Lower bounds on a special type of cyclic sums

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"Entia non sunt multiplicanda praeter necessitatem" (Ockam, W.)

"Dios no juega a los dados con el Universo" (Einstein, Albert)

"Te doy gracias, Padre, porque has ocultado estas cosas a los sabios y entendidos y se las has revelado a la gente sencilla" (Mt 11,25)

Abstract

In this brief paper it is proved a theorem regarding the relative value of the cyclic sums of $f(x) = \frac{a_1^{\alpha+1}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}}$ and the sum of its variables.

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1 Introduction

We define a cyclic sum $\sum_{cyc} f(a_1, a_2, ..., a_n)$ as equal to $f(a_1, a_2, ..., a_n) + f(a_2, a_3, ..., a_n, a_1) + f(a_3, a_4, ..., a_1, a_2) + ... + f(a_n, a_1, ..., a_{n-1})$. Therefore, all the variables are cycled through.

We are interested in studying the sum

$$\sum_{cyc} \frac{a_1^{\alpha+1}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}} = \frac{a_1^{\alpha+1}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}} + \frac{a_2^{\alpha+1}}{k_1 a_2^{\alpha} + k_2 a_3^{\alpha}} + ... + \frac{a_n^{\alpha+1}}{k_1 a_n^{\alpha} + k_2 a_1^{\alpha}}$$

And its relative value compared to

$$\sum_{k=1}^{n} a_k$$

In this regard, in this paper it is proposed and proved the following theorem:

Theorem.

$$\sum_{cyc} \frac{a_1^{\alpha+1}}{k_1 a_1^\alpha + k_2 a_2^\alpha} \geq \frac{\sum_{k=1}^{n} a_k}{k_1 + k_2}$$

2 Proof

2.1 Previous Lemmas

We will need firstly the following

Lemma 1.

$$\sum_{k=1}^{n} a_k^{\alpha+1} \geq \sum_{cyc} a_1 a_2^\alpha$$

Proof.

Let us assume, without loss of generality,

$$a_1 \geq a_2 \geq a_3 \geq .... \geq a_n$$

We have that

$$\sum_{k=1}^{n} a_k^{\alpha+1} - \sum_{cyc} a_1 a_2^\alpha = a_1^\alpha (a_1 - a_n) + a_2^\alpha (a_2 - a_1) + a_3^\alpha (a_3 - a_2) + ... + a_n^\alpha (a_n - a_{n-1})$$

It could be noted that

$$a_1 - a_n = -(a_2 - a_1) - (a_3 - a_2) - ... - (a_n - a_{n-1})$$

Therefore, substituting, we have that

$$\sum_{k=1}^{n} a_k^{\alpha+1} - \sum_{cyc} a_1 a_2^\alpha = a_2^\alpha (a_2 - a_1) + a_3^\alpha (a_3 - a_2) + ... + a_n^\alpha (a_n - a_{n-1}) - a_1^\alpha (a_2 - a_1) - a_1^\alpha (a_3 - a_2) - ... - a_1^\alpha (a$$

As

$$a_1^\alpha \geq a_2^\alpha \geq a_3^\alpha \geq .... \geq a_n^\alpha$$

We have that

$$a_2^\alpha (a_2 - a_1) - a_1^\alpha (a_2 - a_1) \geq 0$$

$$a_3^\alpha (a_3 - a_2) - a_1^\alpha (a_3 - a_2) \geq 0$$

...

$$a_n^\alpha (a_n - a_{n-1}) - a_1^\alpha (a_n - a_{n-1}) \geq 0$$

Then, subsequently,

$$\sum_{k=1}^{n} a_k^{\alpha+1} - \sum_{cyc} a_1 a_2^{\alpha} \geq 0$$

$$\sum_{k=1}^{n} a_k^{\alpha+1} \geq \sum_{cyc} a_1 a_2^{\alpha}$$

Other hand, we need the following

Lemma 2.

$$\sum_{cyc} \frac{a_1^{\alpha+1} + \left(\frac{k_2}{k_1}\right) a_1 a_2^{\alpha}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}} = \frac{\sum_{k=1}^{n} a_k}{k_1}$$

Proof.

If we establish that

$$\frac{a_1^{\alpha+1}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}} + \frac{n}{m} = \frac{a_1}{k_1}$$

Operating, we get that

$$\frac{a_1^{\alpha+1} m + n\left(k_1 a_1^{\alpha} + k_2 a_2^{\alpha}\right)}{m\left(k_1 a_1^{\alpha} + k_2 a_2^{\alpha}\right)} = \frac{a_1}{k_1}$$

$$k_1\left(a_1^{\alpha+1} m + n\left(k_1 a_1^{\alpha} + k_2 a_2^{\alpha}\right)\right) = a_1 m\left(k_1 a_1^{\alpha} + k_2 a_2^{\alpha}\right)$$

$$k_1 a_1^{\alpha+1} m + k_1 n\left(k_1 a_1^{\alpha} + k_2 a_2^{\alpha}\right) = a_1 m k_1 a_1^{\alpha} + a_1 m k_2 a_2^{\alpha}$$

$$k_1 n\left(k_1 a_1^{\alpha} + k_2 a_2^{\alpha}\right) = a_1 m k_2 a_2^{\alpha}$$

$$\frac{n}{m} = \left(\frac{k_2}{k_1}\right) \frac{a_1 a_2^{\alpha}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}}$$

Therefore, we have that

$$\frac{a_1^{\alpha+1} + \left(\frac{k_2}{k_1}\right) a_1 a_2^{\alpha}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}} = \frac{a_1}{k_1}$$

And subsequently, repeating the process for each variable, we get that

$$\sum_{cyc} \frac{a_1^{\alpha+1} + \left(\frac{k_2}{k_1}\right) a_1 a_2^{\alpha}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}} = \frac{\sum_{k=1}^{n} a_k}{k_1}$$

2.2 Proof

Operating, we have that

$$\sum_{cyc} \frac{a_1^{\alpha+1}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}} = \frac{\sum_{k=1}^{n} a_k}{k_1} - \sum_{cyc} \frac{\left(\frac{k_2}{k_1}\right) a_1 a_2^{\alpha}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}}$$

Applying Lemma 1, we derive that

$$\sum_{k=1}^{n} a_k^{\alpha+1} \geq \left(\frac{k_1}{k_1}\right) \sum_{cyc} a_1 a_2^{\alpha}$$

As a result, we can affirm that

$$\sum_{cyc} \frac{a_1^{\alpha+1}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}} \geq \sum_{cyc} \frac{\left(\frac{k_1+k_2}{k_1}\right) a_1 a_2^{\alpha}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}} - \sum_{cyc} \frac{\left(\frac{k_2}{k_1}\right) a_1 a_2^{\alpha}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}}$$

Thus,

$$\sum_{cyc} \frac{\left(\frac{k_1+k_2}{k_1}\right) a_1 a_2^{\alpha}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}} \leq \frac{\sum_{k=1}^{n} a_k}{k_1}$$

$$\sum_{cyc} \frac{a_1 a_2^{\alpha}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}} \leq \frac{\sum_{k=1}^{n} a_k}{\left(\frac{k_1+k_2}{k_1}\right) k_1}$$

$$\sum_{cyc} \frac{a_1 a_2^{\alpha}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}} \leq \frac{\sum_{k=1}^{n} a_k}{k_1 + k_2}$$

And subsequently,

$$\sum_{cyc} \frac{a_1^{\alpha+1}}{k_1 a_1^{\alpha} + k_2 a_2^{\alpha}} \geq \frac{\sum_{k=1}^{n} a_k}{k_1 + k_2}$$

As we wanted to prove.