An Equivalent of the Riemann Hypothesis

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1 Abstract

The Riemann Zeta function is defined as the Analytic Continuation of the Dirichlet series

$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s, \ Re(s) > 1$

$The\ Riemann\ Zeta\ function\ is\ holomorphic\ in\ the\ complex\ plane\ except\ for\ a\ simple\ pole\ at\ s = 1$

$The\ non\ trivial\ zeroes(i.e\ those\ not\ at\ negative\ even\ integers)\ of\ the$

$Riemann\ Zeta\ function\ lie\ in\ the\ critical\ strip$

$0 \leq Re(s) \leq 1$

$Riemann's\ Xi\ function\ is\ defined\ as[4,\ p.1],$

$\epsilon(s) \ = \ s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)/2$

$The\ zero\ of\ (s{-}1)\ cancels\ the\ pole\ of\ \zeta(s)\ , and\ the\ real\ zeroes\ of\ s\ \zeta(s)\ are\ cancelled$

$by\ the\ simple\ poles\ of\ \Gamma(s/2)\ which\ never\ vanishes.$

$Thus,\ \epsilon(s)\ is\ an\ entire\ function\ whose\ zeroes\ are\ the\ non\ trivial\ zeroes\ of\ \zeta(s)([3,p.2])$

$Further,\ \epsilon(s)\ satisfies\ the\ functional\ equation$

$\epsilon(1-s)\ =\ \epsilon(s)$

2 Statement of the Riemann Hypothesis

The Riemann Hypothesis states that all the non trivial zeroes of the

Riemann Zeta function lie on the critical line with real part equal

to 1/2

3 Proof

The Riemann Xi function [1, p.39] is defined as

$$\epsilon(s) = \epsilon(0)\prod_{\rho}(1-\frac{s}{\rho}) \qquad ... \qquad (1)$$

$where\ \rho\ ranges\ over\ all\ the\ roots\ \rho\ of\ \epsilon(\rho)=0.$

$If\ we\ combine\ the\ factors$

$(1-\frac{s}{\rho})\ and\ (1-\frac{s}{1-\rho}),\ then\ \epsilon(s)\ is\ Absolutely\ convergent\ infinite\ product\ [1,p.42].$

$Also,\ \epsilon(0)=1/2$

$From,\ [1,\ p.42,\ Section\ 2.5]$

$$\prod_{\rho}(1-\frac{s}{\rho}) = \prod_{Im(\rho)>0}[1-\frac{s(1-s)}{\rho(1-\rho)}] = \prod_{Im(\rho)>0}(1-\frac{s}{\rho})(1-\frac{s}{1-\rho}). \qquad ... \qquad (2)$$

$Let,\ \rho_0\ be\ a\ zero\ of\ the\ Riemann's\ Xi\ function.$

$\Rightarrow \epsilon(\rho_0) = 0.$

$\Rightarrow \epsilon(\overline{\rho_0}) = 0.$

$From\ (2)\ we\ have,$

$$\epsilon(s) = \epsilon(0) \prod_{Im(\rho)>0} (1 - \frac{s}{\rho})(1 - \frac{s}{1-\rho}).$$

$\epsilon(\bar{\rho}_0) = 0$

$$\Rightarrow \epsilon(0) \prod_{Im(\rho)>0} (1 - \frac{\bar{\rho}_0}{\rho})(1 - \frac{\bar{\rho}_0}{1-\rho}) = 0.$$

$\epsilon(0) = 1/2.$

$$\Rightarrow \prod_{Im(\rho)>0} (1 - \frac{\bar{\rho}_0}{\rho})(1 - \frac{\bar{\rho}_0}{1-\rho}) = 0.$$

$Suppose\ \rho_0\ such\ that\ Im(\rho_0) > 0\ is\ a\ zero\ with\ multiplicity\ k.$

$$\Rightarrow [(1 - \tfrac{\bar{\rho}_0}{\rho_0})(1 - \tfrac{\bar{\rho}_0}{1-\rho_0})]^k \prod_{Im(\rho)>0,\ \rho\neq\rho_0} (1 - \frac{\bar{\rho}_0}{\rho})(1 - \frac{\bar{\rho}_0}{1-\rho}) = 0.$$

$$\Rightarrow [\tfrac{(\rho_0-\bar{\rho}_0)(1-\rho_0-\bar{\rho}_0)}{\rho_0(1-\rho_0)}]^k \prod_{Im(\rho)>0,\ \rho\neq\rho_0} (1 - \frac{\bar{\rho}_0}{\rho})(1 - \frac{\bar{\rho}_0}{1-\rho}) = 0.$$

$$\Rightarrow [\tfrac{(2iIm(\rho_0))(1-2Re(\rho_0))}{\rho_0(1-\rho_0)}]^k \prod_{Im(\rho)>0,\ \rho\neq\rho_0} (1 - \frac{\bar{\rho}_0}{\rho})(1 - \frac{\bar{\rho}_0}{1-\rho}) = 0. \quad ... \qquad (3)$$

$$Let,\ I_\rho = \prod_{Im(\rho)>0,\ \rho\neq\rho_0} (1 - \frac{\bar{\rho}_0}{\rho})(1 - \frac{\bar{\rho}_0}{1-\rho})$$

$From(3),$

$$\Rightarrow [\tfrac{(2iIm(\rho_0))(1-2Re(\rho_0))}{\rho_0(1-\rho_0)}]^k I_\rho = 0. \quad ... \qquad (4)$$

$From\ (2),\ \rho_0\ is\ a\ zero\ of\ Riemann's\ Xi\ function, so\ Im(\rho_0) > 0$

$\Rightarrow [\frac{(1-2Re(\rho_0))}{(\rho_0)(1-\rho_0)}]^k I_\rho = 0.. \quad ... \qquad (5)$

$If\ I_\rho \neq 0\ then,\ Re(\rho_0) = 1/2.$

$From\ (5)\ Riemann\ Hypothesis\ is\ true\ if\ I_\rho \neq 0$

$$where\ I_\rho = \prod_{Im(\rho)>0, \rho\neq\rho_0} (1 - \frac{\bar{\rho}_0}{\rho})(1 - \frac{\bar{\rho}_0}{1-\rho})$$

$Thus,\ an\ equivalent\ of\ the\ Riemann\ Hypothesis\ is\ that\ I_\rho \neq 0.$

4 References:-

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