THE AREA METHOD AND APPLICATIONS

T. AGAMA

Abstract. In this paper we develop a general method for estimating correlations of the forms

$$\sum_{n\leq x}G(n)G(x-n),$$

and

$$\sum_{n\leq x}G(n)G(n+l)$$

for a fixed $1\leq l\leq x$ and where $G:\mathbb{N}\longrightarrow \mathbb{R}^{+}$. To distinguish between the two types of correlations, we call the first **type** 2 correlation and the second **type** 1 correlation. As an application we estimate the lower bound for the **type** 2 correlation of the master function given by

$$\sum_{n\leq x}\Upsilon(n)\Upsilon(n+l_0)\geq (1+o(1))\frac{x}{2\mathcal{C}(l_0)}\log\log^2 x,$$

provided $\Upsilon(n)\Upsilon(n+l_0)>0$. We also use this method to provide a first proof of the twin prime conjecture by showing that

$$\sum_{n\leq x}\Lambda(n)\Lambda(n+2)\geq (1+o(1))\frac{x}{2\mathcal{C}(2)}$$

for some $\mathcal{C}:=\mathcal{C}(2)>0$.

1. **Introduction and statement**

Consider the sum

$$\sum_{n\leq x}G(n)G(x-n)$$

and

$$\sum_{n\leq x}G(n)G(n+l)$$

where $1\leq l\leq x$. It is generally not easy to controll sums of these forms, and unfortunately many of the open problems in number theory can be phrased in this manner. What is often required is an estimate for these sums. There are a good number of techniques in the literature for studying such sums, like the circle method of Hardy and littlewood, the sieve method and many others.

In this paper, we introduce the area method. This method can also be used to controll correlated sums of the form above. The novelty of this method is that it

Date: April 27, 2019.

2000 *Mathematics Subject Classification.* Primary 54C40, 14E20; Secondary 46E25, 20C20.

Key words and phrases. master function; correlation; two-point.

allows us to write any of these correlated sums as a double sum, which is much easier to estimate using existing tools such as the summation by part formula.

2. **The area method**

In this section we introduce and develop a fundamental method for solving problems related to correlations of arithmetic functions. This method is fundamental in the sense that it uses the properties of four main geometric shapes, namely the triangle, the trapezium, the rectangle and the square. The basic identity we will derive is an outgrowth of exploiting the areas of these shapes and putting them together in a unified manner.

Theorem 2.1. *Let* $\{r_j\}_{j=1}^{n}$ *and* $\{h_j\}_{j=1}^{n}$ *be any sequence of real numbers, and let* r *and* h *be any real numbers satisfying* $\sum\limits_{j=1}^{n} r_j = r$ and $\sum\limits_{j=1}^{n} h_j = h$, *and*

$$(r^2+h^2)^{1/2} = \sum_{j=1}^{n}(r_j^2+h_j^2)^{1/2},$$

then

$$\sum_{j=2}^{n} r_j h_j = \sum_{j=2}^{n} h_j\left(\sum_{i=1}^{j} r_i + \sum_{i=1}^{j-1} r_i\right) - 2\sum_{j=1}^{n-1} r_j \sum_{k=1}^{n-j} h_{j+k}.$$

Proof. Consider a right angled triangle, say $< ABC$ in a plane, with height h and base r. Next, let us partition the height of the triangle into n parts, not neccessarily equal. Now, we link those partitions along the height to the hypothenus, with the aid of a parallel line. At the point of contact of each line to the hypothenus, we drop down a vertical line to the next line connecting the last point of the previous partition, thereby forming another right-angled triangle, say $< A_1B_1C_1$ with base and height r_1 and h_1 respectively. We remark that this triangle is covered by the triangle $< ABC$, with hypothenus constituting a proportion of the hypothenus of triangle $< ABC$. We continue this process until we obtain n right-angled triangles $< A_jB_jC_j$, each with base and height r_j and h_j for $j = 1, 2, \ldots n$. This construction satisfies

$$h = \sum_{j=1}^{n} h_j \text{ and } r = \sum_{j=1}^{n} r_j$$

and

$$(r^2+h^2)^{1/2} = \sum_{j=1}^{n}(r_j^2+h_j^2)^{1/2}.$$

Now, let us deform the original triangle $< ABC$ by removing the smaller triangles $< A_jB_jC_j$ for $j = 1, 2, \ldots n$. Essentially we are left with rectangles and squares piled on each other with each end poking out a bit further than the one just above, and we observe that the total area of this portrait is given by the relation

$$\begin{aligned}\mathcal{A}_1 &= r_1h_2 + (r_1+r_2)h_3 + \cdots (r_1+r_2+\cdots+r_{n-2})h_{n-1} + (r_1+r_2+\cdots+r_{n-1})h_n \\ &= r_1(h_2+h_3+\cdots h_n) + r_2(h_3+h_4+\cdots+h_n) + \cdots + r_{n-2}(h_{n-1}+h_n) + r_{n-1}h_n \\ &= \sum_{j=1}^{n-1} r_j \sum_{k=1}^{n-j} h_{j+k}.\end{aligned}$$

On the other hand, we observe that the area of this portrait is the same as the difference of the area of triangle $< ABC$ and the sum of the areas of triangles $< A_jB_jC_j$ for $j = 1, 2, \ldots, n$. That is

$$\mathcal{A}_1 = \frac{1}{2}rh - \frac{1}{2}\sum_{j=1}^{n} r_jh_j.$$

This completes the first part of the argument. For the second part, along the hypothenus, let us construct small pieces of triangle, each of base and height (r_i, h_i) $(i = 1, 2\ldots, n)$ so that the trapezoid and the one triangle formed by partitioning becomes rectangles and squares. We observe also that this construction satisfies the relation

$$(r^2+h^2)^{1/2} = \sum_{i=1}^{n}(r_i^2+h_i^2)^{1/2},$$

Now, we compute the area of the triangle in two different ways. By direct strategy, we have that the area of the triangle, denoted $\mathcal{A}$, is given by

$$\mathcal{A} = 1/2\bigg(\sum_{i=1}^{n} r_i\bigg)\bigg(\sum_{i=1}^{n} h_i\bigg).$$

On the other hand, we compute the area of the triangle by computing the area of each trapezium and the one remaining triangle and sum them together. That is,

$$\mathcal{A} = h_n/2\bigg(\sum_{i=1}^{n} r_i + \sum_{i=1}^{n-1} r_i\bigg) + h_{n-1}/2\bigg(\sum_{i=1}^{n-1} r_i + \sum_{i=1}^{n-2} r_i\bigg) + \cdots + 1/2r_1h_1.$$

By comparing the area of the second argument, and linking this to the first argument, the result follows immediately. □

Remark 2.2. Next we state a result for a general lower bound for any two-point correlation that captures all real arithmetic function.

Theorem 2.3. *Let $f : \mathbb{N} \longrightarrow \mathbb{R}^+$, a real-valued function. If*

$$\sum_{n\leq x} f(n)f(n+l_0) > 0$$

then there exist some constant $\mathcal{C} := \mathcal{C}(l_0) > 0$ such that

$$\sum_{n\leq x} f(n)f(n+l_0) \geq \frac{1}{\mathcal{C}(l_0)x}\sum_{2\leq n\leq x} f(n)\sum_{m\leq n-1} f(m).$$

Proof. By Theorem 2.1, we obtain the identity by taking $f(j) = r_j = h_j$

$$\sum_{n\leq x-1}\sum_{j\leq x-n} f(n)f(n+j) = \sum_{2\leq n\leq x} f(n)\sum_{m\leq n-1} f(m).$$

It follows that

$$\begin{aligned}
\sum_{n\leq x-1}\sum_{j\leq x-n}f(n)f(n+j) &\leq \sum_{n\leq x-1}\sum_{j<x}f(n)f(n+j)\\
&=\sum_{n\leq x}f(n)f(n+1)+\sum_{n\leq x}f(n)f(n+2)\\
&+\cdots\sum_{n\leq x}f(n)f(n+l_0)+\cdots\sum_{n\leq x}f(n)f(n+x)\\
&\leq |\mathcal{M}(l_0)|\sum_{n\leq x}f(n)f(n+l_0)\\
&+|\mathcal{N}(l_0)|\sum_{n\leq x}f(n)f(n+l_0)\\
&+\cdots+\sum_{n\leq x}f(n)f(n+l_0)+\cdots+|\mathcal{R}(l_0)|\sum_{n\leq x}f(n)f(n+l_0)\\
&=\bigg(|\mathcal{M}(l_0)|+|\mathcal{N}(l_0)|+\cdots+1\\
&+\cdots+|\mathcal{R}(l_0)|\bigg)\sum_{n\leq x}f(n)f(n+l_0)\\
&\leq \mathcal{C}(l_0)x\sum_{n<x}f(n)f(n+l_0).
\end{aligned}$$

where $\max\{|\mathcal{M}(l_0)|,|\mathcal{N}(l_0)|,\ldots,|\mathcal{R}(l_0)|\}=\mathcal{C}(l_0)$. By inverting this inequality, the result follows immediately. □

The nature of the implicit constant $\mathcal{C}(l_0)$ could also depend on the structure of the function we are being given. The von mangoldt function, contrary to many class of arithmetic functions, has a relatively small such constant. This behaviour stems from the fact that the Von-mangoldt function is defined on the prime powers. Thus one would expect most terms of sums of the form

$$\sum_{n\leq x-1}\sum_{j\leq x-n}\Lambda(n)\Lambda(n+j)$$

to fall off when j is odd for any prime power $n=p^k$ such that $j+p^k\neq 2^s$.

Theorem 2.4. *Let $f:\mathbb{N}\longrightarrow\mathbb{R}^+$. Suppose there exist some constant $x>\mathcal{C}(x)>0$ such that*

$$\sum_{n\leq x}\sum_{\substack{j\leq x-n\\ j\neq x-2n}}f(n)f(n+j)=\frac{\mathcal{C}(x)}{x}\sum_{n\leq x}\sum_{j\leq x-n}f(n)f(n+j).$$

Then for any $x\geq 2$

$$\sum_{n\leq \frac{x}{2}}f(n)f(x-n)=\frac{\mathcal{D}(x)}{x}\sum_{2\leq n\leq x}f(n)\sum_{m\leq n-1}f(m),$$

where $x-\mathcal{D}(x)=\mathcal{C}(x)$.

Proof. By Theorem 2.1, we obtain

$$\sum_{n\leq x} f^2(n) = f^2(1) + \sum_{2\leq n\leq x} f(n)\Big(\sum_{m\leq n-1} f(m) + \sum_{m\leq n} f(m)\Big) - 2\sum_{n\leq x-1} f(n)\sum_{s\leq x-n} f(n+s)$$

for $f:\mathbb{N}\longrightarrow \mathbb{R}^+$ by taking $r_j = h_j = f(j)$. By rearranging this identity, we obtain the identity

$$\sum_{n\leq x-1}\sum_{j\leq x-n} f(n)f(n+j) = \sum_{2\leq n\leq x} f(n)\sum_{m\leq n-1} f(m).$$

Let $n+j = x-n$, then it follows that $x-2n = j$. It follows that $j\leq x-2$ if and only if $1\leq n<\frac{x}{2}$. Then we can rewrite the sum on the left-hand side as

$$\begin{aligned}\sum_{n\leq x-1}\sum_{j\leq x-n} f(n)f(n+j) &= \sum_{n\leq x-1}\sum_{x-2n=j} f(n)f(n+j) + \sum_{n\leq x-1}\sum_{\substack{j\leq x-n\\ x-2n\neq j}} f(n)f(n+j)\\ &= \sum_{n<\frac{x}{2}} f(n)f(x-n) + \frac{\mathcal{C}(x)}{x}\sum_{n\leq x-1}\sum_{j\leq x-n} f(n)f(n+j)\end{aligned}$$

where $0<\frac{\mathcal{C}(x)}{x}<1$. It follows from this relation

$$\frac{\mathcal{D}(x)}{x}\sum_{n\leq x-1}\sum_{j\leq x-n} f(n)f(n+j) = \sum_{n<\frac{x}{2}} f(n)f(x-n)$$

where $0<\frac{\mathcal{D}(x)}{x} = 1-\frac{\mathcal{C}(x)}{x}<1$. Using Theorem 2.1, we can write

$$\sum_{n<\frac{x}{2}} f(n)f(x-n) = \frac{\mathcal{D}(x)}{x}\sum_{2\leq n\leq x} f(n)\sum_{m\leq n-1} f(m)$$

and the result follows immediately. □

3. Application to the twin prime conjecture

Theorem 3.1. *There exist some constant $\mathcal{C} := \mathcal{C}(2) > 0$, such that*

$$\sum_{n\leq x}\Lambda(n)\Lambda(n+2) \geq (1+o(1))\frac{x}{2\mathcal{C}(2)}.$$

Proof. By invoking Theorem 2.3, we can write

$$\sum_{n\leq x}\Lambda(n)\Lambda(n+2) \geq \frac{1}{\mathcal{C}(2)x}\sum_{2\leq n\leq x}\Lambda(n)\sum_{m\leq n-1}\Lambda(m).$$

Using the prime number theorem [2] of the form

$$\sum_{n\leq x}\Lambda(n) = (1+o(1))x,$$

the result follows immediately by using partial summation. □

Remark 3.2. It is important to remark that with the lower bound in Theorem 3.1, we have solved the twin prime conjecture. This method not only does it solve the twin prime conjecture, but is good in terms of its generality, for it can be used to obtain lower bounds for a general class of correlated sums of the form

$$\sum_{n\leq x} f(n)f(n+k)$$

for a uniform $1 \leq k \leq x$.

4. Application to other correlated sums of type 1

In this section we apply Theorem 2.3 to provide lower estimates of other correlated sums, but with the price of an implicit constant depending on the range of shift.

Corollary 1. For a fixed $l_0 > 0$, there exist some constant $\mathcal{C} := \mathcal{C}(l_0) > 0$ such that

$$\sum_{n\leq x} d(n)d(n+l_0) \geq (1+o(1))\frac{x\log^2 x}{2\mathcal{C}(l_0)}.$$

Proof. The result follows by using Theorem 2.3, using the crude estimate [3]

$$\sum_{n\leq x} d(n) = (1+o(1))x\log x$$

together with partial summation. □

Corollary 2. For a fixed $k_0 > 0$ and for $l \geq 2$, there exist some constant $\mathcal{C} := \mathcal{C}(k) > 0$ such that

$$\sum_{n\leq x} d_l(n)d_l(n+k) \geq (1+o(1))\left(\frac{1}{(l-1)!}\right)\left(1-\frac{1}{2(l-1)!}\right)\frac{x\log^{2(l-1)} x}{\mathcal{C}(k)}.$$

Proof. We recall the weaker estimate for the l th divisor function [3]

$$\sum_{n\leq x} d_l(n) = (1+o(1))\frac{1}{(l-1)!}x\log^{l-1} x,$$

where

$$d_l(n) = \sum_{n_1\cdot n_2\cdots n_l = n} 1.$$

By leveraging Theorem 2.3 and using partial summation, the lower bound follows naturally. □

Corollary 3. For a fixed $l_0 > 0$, there exist some constant $\mathcal{C} := \mathcal{C}(l_0) > 0$ such that

$$\sum_{n\leq x} \phi(n)\phi(n+l_0) \geq (1+o(1))\frac{9}{2\pi^4}\frac{x^3}{\mathcal{C}(l_0)}.$$

Proof. The result follows by applying Theorem 2.3, using the estimate [4]

$$\sum_{n\leq x}\phi(n)=(1+o(1))\frac{3}{\pi^2}x^2$$

together with partial summation. □

Corollary 4. For a fixed $l_0>0$, there exist some constant $\mathcal{C}:=\mathcal{C}(l_0)>0$ such that

$$\sum_{n\leq x}\mu^2(n)\mu^2(n+l_0)\geq(1+o(1))\frac{18}{\pi^4}\frac{x}{\mathcal{C}(l_0)}.$$

Proof. The result follows by applying Theorem 2.3, using the estimate [4]

$$\sum_{n\leq x}\mu^2(n)=(1+o(1))\frac{6}{\pi^2}x$$

together with the use of partial summation. □

5. Application to lower bound for two-point correlation of the master function of type 1 and type 2

In this section we apply the area method developed to establish a lower bound for the two-point **type** 1 correlation and an estimate for the **type** 2 correlation of the master function. We begin with the following result:

Lemma 5.1. *Let* Υ *denotes the master function, then*

$$\sum_{n\leq x}\Upsilon(n)=x\log\log x+O(x).$$

Proof. For a proof, See [1]. □

Theorem 5.2. *The estimate is valid*

$$\sum_{n\leq x}\Upsilon(n)\Upsilon(n+l_0)\geq(1+o(1))\frac{x}{2\mathcal{C}(l_0)}\log\log^2 x,$$

provided $\Upsilon(n)\Upsilon(n+l_0)>0$.

Proof. Applying Theorem 2.3 and Lemma 5.1, we can write

$$\begin{aligned}\sum_{n\leq x}\Upsilon(n)\Upsilon(n+l_0)&\geq\frac{1}{x\mathcal{C}(l_0)}\sum_{2\leq n\leq x}\Upsilon(n)\sum_{m\leq n-1}\Upsilon(m)\\&=\frac{1}{x\mathcal{C}(l_0)}(1+o(1))\sum_{2\leq n\leq x}\Upsilon(n)n\log\log n.\end{aligned}$$

By partial summation, we can write

$$\sum_{2\leq n\leq x}\Upsilon(n)n\log\log n = x\log\log x\sum_{n\leq x}\Upsilon(n)-\int\limits_{2}^{x}(1+o(1))t(\log\log t)\bigg(\log\log t+\frac{1}{\log t}\bigg)dt$$
$$=(1+o(1))x^2\log\log^2 x-(1+o(1))\int\limits_{2}^{x}t(\log\log t)\bigg(\log\log t+\frac{1}{\log t}\bigg)dt$$
$$=(1+o(1))\frac{x^2}{2}\log\log^2 x.$$

The lower bound follows immediately from this estimate. □

Theorem 5.3. *Under the assumption*

$$\frac{\sum\limits_{n\leq x}\sum\limits_{\substack{j\leq x-n\\ j\neq x-2n}}\Upsilon(n)\Upsilon(n+j)}{\sum\limits_{n\leq x}\sum\limits_{j\leq x-n}\Upsilon(n)\Upsilon(n+j)}<1,$$

then

$$\sum_{n\leq \frac{x}{2}}\Upsilon(n)\Upsilon(x-n)=(1+o(1))\frac{x}{2}\mathcal{D}(x)\log\log^2 x$$

where $\mathcal{D}:=\mathcal{D}(x)>0$.

Proof. The result follows by applying the area method. □

6. Application to estimates of the number of representations of an even number as a sum of two primes

In this section we apply the area method developed 2.4 to obtain a weaker estimate for the number of representations of an even number as a sum of two primes, under the assumption that the Goldbach conjecture is true.

Theorem 6.1. *Assuming the Goldbach conjecture is true, then for any even* $x\geq 6$

$$\sum_{n\leq \frac{x}{2}}\Lambda(n)\Lambda(x-n)=(1+o(1))\frac{x}{2}\mathcal{D}(x)$$

where $\mathcal{D}:=\mathcal{D}(x)>0$.

Proof. Under the assumption that the Goldbach conjecture is true, it follows that

$$\frac{\sum\limits_{n\leq x}\sum\limits_{\substack{j\leq x-n\\ j\neq x-2n}}\Lambda(n)\Lambda(n+j)}{\sum\limits_{n\leq x}\sum\limits_{j\leq x-n}\Lambda(n)\Lambda(n+j)}<1.$$

Applying the area method, there exist some constant $\mathcal{D}(x) > 0$ with $\mathcal{D}(x) < x$, such that

$$\sum_{n\leq \frac{x}{2}} \Lambda(n)\Lambda(x-n) = \frac{\mathcal{D}(x)}{x} \sum_{2\leq n\leq x} \Lambda(n) \sum_{m\leq n-1} \Lambda(m).$$

Using the prime number theorem [2] in the form

$$\sum_{n\leq x} \Lambda(n) = (1+o(1))x$$

and partial summation, we obtain

$$\sum_{2\leq n\leq x} \Lambda(n) \sum_{m\leq n-1} \Lambda(m) = (1+o(1))\frac{x^2}{2}.$$

The result follows immediately from this rudimentary estimates. □

7. Application to other correlated sums of type 2

In this section, we apply the area method 2.4 to obtain estimates for various correlated sums, in the following sequel. The area method is perfectly suited for functions of these forms, since they are non-vanishing on the integers.

Theorem 7.1. *The estimate holds*

$$\sum_{n\leq \frac{x}{2}} d(n)d(x-n) = \mathcal{D}(1+o(1))\frac{x\log^2 x}{2}$$

where $\mathcal{D} := \mathcal{D}(x) > 0$.

Proof. Since the divisor function is non-vanishing on the integers, we observe that

$$\frac{\sum\limits_{n\leq x}\sum\limits_{\substack{j\leq x-n\\ x-2n\neq j}} d(n)d(n+j)}{\sum\limits_{n\leq x}\sum\limits_{j\leq x-n} d(n)d(n+j)} < 1.$$

Thus by the area method 2.4, there exist some constant $0 < \mathcal{D}(x) < x$ such that

$$\sum_{n\leq \frac{x}{2}} d(n)d(x-n) = \frac{\mathcal{D}(x)}{x} \sum_{2\leq n\leq x} d(n) \sum_{m\leq n-1} d(m).$$

Using the weaker estimate [3]

$$\sum_{n\leq x} d(n) = (1+o(1))x\log x$$

we obtain by partial summation

$$\sum_{2\leq n\leq x} d(n) \sum_{m\leq n-1} d(m) = (1+o(1))\frac{x^2\log^2 x}{2}$$

and the result follows immediately. □

Theorem 7.2. *The estimate holds*

$$\sum_{n\leq \frac{x}{2}}\phi(n)\phi(x-n)=(1+o(1))\mathcal{D}\frac{9}{2\pi^4}x^3$$

where $\mathcal{D}:=\mathcal{D}(x)>0$.

Proof. Since $\phi(n)$ is non-vanishing on the integers, we observe that

$$\frac{\sum\limits_{n\leq x}\sum\limits_{\substack{j\leq x-n\\ x-2n\neq j}}\phi(n)\phi(n+j)}{\sum\limits_{n\leq x}\sum\limits_{j\leq x-n}\phi(n)\phi(n+j)}<1.$$

Then by the area method 2.4, there exist some constant $\mathcal{D}(x)>0$ with $\mathcal{D}(x)<x$ such that

$$\sum_{n\leq \frac{x}{2}}\phi(n)\phi(x-n)=\frac{\mathcal{D}(x)}{x}\sum_{2\leq n\leq x}\phi(n)\sum_{m\leq n-1}\phi(m).$$

Now using the estimate [3]

$$\sum_{n\leq x}\phi(n)=(1+o(1))\frac{3}{\pi^2}x^2$$

we obtain by partial summation

$$\sum_{2\leq n\leq x}\phi(n)\sum_{m\leq n-1}\phi(m)=(1+o(1))\frac{9}{2\pi^4}x^4,$$

and the result follows immediately. □

Theorem 7.3. *The estimate holds*

$$\sum_{n\leq \frac{x}{2}}d_l(n)d_l(x-n)=(1+o(1))\mathcal{D}\bigg(\frac{1}{(l-1)!}\bigg)\bigg(1-\frac{1}{2(l-1)!}\bigg)x\log^{2(l-1)}x$$

where $\mathcal{D}:=\mathcal{D}(x)>0$ *and where*

$$d_l(n):=\sum_{n_1n_2\cdots n_l=n}1.$$

Proof. We observe that

$$\frac{\sum\limits_{n\leq x}\sum\limits_{\substack{j\leq x-n\\ j\neq x-2n}}d_l(n)d_l(n+j)}{\sum\limits_{n\leq x}\sum\limits_{j\leq x-n}d_l(n)d_l(n+j)}<1.$$

It follows from the area method 2.4, there exists some constant $\mathcal{D}(x)>0$ with $\mathcal{D}(x)<x$ such that

$$\sum_{n\leq \frac{x}{2}}d_l(n)d_l(x-n)=\frac{\mathcal{D}(x)}{x}\sum_{2\leq n\leq x}d_l(n)\sum_{m\leq n-1}d_l(m).$$

Using the estimate [3]

$$\sum_{n\leq x} d_l(n) = (1+o(1))\frac{1}{(l-1)!}x\log^{l-1}x,$$

It follows by partial summation

$$\sum_{2\leq n\leq x} d_l(n)\sum_{m\leq n-1} d_l(m) = (1+o(1))\left(\frac{1}{(l-1)!}\right)\left(1-\frac{1}{2(l-1)!}\right)x^2\log^{2(l-1)}x.$$

The claimed estimate follows immediately. □

8. Application to the global distribution of integers with $\Omega(n)=2$

The lower bounds of correlations of arithmetic functions tells us a lot about their local distributions as well as their global distribution. Theorem 5.2 gives

$$\sum_{n\leq x}\Upsilon(n)\Upsilon(n+l_0)\geq (1+o(1))\frac{x}{2\mathcal{C}(l_0)}\log\log^2 x,$$

provided $\sum\limits_{n\leq x}\Upsilon(n)\Upsilon(n+l_0)>0$. Thus for some shift in the range $[1,x]$ the correlation can be made arbitrarily large by taking the right hand side arbitrarily large. This follows that there are infinitely many pairs of the form $(n,n+l_0)$ such that n and $n+l_0$ each has exactly two prime factors.

9. Final remarks

The area method seems not particularly suited for arithmetic functions defined on a certain subsequence of the integers. As such it's current form cannot be applied directly to important open problems like the Goldbach conjecture, since the implicit constant in Theorem 6.1 relies on the condition

$$\frac{\sum\limits_{n\leq x}\sum\limits_{\substack{j\leq x-n\\ j\neq x-2n}}\Lambda(n)\Lambda(n+j)}{\sum\limits_{n\leq x}\sum\limits_{j\leq x-n}\Lambda(n)\Lambda(n+j)}<1.$$

Add to this, even if this condition were to be satisfied, we would certainly not have much information about the constant, although at the barest minimum $0<\mathcal{D}(x)<x$. However, we believe this method can be refined to the form applicable to functions defined on a subsequence of the integers like the primes.

[1].

[1] .

References

1. T. Agama *The master function and applications*, arXiv preprint arXiv:1707.05679, 2017.
2. Montgomery, H.L, and Vaughan, R.C, *Multiplicative number theory 1*:Classical theory. vol.97, Cambridge university press, 2006.
3. Nathanson, M.B, *Graduate Texts in Mathematics*, New York, NY: Springer New York, 2000.
4. Gérald Tenenbaum, *Introduction to analytic and probabilistic number theory*, vol. 163, American Mathematical Soc., 2015.

Department of Mathematics, African Institute for Mathematical science, Ghana

E-mail address: theophilus@aims.edu.gh/emperordagama@yahoo.com