Electromagnetic Duality From Biot-Savart Law

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A stationary charge distribution in one inertial reference frame becomes a charge current in another inertial reference frame. The magnetic field described by Biot-Savart Law becomes a representation of the relative velocity between two inertial reference frames and the electric field described by Coulomb's law. The charge current in dielectric medium exhibits property similar to the light in the vacuum. Both are characterized by a pair of electric field and magnetic field in the transverse direction. The electromagnetic duality from Maxwell's equations is indeed a representation of Biot-Savart law.

I. INTRODUCTION

Coulomb's law[1] describes a stationary charge distribution. Biot-Savart law[2] describes a dynamic charge distribution. The equations from both laws are closely related mathematically.

Lorentz force describes a magnetic force on a charged particle in motion. The magnetic force vanishes if the particle becomes stationary. The magnetic force depends on the relative motion between the rest frames of the charged particle and the magnetic source.

In the rest frame of a charge distribution, there exists only electric field. In another inertial reference frame, there are both magnetic field and electric field from the motion of the charge distribution.

The magnetic field is exactly a representation of the electric field in a different inertial reference frame.

II. PROOF

A. Electromagnetic Field

Define $\vec{Y}$ as

$$\vec{Y}(\vec{R},\vec{r}) = \frac{\vec{R}-\vec{r}}{\mid \vec{R}-\vec{r}\mid^{3}} \tag{1}$$

Coulomb's law states that the electric field is represented as

$$\vec{E}(\vec{R}) = \frac{1}{4\pi\epsilon_0}\iiint \rho(\vec{r})\vec{Y}\,dr^3 \tag{2}$$

$$= \frac{1}{4\pi\epsilon_0}\int \vec{Y}dQ(\vec{r}) \tag{3}$$

Biot-Savart law states that the magnetic field is represented as

$$\vec{B}(\vec{R}) = \frac{\mu_0}{4\pi}\int I d\vec{r}\times\vec{Y} \tag{4}$$

$$= \frac{\mu_0}{4\pi}\int \frac{dQ}{dt}d\vec{r}\times\vec{Y} \tag{5}$$

$$= \frac{\mu_0}{4\pi}\int \vec{v}\times\vec{Y}dQ(\vec{r}) \tag{6}$$

The infinitesimal charge moves at the velocity of $\vec{v}$.

B. Inertial Reference Frame

Let F_0 be the rest frame of a charge distribution corresponds to equation (3). Let another inertial reference frame F_1 move at the constant velocity of $-\vec{v}$ relative to F_0.

Let t_0 be the time of F_0. The elapsed time is related to the displacement by the velocity.

$$\frac{d}{dt_0} = -\vec{v}\bullet\vec{\nabla} \tag{7}$$

Let t_1 be the time of F_1. The elapsed time is conserved in both F_0 and F_1[3].

$$dt_0 = dt_1 = dt \tag{8}$$

The static electric field in F_0 becomes dynamic in F_1. From equation (3),

$$\frac{d}{dt}\vec{E}(\vec{R}) = \frac{1}{4\pi\epsilon_0}\frac{d}{dt}\int\vec{Y}dQ \tag{9}$$

The charge distribution moves at a constant velocity of $\vec{v}$ in F_1 and represents the current I. From equation (6), the magnetic field produced by the current is

$$\vec{B}(\vec{R}) = \frac{\mu_0}{4\pi}\vec{v}\times\int\vec{Y}dQ \tag{10}$$

From equations (3,10),

$$\vec{B}(\vec{R}) = \mu_0\epsilon_0\vec{v}\times\vec{E}(\vec{R}) \tag{11}$$

The magnetic field in F_1 is a representation of electric field in F_0 and the relative velocity between F_0 and F_1.

Apply the vector identity:

$$\vec{\nabla}\times(\vec{a}\times\vec{b}) = (\vec{\nabla}\bullet\vec{b}+\vec{b}\bullet\vec{\nabla})\vec{a} - (\vec{\nabla}\bullet\vec{a}+\vec{a}\bullet\vec{\nabla})\vec{b} \tag{12}$$

From equation (11,12),

$$\vec{\nabla}\times\vec{B}=\mu_0\epsilon_0((\vec{\nabla}\bullet\vec{E}+\vec{E}\bullet\vec{\nabla})\vec{v}-(\vec{\nabla}\bullet\vec{v}+\vec{v}\bullet\vec{\nabla})\vec{E}) \quad (13)$$

$\vec{v}$ is a constant vector. Derivative of $\vec{v}$ is zero.

$$\vec{\nabla}\times\vec{B}=\mu_0\epsilon_0((\vec{\nabla}\bullet\vec{E})\vec{v}-(\vec{v}\bullet\vec{\nabla})\vec{E}) \quad (14)$$

Divergence of Coulomb field is zero[4].

$$\vec{\nabla}\times\vec{B}=\mu_0\epsilon_0(-\vec{v}\bullet\vec{\nabla})\vec{E} \quad (15)$$

From equations (7,8,15),

$$\vec{\nabla}\times\vec{B}=\mu_0\epsilon_0\frac{d}{dt}\vec{E} \quad (16)$$

C. Non-inertial Reference frame

Let a constant current flow in a non-inertial reference frame. From equation (4),

$$\vec{\nabla}_R\times\vec{B}(\vec{R})=\frac{\mu_0 I}{4\pi}\vec{\nabla}_R\times\int d\vec{r}\times\vec{Y} \quad (17)$$

Apply the vector identity from equation (12),

$$=\frac{\mu_0 I}{4\pi}\int(\vec{\nabla}_R\bullet\vec{Y}+(\vec{Y}\bullet\vec{\nabla}_R)d\vec{r} \quad (18)$$

$$-\frac{\mu_0 I}{4\pi}\int(d\vec{r}\bullet\vec{\nabla}_R+\vec{\nabla}_R\bullet d\vec{r})\vec{Y} \quad (19)$$

Both divergence and gradient of $\vec{R}$ on $\vec{r}$ are zero.

$$=\frac{\mu_0 I}{4\pi}\int((\vec{\nabla}_R\bullet\vec{Y})d\vec{r}-(d\vec{r}\bullet\vec{\nabla}_R)\vec{Y}) \quad (20)$$

Divergence of $\vec{Y}$ is zero[4].

$$=\frac{\mu_0 I}{4\pi}\int(-d\vec{r}\bullet\vec{\nabla}_R)\vec{Y} \quad (21)$$

$$=\frac{\mu_0}{4\pi}\int\frac{dQ}{dt}(-d\vec{r}\bullet\vec{\nabla}_R)\vec{Y} \quad (22)$$

$$=\frac{\mu_0}{4\pi}\int dQ(-\vec{v}\bullet\vec{\nabla}_R)\vec{Y} \quad (23)$$

From equations (7,8,9,23),

$$\vec{\nabla}\times\vec{B}=\mu_0\epsilon_0\frac{d}{dt}\vec{E} \quad (24)$$

D. Monopole Antenna

let S be a spherical surface with its center located at the end of an antenna.

From equation (16,24),

$$\iint_S\vec{\nabla}\times\vec{B}\bullet d\vec{S}=\mu_0\epsilon_0\frac{d}{dt}\iint_S\vec{E}\bullet d\vec{S} \quad (25)$$

Guass' law[5] states

$$\iint\vec{E}\bullet d\vec{S}=\frac{Q}{\epsilon_0} \quad (26)$$

From equations (25,26),

$$\iint_S\vec{\nabla}\times\vec{B}\bullet d\vec{S}=\mu_0\frac{dQ}{dt}=\mu_0 I \quad (27)$$

Equation (27) is different from Ampere's law[5] which requires an open surface.

E. Infinite Charge distribution

Let a uniform charge distribution follow a linear path infinitely along the direction of $\vec{v}$. The charge density per unit length, λ, in F_0 is independent of location. The electric field of equation (3) is represented by

$$\vec{E}=\frac{\lambda}{2\pi\epsilon_0 R^2}\vec{R} \quad (28)$$

$$\vec{R}\bullet\vec{v}=0 \quad (29)$$

From equations (11,28), the magnetic field is represented by

$$\vec{B}=\mu_0\epsilon_0\vec{v}\times\vec{E}=\frac{\mu_0\lambda}{2\pi R^2}\vec{v}\times\vec{R} \quad (30)$$

Both magnetic field and electric field are perpendicular to $\vec{v}$.

F. Electromagnetic Radiation

Equation (30) can be used to describe an infinite distribution of charges in dielectric medium moving at the velocity of $\vec{v}$. Let $\vec{n}$ be the direction vector.

$$\vec{v}=v\vec{n} \quad (31)$$

From equations (30,31),

$$\vec{B}=\mu_0\epsilon_0 v\vec{n}\times\vec{E}=\frac{v}{c^2}\vec{n}\times\vec{E} \quad (32)$$

Equation (32) can represent the unpolarized light in the vacuum if the velocity becomes the speed of light.

$$\vec{B}=\frac{c}{c^2}\vec{n}\times\vec{E} \quad (33)$$

G. Electromagnetic Symmetry

The uniform charge distribution along the direction of $\vec{v}$ produces a special symmetry between electric field and magnetic field.

From equation (11),

$$\vec{v} \times \vec{B} = \mu_0 \epsilon_0 \vec{v} \times (\vec{v} \times \vec{E}) \tag{34}$$

From equation (29),

$$\vec{v} \bullet \vec{E} = 0 \tag{35}$$

Apply the vector triple product identity.

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \bullet \vec{c})\vec{b} - (\vec{a} \bullet \vec{b})\vec{c} \tag{36}$$

From equations (34,35,36),

$$\vec{v} \times \vec{B} = -\mu_0 \epsilon_0 (\vec{v} \bullet \vec{v})\vec{E} \tag{37}$$

$$= -\frac{v^2}{c^2}\vec{E} \tag{38}$$

From equation (38),

$$\vec{E} = -\frac{c^2}{v^2}\vec{v} \times \vec{B} \tag{39}$$

A transverse electric field can be represented by the magnetic field in a similar equation to equation (11) which shows the magnetic field is represented by the electric field from any charge distribution in constant motion.

Apply curl to equation (39),

$$\vec{\nabla} \times \vec{E} = -\frac{c^2}{v^2}\vec{\nabla} \times (\vec{v} \times \vec{B}) \tag{40}$$

Apply the vector identity:

$$\vec{\nabla} \times (\vec{a} \times \vec{b}) = (\vec{\nabla} \bullet \vec{b} + \vec{b} \bullet \vec{\nabla})\vec{a} - (\vec{\nabla} \bullet \vec{a} + \vec{a} \bullet \vec{\nabla})\vec{b} \tag{41}$$

Apply the property of divergence and constant motion.

$$\vec{\nabla} \bullet \vec{B} = 0 = \vec{\nabla} \bullet \vec{v} \tag{42}$$

From equations (40,41,42),

$$\vec{\nabla} \times \vec{E} = -\frac{c^2}{v^2}(-\vec{v} \bullet \vec{\nabla})\vec{B} \tag{43}$$

From equations (7,8,43),

$$\vec{\nabla} \times \vec{E} = -\frac{c^2}{v^2}\frac{d}{dt}\vec{B} \tag{44}$$

As a summary, the symmetry between the electric field of Coulomb's law and the magnetic field of Biot-Savart law can be represented by the following 4 equations.

From equation (11),

$$\vec{B} = \frac{1}{c^2}\vec{v} \times \vec{E} \tag{45}$$

From equation (39),

$$\vec{E} = -\frac{c^2}{v^2}\vec{v} \times \vec{B} \tag{46}$$

From equations (16,24),

$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{d}{dt}\vec{E} \tag{47}$$

From equation (44),

$$\vec{\nabla} \times \vec{E} = -\frac{c^2}{v^2}\frac{d}{dt}\vec{B} \tag{48}$$

H. Electromagnetic Duality

The symmetry can be made more distinctive with the identity for curl of curl.

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{a}) = \vec{\nabla}(\vec{\nabla} \bullet \vec{a}) - \vec{\nabla}^2\vec{a} \tag{49}$$

From equation (47),

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \frac{1}{c^2}\frac{d}{dt}\vec{\nabla} \times \vec{E} \tag{50}$$

From equations (48,49,50),

$$-\vec{\nabla}^2\vec{B} = \frac{1}{c^2}\frac{d}{dt}(-\frac{c^2}{v^2}\frac{d}{dt}\vec{B}) \tag{51}$$

$$\vec{\nabla}^2\vec{B} = \frac{1}{v^2}\frac{d^2}{dt^2}\vec{B} \tag{52}$$

From equation (48),

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{c^2}{v^2}\frac{d}{dt}\vec{\nabla} \times \vec{B} \tag{53}$$

From equations (47,49,53),

$$-\vec{\nabla}^2\vec{E} = -\frac{c^2}{v^2}\frac{d}{dt}(\frac{1}{c^2}\frac{d}{dt}\vec{E}) \tag{54}$$

$$\vec{\nabla}^2\vec{E} = \frac{1}{v^2}\frac{d^2}{dt^2}\vec{E} \tag{55}$$

The symmetry from equations (52,55) demonstrates a special duality between the electric field of Coulomb's law and the magnetic field of Biot-Savart law.

III. CONCLUSION

A stationary charge distribution in one inertial reference frame is represented by a charge current in another inertial reference frame. The magnetic field is represented by an electric field in another inertial reference frame.

For any dynamic charge distribution that generates electric field in the transverse direction, the magnetic field is also present in the transverse direction in another inertial reference frame. Both fields form a duality and a special symmetry.

Unknown to J. B. Biot, F. Savart and L. C. Maxwell[6], Biot-Savart law describes not only magnetic field but also electromagnetic duality.

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