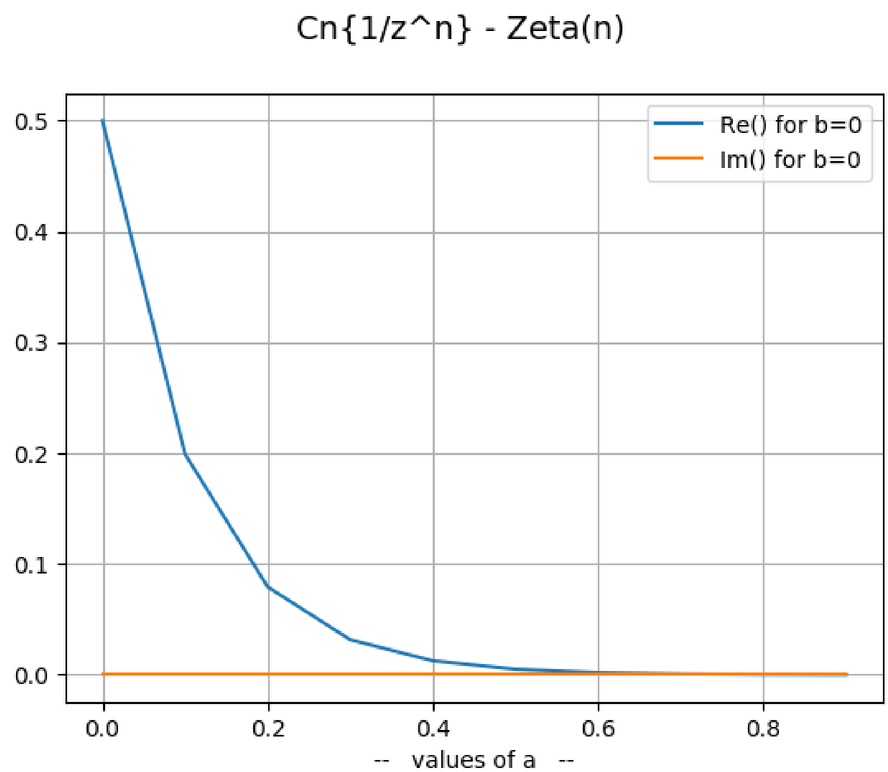


Figure 1 where a=Re(z) and b=Im(z)

And plot the difference for variable values of ß ∈ [0,1) and $\alpha = 0$: (Similar sine charts occur for all values of ß ∈ [0,1) for any given value of α)

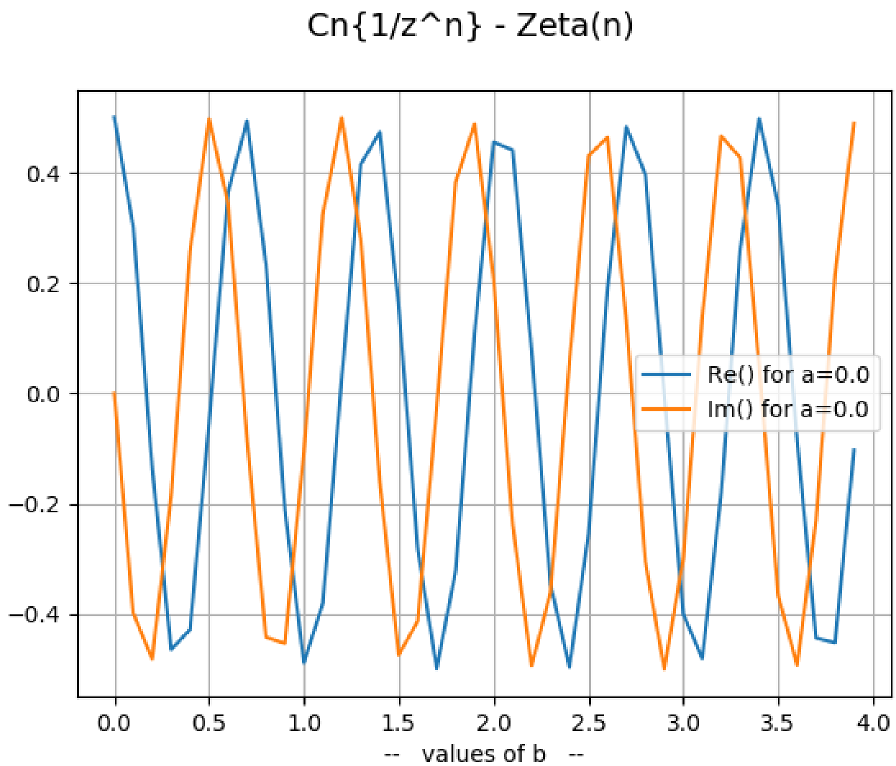


Figure 2 where a=Re(z) and b=Im(z)

These charts lead to the following calculation of the difference $C_n\left\{\frac{1}{k^z}\right\} - \zeta(z)$:

$$\text{Re}[C_n\left\{\frac{1}{k^z}\right\} - \zeta(z)] = \frac{1}{2}\, n^{-a} * \cos\,(ß * \ln\,(\text{n})) + \text{O}(\tfrac{1}{\text{n}}) \tag{26}$$

$$\text{Im}[C_n\left\{\frac{1}{k^z}\right\} - \zeta(z)] = \frac{1}{2}\, n^{-a} * \sin\,(ß * \ln\,(\text{n})) + \text{O}(\tfrac{1}{\text{n}}) \tag{27}$$

With $O(1/n\,) -> 0$ when $n -> \infty$.

And one can finally write:

$$Re(C_n\{f\}) = \sum_{k=1}^{n} k^{-\alpha}\,(\cos\,(ß * \ln(k)) +$$

$$+ \frac{1}{[(1-\alpha)^2 + ß^2]}\,(n^{(1-\alpha)}\ [(1\text{-}\alpha)*\cos(ß*\ln(\text{n}))+ß*\ \sin(ß*\ln(\text{n}))])$$

$$+\frac{1}{2}\, n^{-a} * \cos(ß * \ln(\text{n})) \tag{28}$$

$$Im(C_n\{f\}) = -\sum_{k=1}^{n} k^{-\alpha}\,(\sin\,(ß * \ln(k)) +$$

$$+ \frac{1}{[(1-\alpha)^2 + ß^2]}\,(n^{(1-\alpha)}\ [ß*\cos(ß*\ln(\text{n}))\ \text{-}\ (1\text{-}\alpha)*\sin(ß*\ln(\text{n}))])$$

$$+\frac{1}{2}\, n^{-a} * \sin(ß * \ln(\text{n})) \tag{29}$$

and the C-value of $f(x) = \frac{1}{x^z}$ for $z \in C, Re(z) \geq 0, z \neq 1$ is the Riemann Zeta function ζ(z).

1.7. A decomposition of ζ(z) based on the C-transformation of $f(x) = \frac{1}{x^z}$ for $z \in C, Re(z) \geq 0, z \neq 1$

One can rewrite [28] and [29] creating the $X(z,n)$ and $Y(z,n)$ functions:

$$\zeta(z) = \lim_{n\to\infty} [X(z,n) - Y(z,n)], \text{ where:} \tag{30}$$

$$X(z,n) = (\textstyle\sum_{k=1}^{n} k^{-\alpha} (cos\ (ß * ln(k)) + \frac{1}{2}\ n^{-\alpha} cos\ (ß\ ln(n)) + \\ + i * (\textstyle\sum_{k=1}^{n} k^{-\alpha}\ (sin\ (ß * ln(k)) + \frac{1}{2}\ n^{-\alpha}\ sin(ß\ ln(n)))) \tag{31}$$

$$Y(z,n) = \mathrm{n}^{(1-\alpha)} \frac{1}{[(1-\alpha)^2 + ß^2]} [((1-\alpha) * \cos(ß \ln(n)) + ß * \sin(ß \ln(n))) + \\ + \mathrm{i}\ \ (ß * \cos(ß \ln(n)) - (1-\alpha) * \sin(ß \ln(n)))] \tag{32}$$

We define:

$$X(z) = \lim_{n\to\infty} \mathrm{X(z,n)} \text{ and} \tag{33}$$

$$Y(z) = \lim_{n\to\infty} \mathrm{Y(z,n)} \tag{34}$$

Then, one can write:

$$\zeta(z) = X(z) - Y(z) \tag{35}$$

The following table shows values for [30]:

z= 0 +j* 0   and n=500
Zeta(z)   = -0.5 + i* 0.0 X(z)-Y(z) = -0.5 +i* 0.0 ---> Error = 0.0 +i* 0.0
z= 0.2 +j* 2   and n=500
Zeta(z)   = 0.360102590022591 + i* -0.266246199765574 X(z)-Y(z) = 0.360102741838091 +i* -0.266246128959438 ---> Error= -1.5181550 e-7 +i* -7.080613 e-8
z= 0.4 +j* 0   and n=500
Zeta(z)   = -1.13479778386698 + i* 0.0 X(z)-Y(z) = -1.1347977871726 +i* 0.0 ---> Error= 3.305619 e-9 +i* 0.0

Table 3. ζ(z) compared to X(z) - Y(z)

The highest error for α∈ [0,1), ß ∈ [0,100], n=1000 is $8x10^{-6}$.

2. Representation of the function $X(z,n)$

The following chart represents $X(z,n)$ for $a = 1/2$ and $b \in [1,6]$ and $n = 250$

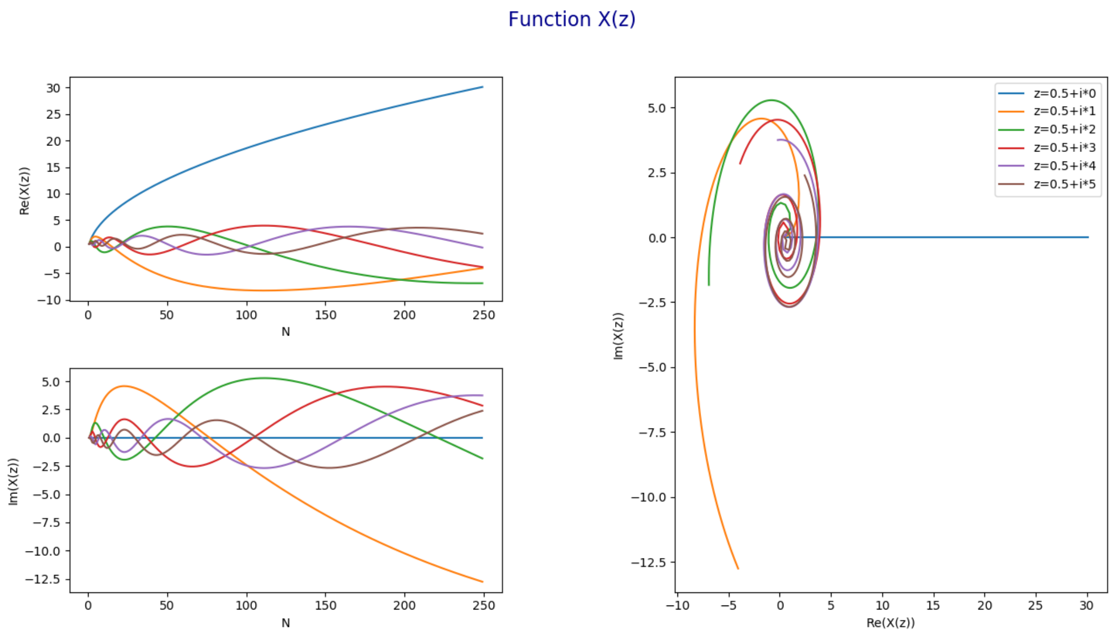


Fig. 3: $X(z,n)$

The following chart represents $X(z,n)$ for $a \in [1,6]$ and $b = 1$ and $n = 250$

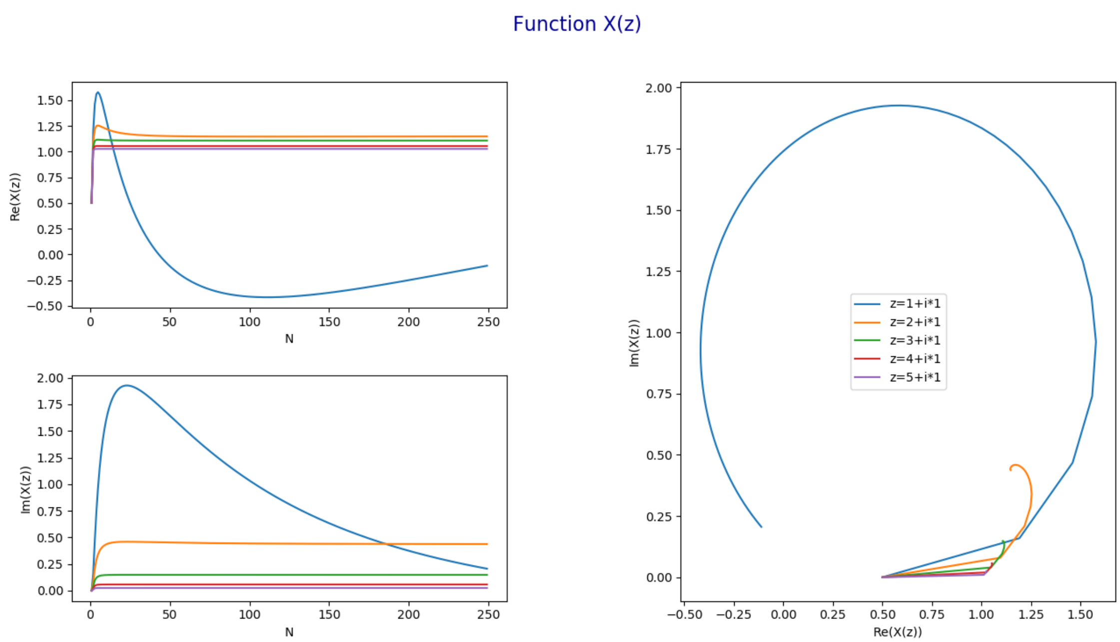


Fig. 4: $X(z,n)$

3. Representation of the function $Y(z,n)$

The following chart represents $Y(z,n)$ for $a = 1/2$ and $b \in [1,6]$ and $n = 250$

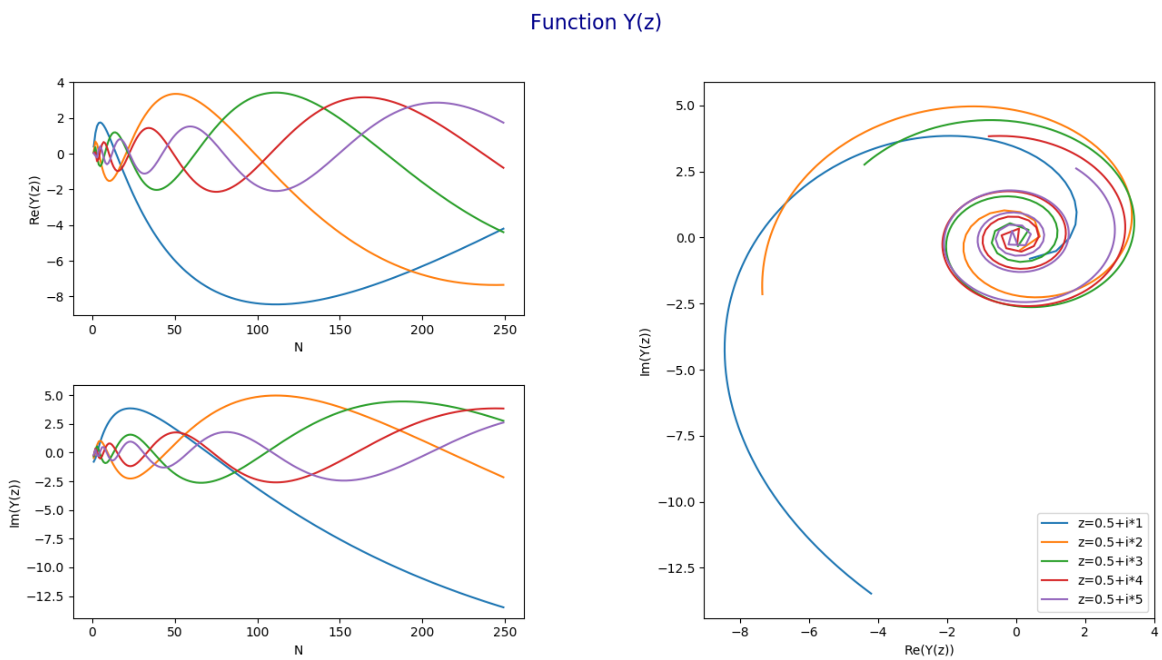


Fig. 5: $Y(z,n)$

The following chart represents $Y(z,n)$ for $a \in [1,6]$ and $b = 1$ and $n = 250$

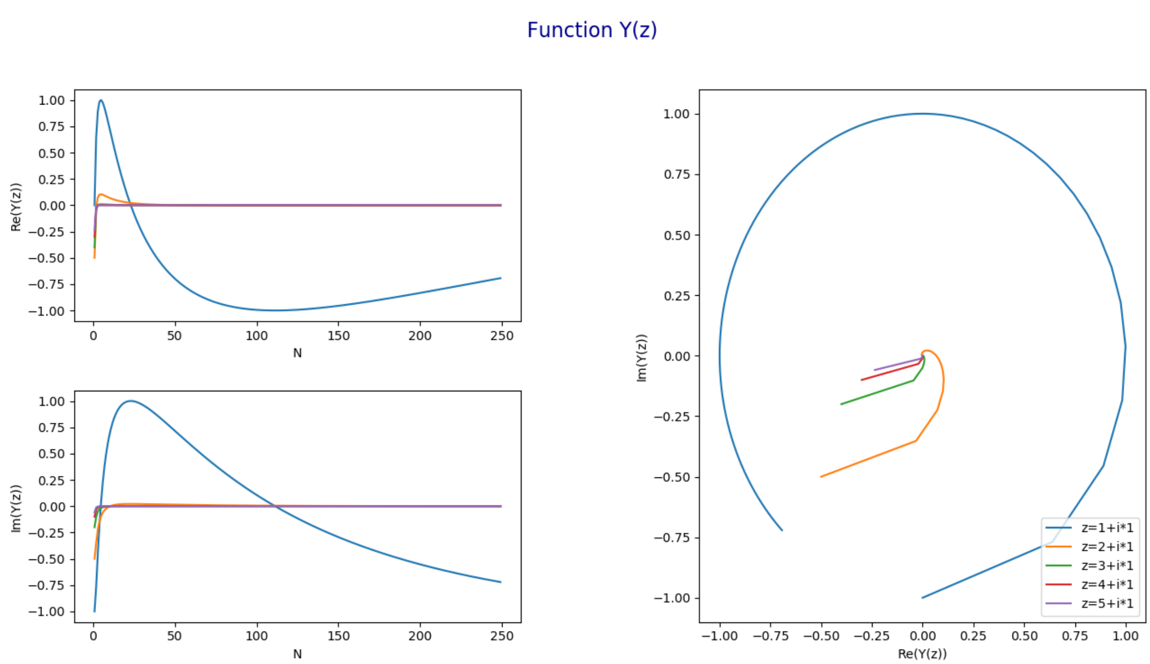
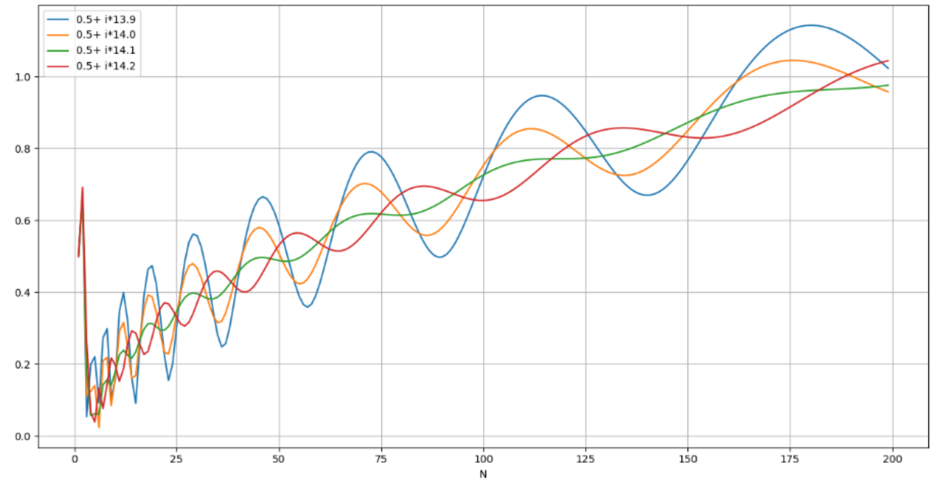


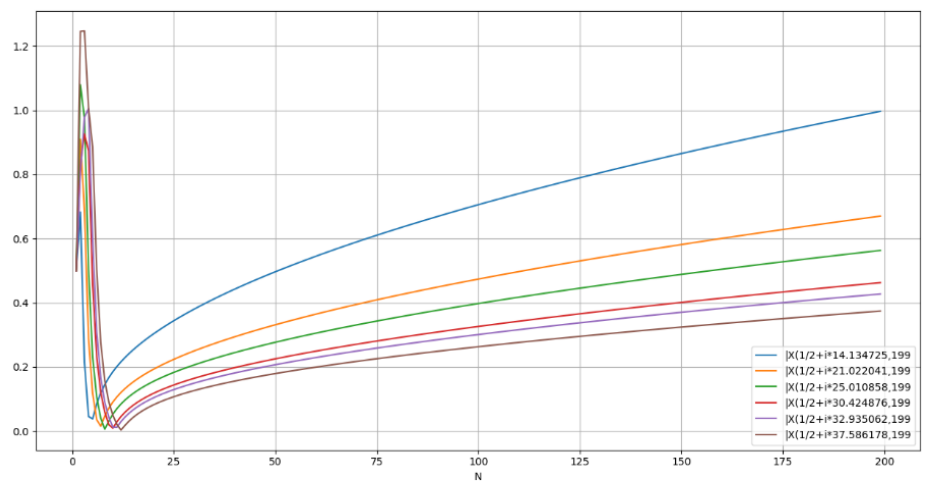
Fig. 6: $Y(z,n)$

4. Representation of $|X(z,n)|$

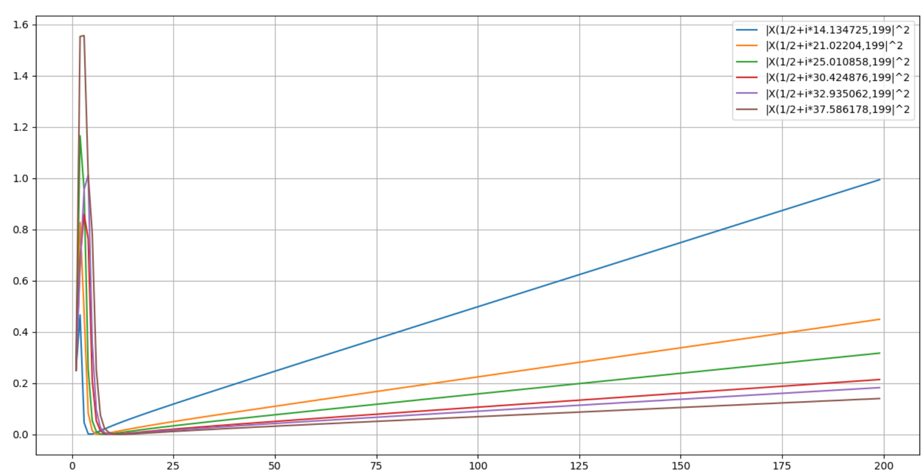
Wave representation for $|X(z,n)|$ $for\ Re(z) = 1/2\ and\ Im(z)\ variable$.



Parabolic representation for $|X(z,n)|$ for (z) a nontrivial zero of Riemann Zeta.

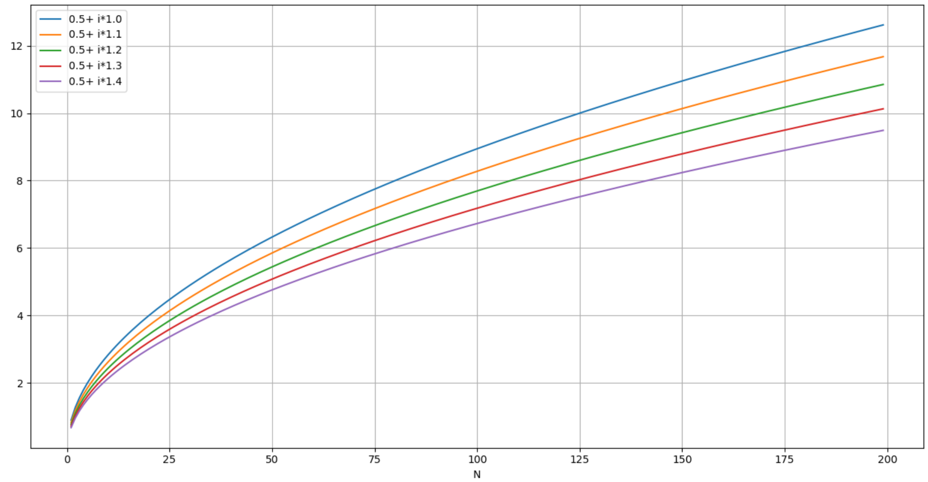


Linear representation for $|X(z,n)|^2$ for (z) a nontrivial zero of Riemann Zeta.

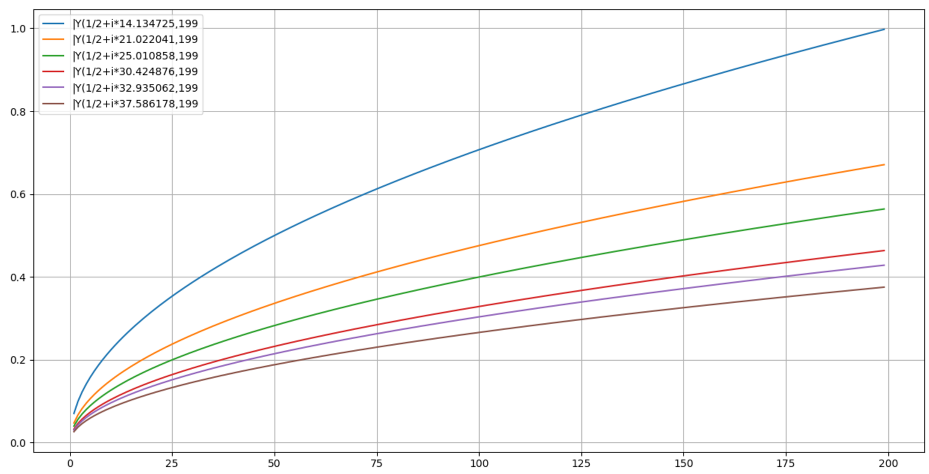


5. Representation of $|Y(z,n)|$

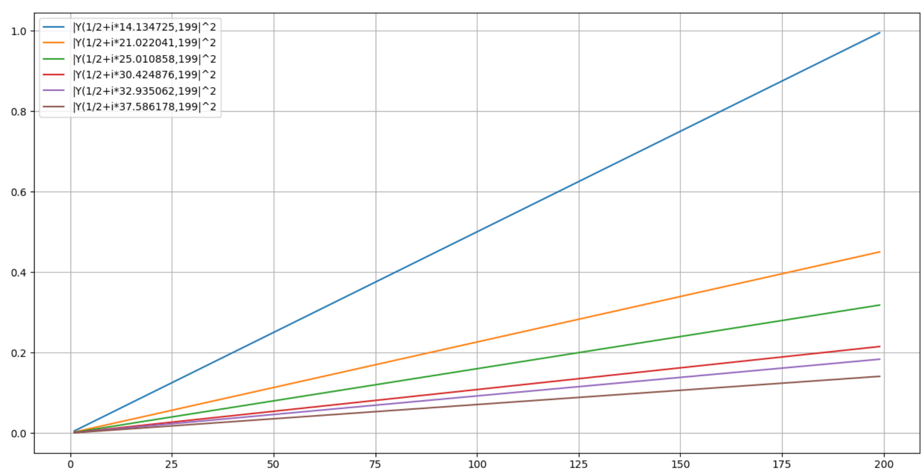
Polynomial representation for $|Y(z,n)|\ for\ Re(z) = 1/2\ and\ Im(z)\ variable.$



Parabolic representation for $|Y(z,n)|\ for\ (z)$a nontrivial zero of Riemann Zeta.



Linear representation for $|Y(z,n)|^2\ for\ (z)$ a nontrivial zero of Riemann Zeta.



6. Representation of the function $\zeta(z) = X(z) - Y(z)$ for Re(z)=1/2

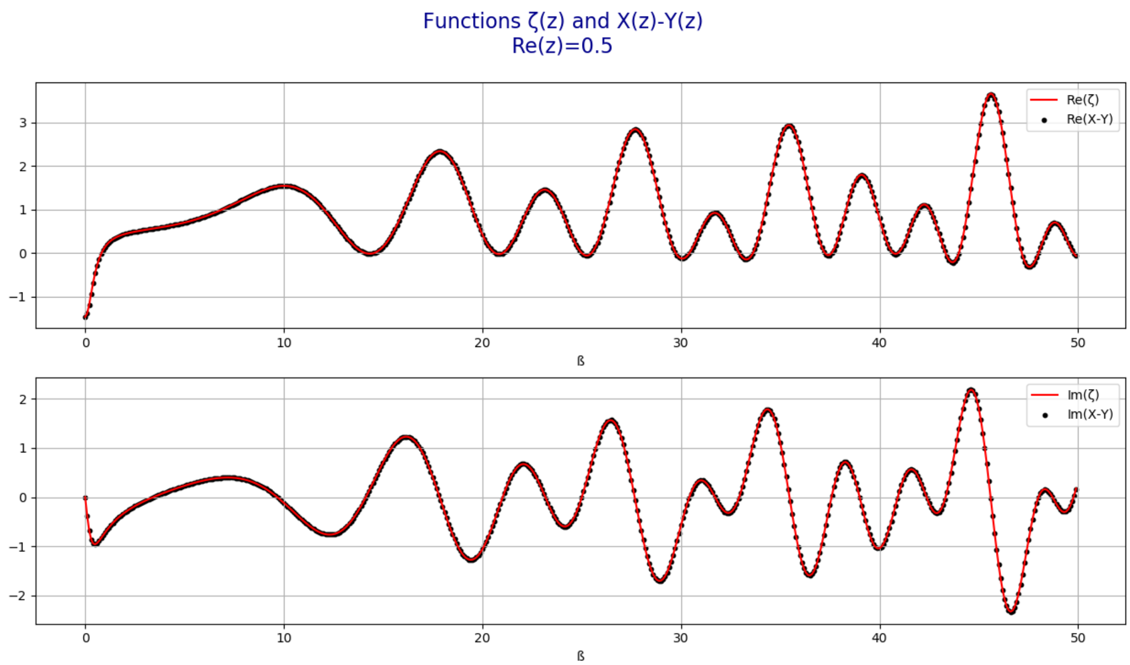


Fig. 7: ζ(z) = X(z) - Y(z)

7. Representation of the function $|\zeta(z)| = |X(z) - Y(z)|$ for Re(z)=1/2

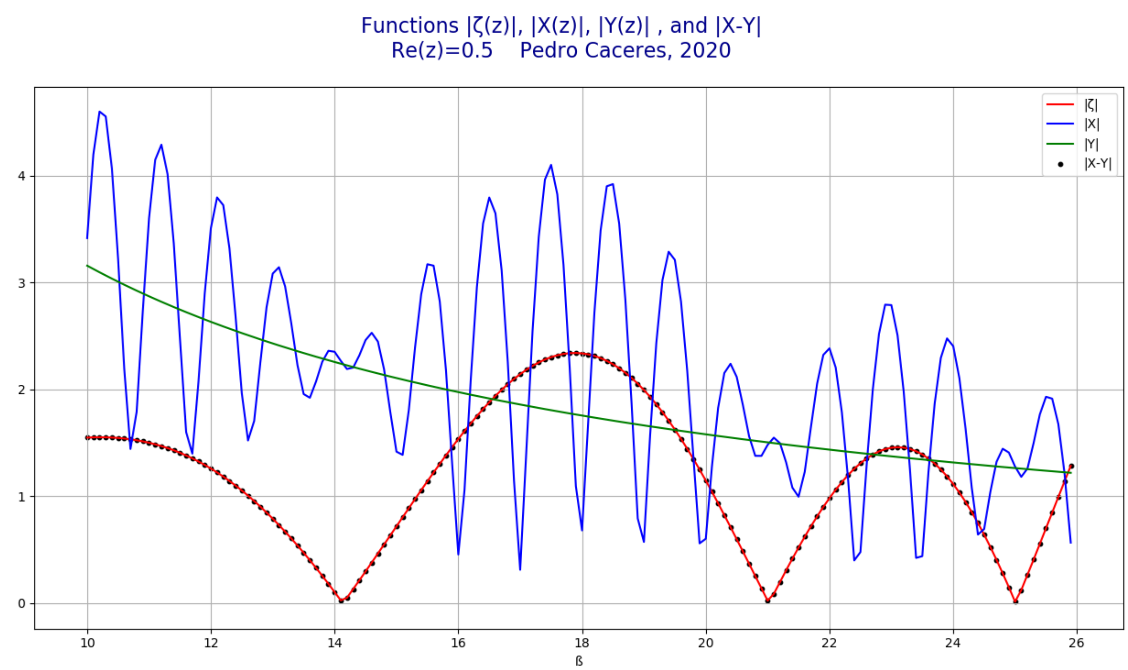


Fig. 8: |ζ(z)| = |X(z) - Y(z)|

8. Conclusion

Using the defined C-transformation, one can write the Riemann Zeta function as the difference of two functions X(z) and Y(z). This will provide a new way of analyzing the zeros of the Zeta function, and a new approach to the Riemann Hypothesis.

The decomposition is as follows:

$\zeta(z) \;=\; X(z) \;-\; Y(z)$, where:

$$X(z,n) \;= (\textstyle\sum_{k=1}^{n} k^{-\alpha}\; (\cos\,(ß * \ln(k)) + \frac{1}{2}\; n^{-\alpha} \cos\,(ß \ln(n)) +$$
$$+\; i * (\textstyle\sum_{k=1}^{n} k^{-\alpha}\; (\sin\,(ß * \ln(k)) + \frac{1}{2}\; n^{-\alpha}\, \sin(ß \ln(n))))$$

and: $X(z) \;=\; \lim_{n\to\infty} X(z,n)$

$$Y(z,n) =\; \mathrm{n}^{(1-\alpha)}\; \frac{1}{[(1-\alpha)^2+ß^2]}\; [\big((1-\alpha) * \cos(ß \ln(\mathrm{n})) + ß * \sin(ß \ln(\mathrm{n}))\big) +$$
$$+\; \mathrm{i}\;\; (ß * \cos(ß \ln(\mathrm{n})) - (1-\alpha) * \sin(ß \ln(\mathrm{n})))]$$

and: $Y(z) \;=\; \lim_{n\to\infty} Y(z,n)$

Observations:

1. $|X(z,n)|$ has a wave representation
2. $|X(z,n)|$ becomes a parable when z is a nontrivial zero of Riemann Zeta
3. $|X(z,n)|^2$ becomes a line when z is a nontrivial zero of Riemann Zeta with slope equal $\frac{1}{ß^2+¼}$

4. $|Y(z,n)|$ has a polynomial representation
5. $|Y(z,n)|$ becomes a parable when z is a nontrivial zero of Riemann Zeta
6. $|Y(z,n)|^2$ becomes a line when z is a nontrivial zero of Riemann Zeta with slope equal $1/(ß^2 + ¼)$

The only common representation for |X(z)| and |Y(z)| occurs when Re(z)=1/2, so

$X(z) \;-\; Y(z) = 0$ if and only if Re(z)=1/2

****

REFERENCES

(1) B. Riemann, “Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse”, Monat. der Königl. Preuss. Akad. der Wissen. zu Berlin aus der Jahre 1859 (1860), 671–680; also, Gesammelte math. Werke und wissensch. Nachlass, 2. Aufl. 1892, 145–155.

(2) Kopp, Konrad. “Theory and Application of Infinite Series.” Blackie & Son Limited, London, 1928

(3) Sondow, Jonathan and Weisstein, Eric W. "Riemann Zeta Function." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/RiemannZetaFunction.html

(4) WolframAlpha.com used for calculations, accessed March 2020.