A Theory of Twin Prime Generators

By *Berndt Gensel* at Spittal an der Drau

Release: March 10, 2020

Abstract. It's well known that every prime number $p \geq 5$ has the form $6k-1$ or $6k+1$. We'll call k the **generator** of p. Twin primes are distinghuished due to a common generator for each pair. Therefore it makes sense to search for the twin primes on the level of their generators. The present paper developes a sieve method to extract all twin primes on the level of their generators up to any limit. On this basis importend properties of the set of the twin prime generators will be studied. Finally the **Twin Prime Conjecture** is proved based on the studied properties.

Keywords. twin primes, primes, number theory - MSC2010: 11A41

Notations

We'll use the following notations:

$\mathbb{N}$ the set of the positve integers,	
$\mathbb{P}$ the set of the primes,	$\mathbb{P}^*$ primes ≥ 5,
$\mathbb{P}_- = \{p \in \mathbb{P}^* \mid p \equiv -1 (\bmod 6)\}$	and $\mathbb{E}_- = \{n \in \mathbb{N} \mid 6n-1 \in \mathbb{P}_-\}$
$\mathbb{P}_+ = \{p \in \mathbb{P}^* \mid p \equiv +1 (\bmod 6)\}$	and $\mathbb{E}_+ = \{n \in \mathbb{N} \mid 6n+1 \in \mathbb{P}_+\}$
	and $\mathbb{E} = \mathbb{E}_- \cap \mathbb{E}_+$
	and $\mathbb{E}^* = \mathbb{E}_- \cup \mathbb{E}_+$.

1. Twin Prime Generators

It's well known that every prime number $p \geq 5$ has the form $6k-1$ or $6k+1$. We'll call k the **generator** of p. Twin primes are distinghuished due to a common generator for each pair. Therefore it makes sense to search for the twin primes on the level of their generators.

Lemma 1.1. *All squares p^2 of primes have the form $p^2 = 6k + 1$ with a generator divisible by* 4*.*

Proof. With $p = 6u \pm 1$ we have

$$\begin{aligned}
(6u \pm 1)^2 &= 36u^2 \pm 12u + 1 \\
&= 6u(6u \pm 2) + 1 \\
&= 6k + 1, \text{ with } k = u(6u \pm 2) \\
\Rightarrow k &= 2u(3u \pm 1).
\end{aligned}$$

If u is even then $2u$ is divisible by 4. If u is odd then $3u \pm 1$ is even and k is divisible by 4 too. □

Let be

$$\kappa(p) = \begin{cases} \dfrac{p+1}{6} & \text{for } p \in \mathbb{P}_- \\ \dfrac{p-1}{6} & \text{for } p \in \mathbb{P}_+ \end{cases} \tag{1.1}$$

an in $\mathbb{P}^*$ defined function, the **generator** of the pair $(6\kappa(p) - 1, 6\kappa(p) + 1)$.

A number x is a member of $\mathbb{E}$ if $6x - 1$ as well as $6x + 1$ are primes. This is true if the following statement holds.

Theorem 1. *A number x is a member of $\mathbb{E}$ if and only if there is* ***no*** *$p \in \mathbb{P}^*$ with $p < 6x - 1$ where one of the following congruences holds:*

$$x \equiv -\kappa(p) (\operatorname{mod} p) \tag{1.2}$$

$$x \equiv +\kappa(p) (\operatorname{mod} p) \tag{1.3}$$

Proof.

A. $p \in \mathbb{P}_-$, therefore is $p = 6\kappa(p) - 1$:

If (1.2) is true then there is a $n \in \mathbb{N}$:

$$\begin{aligned}
x &= -\kappa(p) + n \cdot (6\kappa(p) - 1) \\
6x &= -6\kappa(p) + 6n \cdot (6\kappa(p) - 1) \\
6x + 1 &= -6\kappa(p) + 6n \cdot (6\kappa(p) - 1) + 1 \\
&= (6n - 1)(6\kappa(p) - 1) \\
\Rightarrow 6x + 1 &\equiv 0(\operatorname{mod}(6\kappa(p) - 1)) \Rightarrow x \notin \mathbb{E}
\end{aligned}$$

For (1.3) the proof will be done with $6x - 1$:

$$\begin{aligned}
6x - 1 &= 6\kappa(p) + 6n \cdot (6\kappa(p) - 1) - 1 \\
&= (6n + 1)(6\kappa(p) - 1) \\
\Rightarrow 6x - 1 &\equiv 0(\operatorname{mod}(6\kappa(p) - 1)) \Rightarrow x \notin \mathbb{E}
\end{aligned}$$

B. $p \in \mathbb{P}_+$, therefore is $p = 6\kappa(p) + 1$: We go the same way with (1.2) and $6x - 1$ as well as (1.3) and $6x + 1$:

$$6x - 1 = (6n - 1)(6\kappa(p) + 1) \Rightarrow 6x - 1 \equiv 0(\mathrm{mod}(6\kappa(p) + 1))$$
$$6x + 1 = (6n + 1)(6\kappa(p) + 1) \Rightarrow 6x + 1 \equiv 0(\mathrm{mod}(6\kappa(p) + 1))$$

With these it's shown that $x \notin \mathbb{E}$ if the congruences (1.2) or (1.3) hold. They cannot be true both because they exclude each other.
If on the other hand $x \notin \mathbb{E}$, then is $6x - 1$ or $6x + 1$ no prime. Let be $6x - 1 \equiv 0(\bmod p)$ and $p \in \mathbb{P}_-$. Then we have

$$\begin{aligned} 6x - 1 &\equiv p(\bmod p) \\ &\equiv (6\kappa(p) - 1)(\bmod p) \\ 6x &\equiv 6\kappa(p)(\bmod p) \\ x &\equiv \kappa(p)(\bmod p). \end{aligned}$$

For $p \in \mathbb{P}_+$ we have

$$\begin{aligned} 6x - 1 &\equiv -p(\bmod p) \\ &\equiv -(6\kappa(p) + 1)(\bmod p) \\ 6x &\equiv -6\kappa(p)(\bmod p) \\ x &\equiv -\kappa(p)(\bmod p). \end{aligned}$$

The other both cases we can handle in the same way. Therefore either (1.2) or (1.3) is valid if $x \notin \mathbb{E}$. □

If we consider that the smallest proper divisor of a number $6x - 1$ or $6x + 1$ is less or equal to $\sqrt{6x + 1}$ than p in the congruences (1.2) and (1.3) can be further limited by

$$\hat{p}(x) = \max(p \in \mathbb{P}^* \mid p \leq \sqrt{6x + 1}).$$

Because we consider only primes as modules we have independent congruences. Now we have

$$\left.\begin{array}{l} x \equiv -\kappa(p)(\bmod p) \\ \text{or} \\ x \equiv +\kappa(p)(\bmod p) \end{array}\right\} , p \in \mathbb{P}^*, p \leq \hat{p}(x) \tag{1.4}$$

as a proofable system of criteria to check a number $x \geq 4$ to be not a member of $\mathbb{E}$.

2. The Twin Sieve

The congruences in (1.4) can be combined in the following way:

$$x^2 \equiv \kappa(p)^2(\bmod p) \text{ for } p \in \mathbb{P}^*, p \leq \hat{p}(x), \tag{2.1}$$

because if $x \equiv \pm\kappa(p)(\bmod\, p)$ then there is a number t with $x = \pm\kappa(p) + tp$. Squared this produces $x^2 = \kappa(p)^2 + p(t^2 p \pm 2t\kappa(p))$ and we get $x^2 \equiv \kappa(p)^2(\bmod\, p)$. This is a system of sieves with a sieve function $\psi(x,p)$ for which is

$$\left.\begin{aligned} x^2-\kappa(p)^2 &\equiv \psi(x,p)(\bmod\, p) \\ &\text{or} \\ \psi(x,p) &= (x^2-\kappa(p)^2)\,\mathrm{Mod}\, p \end{aligned}\right\} \quad \text{for } p\in\mathbb{P}^*, p\le\hat{p}(x). \tag{2.2}$$

Obviously $\psi(x,p)$ is a periodical function in x with a period length of p. It is $\psi(x,p) = 0$ if and only if (1.4) is true for p. We'll call the sieve produced by $\psi(x,p)$ as $\boldsymbol{S}_p$. For the system of the sieves $\boldsymbol{S}_5 \times \ldots \times \boldsymbol{S}_p$ we'll build the *aggregate* sieve functions

$$\begin{aligned} \Psi(x,p) &= \prod_{q\in\mathbb{P}^*}^{p} \frac{\psi(x,q)}{q} \\ &\text{and} \\ \hat{\Psi}(x) &= \Psi(x,\hat{p}(x)). \end{aligned} \tag{2.3}$$

Because the value set of $\psi(x,p)$ consists of positive integers between 0 and $p-1$, $\Psi(x,p)$ and $\hat{\Psi}(x)$ have rational values between 0 and < 1.

A number x will be "sieved" by $\boldsymbol{S}_p$ if and only if $\psi(x,p) = 0$. With this the statement of Theorem 1 can be newly expressed:

Theorem 1a . *x is a member of $\mathbb{E}$ if and only if*

$$\hat{\Psi}(x) \neq 0.$$

In contrast to the sieve of ERATOSTHENES in our sieve the exclusion of a number x will be not controlled by $x\,\mathrm{Mod}\, p = 0,$ but by $\left(x^2-\kappa(p)^2\right)\mathrm{Mod}\, p = 0.$
Let be

$$O_p = \min(x\in\mathbb{N} \mid \hat{p}(x) = p). \tag{2.4}$$

For $x \ge O_p$ "works" the sieve $\boldsymbol{S}_p$, i.e. O_p is the **origin point** (in future we will call it: **OP**) of the sieve $\boldsymbol{S}_p$. Every sieve has up from O_p in every ψ–period just $p-2$ positions with $\psi(x,p) \neq 0$ and two positions with $\psi(x,p) = 0$, once if (1.2) and on the other hand if (1.3) is valid. We speak about a– and b–bars of the sieve $\boldsymbol{S}_p$. From (1.2) and (1.3) it is easy to see that the distance between an a– and a b–bar is $2\kappa(p)$.

It is $p \le \hat{p}(x) \le \sqrt{6x+1}$ and therefore $p^2 \le 6x+1$. Then

$$O_p = \frac{p^2-1}{6} \tag{2.5}$$

is the least number which meets this relation. Because of Lemma 1.1 this is a positive integer divisible by 4.

Theorem 2. *Every sieve $\boldsymbol{S}_p \mid p\in\mathbb{P}^*$ starts at position O_p with a sieve bar and we have* $\psi(O_p,p) = 0.$

Proof. We substitute p by $6\kappa(p) \pm 1$. With this and (2.5) holds

$$\begin{aligned} O_p &= \frac{(6\kappa(p) \pm 1)^2 - 1}{6} \\ &= \frac{6\kappa(p)\,(6\kappa(p) \pm 2)}{6} \\ &= \kappa(p)\,(6\kappa(p) \pm 1) \pm \kappa(p) \\ &= \kappa(p) \cdot p \pm \kappa(p) \\ &\equiv \pm\kappa(p) (\operatorname{mod} p) \rightarrow \psi(O_p, p) = 0. \end{aligned}$$

□

It's evident that $\boldsymbol{S}_p$ starts for $p \in \mathbb{P}_-$ with an a–bar and in the other case it starts with a b–bar, $2\kappa(p)$ behind the a–bar.

Because every sieve has two bars per period only, the density of bars of the sieve $\boldsymbol{S}_p$ is

$$\varrho(p) = \frac{2}{p}. \tag{2.6}$$

For every $x \geq O_p$ the local position in the sieve $\boldsymbol{S}_p$ relative to the phase start [1)] can be determined by the *position function* $\tau(x, p)$:

$$\begin{aligned} &x + \kappa(p) \equiv \tau(x, p) (\operatorname{mod} p) \text{ with} \\ &\quad \tau(x, p) = (x + \kappa(p)) \operatorname{Mod} p. \end{aligned} \tag{2.7}$$

Between the sieve function $\psi(x, p)$ and the position function $\tau(x, p)$ there is the following relationship:

$$\begin{aligned} \psi(x, p) &= \tau(x, p) \cdot (x - \kappa(p)) \operatorname{Mod} p \\ &= \tau(x, p) \cdot (\tau(x, p) - 2\kappa(p)) \operatorname{Mod} p. \end{aligned} \tag{2.8}$$

Obviously is $\psi(x, p) = 0$ if and only if $\tau(x, p) = 0$ (a–bar) or $\tau(x, p) = 2\kappa(p)$ (b–bar).

Theorem 3. *The sieve* $\boldsymbol{S}_5$ *has for* $p > 5$ *at all positions* $O_p + 1$ *a sieve bar. It is*

$$\psi(O_p + 1,\ 5) = 0.$$

Proof. It is $\kappa(5) = 1$ and with (2.7) we get

$$\begin{aligned} O_p + 1 &\equiv \tau(O_p, 5) (\operatorname{mod} 5) \quad | \cdot 6 \\ 6O_p + 6 &\equiv 6\tau(O_p, 5) (\operatorname{mod} 5) \quad | -5 \\ 6O_p + 1 &\equiv (6\tau(O_p, 5) - 5) (\operatorname{mod} 5) \\ p^2 = 6O_p + 1 &\equiv (6\tau(O_p, 5) - 5) (\operatorname{mod} 5) \\ &\equiv 6\tau(O_p, 5) (\operatorname{mod} 5) \\ &\equiv (5\tau(O_p, 5) + \tau(O_p, 5)) (\operatorname{mod} 5) \\ &\equiv \tau(O_p, 5) (\operatorname{mod} 5). \end{aligned}$$

[1)] For $p \in \mathbb{P}_-$ the phase start is O_p and else it is $O_p - 2\kappa(p)$.

Therefore is

$$\tau(O_p, 5) = p^2 \,\mathrm{Mod}\, 5.$$

The prime p as odd number ends for $p > 5$ on $1, 3, 7$ or 9 and therefore p^2 on 1 or 9.

$$\begin{aligned}(1 \vee 9)\,\mathrm{Mod}\,5 &= 1 \vee 4 \text{ and therefore}\\ \tau(O_p + 1,\ 5) &= 2 \vee 0 \quad \Rightarrow \psi(O_p + 1,\ 5) = 0.\end{aligned}$$

□

Corollary 2.1. Because of the periodicity of the sieves the Theorem 3 is valid for all positions $O_p + 5t + 1 \mid t = 0, 1, 2, \ldots$ too

$$\psi(O_p + 5t + 1,\ 5) = 0 \mid t = 0, 1, 2, \ldots.$$

3. The Permeability of the Sieves $S_5 \times \ldots \times S_p$

Let's say $p' = \min(t > p \mid t, p \in \mathbb{P}^*)$ is the first prime following on p. Then $\hat{p}(x)$ persists constant on value p in the interval

$$\mathcal{A}_p := \left[O_p, O_{p'} - 1\right]. \tag{3.1}$$

The length of this interval will be notated by d_p.
d_p is depending on the distance between successive primes. Since they only can be even, it is valid with $a = 2, 4, 6, \ldots$

$$\begin{aligned} d_p &= \frac{(p+a)^2 - 1}{6} - \frac{p^2 - 1}{6}\\ &= \frac{2ap + a^2}{6}\\ &= \frac{a}{3}(p + \frac{a}{2})\\ &\geq \frac{2}{3}(p + 1). \end{aligned} \tag{3.2}$$

If $a = 2$ then is $(p, p+2)$ a twin prime. Thus is $p \in \mathbb{P}_-$ and therefore

$$\frac{2}{3}(p + 1) = \frac{2}{3}(6\kappa(p) - 1 + 1) = 4\kappa(p).$$

On the other hand it results because of $p' < 2p$ (see [3], p. 188)

$$\begin{aligned} d_p &= \frac{p'^2 - 1}{6} - \frac{p^2 - 1}{6}\\ &= \frac{p'^2 - p^2}{6}\\ &= \frac{(p' + p)(p' - p)}{6}\\ &< \frac{3p \cdot p}{6} = \frac{p^2}{2}. \end{aligned}$$

p^2 is odd. The last even number is p^2-1. Thus for the upper bound is valid

$$\frac{p^2-1}{2} = 3O_p$$

and therefore

$$4\kappa(p) \le d_p \le 3O_p. \tag{3.3}$$

The congruences from (2.7)

$$x + \kappa(t) \equiv \tau(x,t) (\operatorname{mod} t), \quad t \le p \mid t \in \mathbb{P}^* \tag{3.4}$$

meet the requirements of the Chinese Remainder Theorem [3, p. 27]. Therefore it is $(\operatorname{mod} 5{\cdot}7{\cdot}\ldots{\cdot}p)$ uniquely resolvable. With

$$p\sharp_5 := \prod_{t\in\mathbb{P}^*}^{p} t \tag{3.5}$$

it's $(\operatorname{mod} p\sharp_5)$ [2)] uniquely resolvable. In other words there are $p\sharp_5$ different tuples

$$(\tau(x,5), \tau(x,7), \ldots, \tau(x,p))$$

in the sieves $\boldsymbol{S}_5 \times \ldots \times \boldsymbol{S}_p$ from O_p. Therefore the aggregate sieve function has the period length $p\sharp_5$:

$$\Psi(x + a \cdot p\sharp_5, p) = \Psi(x,p) \mid a \in \mathbb{N}.$$

Let be

$$\mathbb{K}_p = \{x \ge O_p \mid \Psi(x,p) > 0\}. \tag{3.6}$$

The set $\mathbb{K}_p$ contains all numbers which are **not** sieved by the sieves $\boldsymbol{S}_5 \times \ldots \times \boldsymbol{S}_p$. Since from $O_{p'}$ already the sieve $\boldsymbol{S}_{p'}$ is working, the members of $\mathbb{K}_p$ from this point on are not necessary generators of twin primes. On the other hand there is no twin prime generator $(> O_{p'})$ which is not a member of $\mathbb{K}_p$.

Definition 3.1. A positive integer will be called an "ω_p–number" if x is a member of $\mathbb{K}_p$ which means $\Psi(x,p) > 0$.

Let be

$$\mathcal{P}_p := [O_p, O_p + p\sharp_5 - 1]$$

the interval of the period of the sieves $\boldsymbol{S}_5 \times \ldots \times \boldsymbol{S}_p$. Evidently is $d_p \ll p\sharp_5$ and it is for all p

$$\mathcal{A}_p \subset \mathcal{P}_p.$$

---

2) It is $p\sharp_5 = \frac{1}{6}p\sharp$, with the primorial $p\sharp$.

Lemma 3.1. *At the beginnning $O_p + a \cdot p\sharp_5 \mid a \in \mathbb{N}$ of every further period of $\boldsymbol{S}_5 \times \ldots \times \boldsymbol{S}_p$ it cannot be an OP O_q of a "later" sieve S_q.*

Proof. The equation

$$\frac{p^2-1}{6} + a \cdot p\sharp_5 = \frac{q^2-1}{6} \text{ and thus } p^2 + 6a \cdot p\sharp = q^2$$

is for no prime q solvable, because of $\gcd(p,q) = 1$. □

Reversed it concludes that every sieve $\boldsymbol{S}_{p'}$ starts always in the inner of period sections of the "previous" sieves $\boldsymbol{S}_5 \times \ldots \times \boldsymbol{S}_p$. And it is easy to verify that for $p \geq 11$ is

$$O_{p'} < \frac{p\sharp_5 - 1}{2}. \tag{3.7}$$

The values of the function $\tau(x,t)$ are the numbers $0, 1, \ldots, t-1$. Two of them produce the exlcuding of x and $t-2$ don't. Therefore by working of the sieves $\boldsymbol{S}_5 \times \ldots \times \boldsymbol{S}_p$ we have

$$\varphi(p) = \prod_{t \in \mathbb{P}^*}^{p} (t-2) \tag{3.8}$$

ω_p–numbers in $\mathcal{P}_p$. If these are in $\mathcal{A}_p$, they are members of $\mathbb{E}$, consequently generators of twin primes because the sieves $\boldsymbol{S}_5 \times \ldots \times \boldsymbol{S}_p$ here are working only. The relation between (3.8) and the period length of (3.4) results in

$$\eta(p) = \frac{\varphi(p)}{p\sharp_5} = \prod_{t \in \mathbb{P}^*}^{p} \frac{t-2}{t}, \tag{3.9}$$

as a measure of the mean "permeability" of working of the sieves $\boldsymbol{S}_5 \times \ldots \times \boldsymbol{S}_p$ or as the density of the ω_p–numbers in $\mathcal{P}_p$. Obviously $\eta(p)$ is a strong monotonously decreasing function. Its inversion

$$\bar{\delta}(p) = \frac{1}{\eta(p)} \tag{3.10}$$

discribes the mean distance between the ω_p–numbers up from O_p.

Theorem 4.

$$\eta(p) > \frac{3}{p} \quad \textit{for } p \in \mathbb{P}^*, p > 7.$$

Proof. Let $\mathbb{T}_p = \{t \in \mathbb{P}^* \mid t \leq p\}$ and $\mathbb{U}_p = \{u \equiv 1 (\bmod 2) \mid 5 \leq u \leq p\}$. Because all primes > 2 are odd numbers $\mathbb{T}_p \subset \mathbb{U}_p$ for $p > 7$ [3] is valid. All factors of $\eta(p)$ are less than 1. It results

$$\eta(p) > \prod_{u \in \mathbb{U}_p}^{p} \frac{u-2}{u} = \frac{3}{5} \cdot \frac{5}{7} \cdot \frac{7}{9} \cdot \ldots \cdot \frac{p-4}{p-2} \cdot \frac{p-2}{p} = \frac{3}{p}.$$

□

[3] For $p \leq 7$ is $\mathbb{T}_p = \mathbb{U}_p$.

By inversion of this relationship, we get

$$\bar{\delta}(p) < \frac{p}{3}. \tag{3.11}$$

For the number of the $\omega_{p'}$–numbers in $\mathcal{P}_{p'}$ is corresponding with (3.8)

$$\varphi(p') = \varphi(p) \cdot (p' - 2).$$

In $\mathcal{P}_p$ the $\varphi(p)$ ω_p–numbers are distributed on $p\sharp_5$ positions. In $\mathcal{P}_{p'}$ there are $p' \cdot \varphi(p)$ ω_p–numbers, the p'–times. In comparison to the $\omega_{p'}$–numbers we see

$$p' \cdot \varphi(p) - \varphi(p') = p' \cdot \varphi(p) - (p' - 2) \cdot \varphi(p) = 2\varphi(p). \tag{3.12}$$

We loose by the working of $\boldsymbol{S}_{p'}$ in the interval $\mathcal{P}_{p'}$ just $2\varphi(p)$ potential generators of twin primes in comparison to the

$$p'\sharp_5 \cdot \varrho(p') = p'\sharp_5 \cdot \frac{2}{p'} = 2p\sharp_5$$

sieve bars of $\boldsymbol{S}_{p'}$. In other words, the sieve $\boldsymbol{S}_{p'}$ has in $\mathcal{P}_{p'}$ $2\varphi(p)$ "working" bars. At these positions x is

$$\Psi(x, p) > 0 \text{ and } \psi(x, p') = 0. \tag{3.13}$$

4. Quadratic Residues

Theorem 5. *Let be r for $q < p$ the quadratic residue of p modulo q:*

$$p^2 \equiv r (\operatorname{mod} q).$$

Then holds

$$r \cdot \kappa(q) \equiv \begin{cases} \tau(O_p, q)(\operatorname{mod} q) & \textit{for } q \in \mathbb{P}_- \\ (2\kappa(q) - \tau(O_p, q))(\operatorname{mod} q) & \textit{for } q \in \mathbb{P}_+. \end{cases} \tag{4.1}$$

Proof.

a. $q \in \mathbb{P}_-$:

$$\begin{aligned} O_p + \kappa(q) &\equiv \tau(O_p, q)(\operatorname{mod} q) \mid -\kappa(q) \mid \cdot 6 \mid +1 \\ 6O_p + 1 &\equiv (6\tau(q) - 6\kappa(q) + 1))(\operatorname{mod} q) \end{aligned}$$

And because $6O_p + 1 = p^2$ it holds

$$\begin{aligned} p^2 &\equiv (6\tau(O_p, q) - 6\kappa(q) + 1))(\operatorname{mod} q) \mid q = 6\kappa(q) - 1 \\ &\equiv (6\tau(O_p, q) - q)(\operatorname{mod} q) \\ &\equiv 6\tau(O_p, q)(\operatorname{mod} q). \end{aligned}$$

Additionally holds $p^2 \equiv r (\mathrm{mod}\, q)$ and with it

$$\begin{aligned} r &\equiv 6\tau(O_p, q)(\mathrm{mod}\, q) \mid \cdot\kappa(q) \\ r \cdot \kappa(q) &\equiv (6\kappa(q) \cdot \tau(O_p, q))(\mathrm{mod}\, q) \mid 6\kappa(q) = q + 1 \\ &\equiv ((q + 1) \cdot \tau(O_p, q))(\mathrm{mod}\, q) \\ &\equiv \tau(O_p, q)(\mathrm{mod}\, q). \end{aligned}$$

b. $q \in \mathbb{P}_+$:

$$\begin{aligned} 6O_p + 1 = p^2 &\equiv (6\tau(O_p, q) - 6\kappa(q) + 1)(\mathrm{mod}\, q) \mid q - 2 = 6\kappa(q) - 1 \\ \text{und somit} & \\ p^2 &\equiv (6\tau(O_p, q) - q + 2)(\mathrm{mod}\, q) \\ &\equiv (6\tau(O_p, q) + 2)(\mathrm{mod}\, q). \end{aligned}$$

Forward like above:

$$\begin{aligned} r \cdot \kappa(q) &\equiv (6\kappa(q) \cdot \tau(O_p, q) + 2)(\mathrm{mod}\, q) \mid 6\kappa(q) = q - 1 \\ &\equiv ((q - 1) \cdot \tau(O_p, q) + 2)(\mathrm{mod}\, q) \\ &\equiv (2\kappa(q) - \tau(O_p, q))(\mathrm{mod}\, q). \end{aligned}$$

□

From (4.1) it follows immediately

$$\tau(O_p, q) = \begin{cases} r \cdot \kappa(q)\, \mathrm{Mod}\, q & \text{for } q \in \mathbb{P}_- \\ (q + 2 - r)\kappa(q)\, \mathrm{Mod}\, q & \text{for } q \in \mathbb{P}_+. \end{cases} \tag{4.2}$$

If we put this in (2.8) then we obtain finally

$$\psi(O_p, q) = \kappa(q)^2 \cdot r(r - 2)\, \mathrm{Mod}\, q, \text{ for } q \in \mathbb{P}^*. \tag{4.3}$$

Theorem 6. *In all OP's O_q there are exactly $\frac{q-1}{2}$ different τ – values.*

Proof. Modulo to every prime q there are $\frac{q-1}{2}$ quadratic residues (see [2], p. 125, Corollary C). Hence the right side of (4.2) can have exactly $\frac{q-1}{2}$ different values for a fixed q. Then this holds for the left side of the equation too. □

Corollary 4.1. For every positive integer a as an equal offset is for all $p > q$

$$\tau(O_p + a, q) = (\tau(O_p, q) + a)\, \mathrm{Mod}\, q. \tag{4.4}$$

Therefore the range of $\tau(O_p + a, q)$ contains by fixed offset a over all p equally many different values like in the OP's. Corresponding with Theorem 6 this are $\frac{q-1}{2}$.

Corollary 4.2. While a square number is for every modul a quadratic residue [4)] it holds not for 2. This value is quadratic noresidue f.i. for the moduls

$$5, 11, 13, 19, 29, 37, 43, 53, 59, 61, \ldots$$

. How with (4.2) and (4.3) can easy be checked, it holds for $r \neq 2$

$$\psi(O_p, q) \neq 0 \text{ , für } q < p, q \in \mathbb{P}^*.$$

Hence it cannot be a sieve bars in the OP's of all the sieves $\boldsymbol{S}_q$ for their order q the value 2 is a quadratic noresidue and with EULER (see [2], p. 131 below) holds

$$2^{\frac{q-1}{2}} \not\equiv 1 (\operatorname{mod} q).$$

Also like Corollary 4.2 there are sieves $\boldsymbol{S}_q$ whose τ–values in the OP's cannot have the values 0 und $2\kappa(q)$, so there are sieves $\boldsymbol{S}_t$ for which with an equal offset a to all OP's O_p holds

$$\tau(O_p + a, t) \neq 0 \vee 2\kappa(t), \forall p \in \mathbb{P}^*, p \geq t.$$

Hence in these sieves the ψ–function at these positions is not zero. Let be $\mathbb{T}_a$ the set of the orders of these sieves and

$$\eta_a(p) = \prod_{t \in \mathbb{T}_a}^{p} \frac{t-2}{t}$$

the permeability rate of these sieves. We will call them $\mathbb{T}_a$**–sieves**. At all positions $O_s + a \mid \forall s \geq p$, $s \in \mathbb{P}^*$ then there are with "probability"

$$W_a(p) = \frac{\eta(p)}{\eta_a(p)} = \prod_{q \notin \mathbb{T}_a}^{p} \frac{q-2}{q}$$

ω_p–numbers because at these positions there are no sieve bars in the sieves $\boldsymbol{S}_q \mid q \in \mathbb{T}_a$. Therefore only the remaining sieves can make for sieving.

Example. $1, 3, 4, 9, 10$ and 12 are the 6 possible quadratic residues modulo 13. From this with (4.2) we can calculate the corresponding τ–values at all OP's:

$$2, 6, 9, 10, 11 \text{ and } 12$$

and with (4.4) calculate the τ–values of the by 285 shifted positions which are

$$1, 5, 8, 9, 10 \text{ and } 11.$$

Neither 0 nor $2\kappa(13) = 4$ are among the τ–values. Hence the sieve $\boldsymbol{S}_{13}$ has at no position $O_p + 285 \mid \forall p \in \mathbb{P}^*$ a bar. The same holds also for the sieves $\boldsymbol{S}_5$, $\boldsymbol{S}_7$, $\boldsymbol{S}_{11}$, $\boldsymbol{S}_{19}$, $\boldsymbol{S}_{23}$, $\boldsymbol{S}_{37}$, $\boldsymbol{S}_{41}$, $\boldsymbol{S}_{47}$, $\boldsymbol{S}_{53}$, $\boldsymbol{S}_{61}$, $\boldsymbol{S}_{67}$, $\boldsymbol{S}_{71}$, $\boldsymbol{S}_{73}$, $\boldsymbol{S}_{79}$ and ... They are $\mathbb{T}_{285}^{p}$–sieves. In the consecutive sequence up to the sieve $\boldsymbol{S}_{79}$ are only absent the sieves $\boldsymbol{S}_{17}$, $\boldsymbol{S}_{29}$, $\boldsymbol{S}_{31}$, $\boldsymbol{S}_{43}$ und $\boldsymbol{S}_{59}$. In 14 of the 19 sieves with the highest bar density there are no sieve bars at the positions $O_p + 285 \mid p \geq 79$. The "probability" to meet at any position $O_p + 285$ with $p \geq 79$ to an ω_{79}–number is here about 71%. In fact $O_{79} + 285 = 1325$ is an ω_{79}–number. Also in the three subsequent period sections $\mathcal{P}_{83}$, $\mathcal{P}_{89}$ und $\mathcal{P}_{97}$ there are at the positions $O_p + 285$ ω_{79}–numbers. Only $\mathcal{P}_{101}$ breaks this serial.

---

4) This deduces from the Euler-Criterion (see [2], p. 131) and the Little Fermat Theorem.

If $a < d_p$ then $O_p + a$ remains in the interval $\mathcal{A}_p$. Then because of (3.2) all $O_p + a$ are for

$$p > \frac{3}{2} \cdot a - 1$$

with ”probability“ $W_a(p)$ Twin Prime Generators.

5. The Function $\eta(p)$

There is an interesting relation of the η–function to the *Twin Prime Constant* C_2 (see [3], p. 202)

$$C_2 = \prod_{\substack{p \in \mathbb{P} \\ p>2}} \left(1 - \frac{1}{(p-1)^2}\right) = \frac{3}{4} \prod_{p \in \mathbb{P}^*} \frac{p(p-2)}{(p-1)^2}.$$

C_2 is the limes of the function $\chi(p)$

$$\chi(p) = \frac{3}{4} \prod_{q \in \mathbb{P}^*}^{p} \frac{q(q-2)}{(q-1)^2} \text{ with } C_2 = \lim_{p \to \infty} \chi(p).$$

$\chi(p)$ is obviously a strong monotonously decreasing function. Therefore holds

$$\chi(p) > C_2, \text{ for all } p \in \mathbb{P}^*.$$

Let be

$$\eta_1(p) = \prod_{q \in \mathbb{P}^*}^{p} \frac{q-1}{q}.$$

Evidently it holds $\eta(p) < \eta_1(p)$ [5] and

$$\chi(p) = \frac{3}{4} \prod_{q \in \mathbb{P}^*}^{p} \frac{q(q-2)}{(q-1)^2} = \frac{3}{4} \prod_{q \in \mathbb{P}^*}^{p} \frac{q^2(q-2)}{q(q-1)^2} = \frac{3}{4} \cdot \frac{\eta(p)}{\eta_1(p)^2}$$

and hence

$$\eta(p) = \frac{4}{3}\chi(p) \cdot \eta_1(p)^2 \text{ and also } \eta(p) > \frac{4}{3} C_2 \cdot \eta_1(p)^2$$

and

$$\lim_{p \to \infty} \left(\frac{\eta(p)}{\eta_1(p)^2}\right) \geq \frac{4}{3} C_2 \approx 0,880173.$$

It is well known that is

$$\frac{3}{\eta_1(x)} = \prod_{p \in \mathbb{P}}^{x} \left(\frac{p-1}{p}\right)^{-1} > \log x$$

[5] In this logic is our η–function the function $\eta_2(p)$.

(see [2], p. 40). Hence we have

$$\begin{aligned} \eta_1(p) &< \frac{3}{\log p} \text{ and because of } \eta(p) < \eta_1(p) \\ \eta(p) &< \frac{3}{\log p} \end{aligned}$$

and finally with Theorem 4

$$\frac{3}{\log p} > \eta(p) > \frac{3}{p}.$$

Because both bounds for $p \to \infty$ go to 0, it holds also

$$\lim_{p\to\infty} \eta(p) = 0.$$

Let be

$$\lambda(p) = \sum_{q\in\mathbb{P}^*}^{p} \frac{\eta(q)}{q-2}. \tag{5.1}$$

Between this function and $\eta(p)$ there is an amazing connection.

Lemma 5.1. $\eta(p) + 2\lambda(p) = 1.$

Proof. From (3.9) we get

$$\eta(p') = \eta(p) \cdot \frac{p'-2}{p'}.$$

The proof will be done by mathematical induction.

p=5:

$$\begin{aligned} \eta(5) + 2\lambda(5) &= \frac{3}{5} + 2 \cdot \frac{1}{3} \cdot \frac{3}{5} \\ &= \frac{3}{5} + \frac{2}{5} = 1. \end{aligned}$$

p: We assume that holds $\eta(p) + 2\lambda(p) = 1$.

p’: We consider for p' [6]:

$$\begin{aligned} \lambda(p') &= \lambda(p) + \frac{\eta(p')}{p'-2} \\ \eta(p') + 2\lambda(p') &= \eta(p') + 2\left(\lambda(p) + \frac{\eta(p')}{p'-2}\right) \\ &= \eta(p')\left(1 + \frac{2}{p'-2}\right) + 2\lambda(p) \\ &= \eta(p') \cdot \frac{p'}{p'-2} + 2\lambda(p) \\ &= \eta(p) + 2\lambda(p) = 1. \end{aligned}$$

[6] Note that p' is the on p immediately subsequent prime number.

□

The with Theorem 4 found lower bound for $\eta(p)$ can be improved.

Theorem 7. $\eta(p) > \dfrac{1}{\sqrt{p}} \mid p \geq 19.$

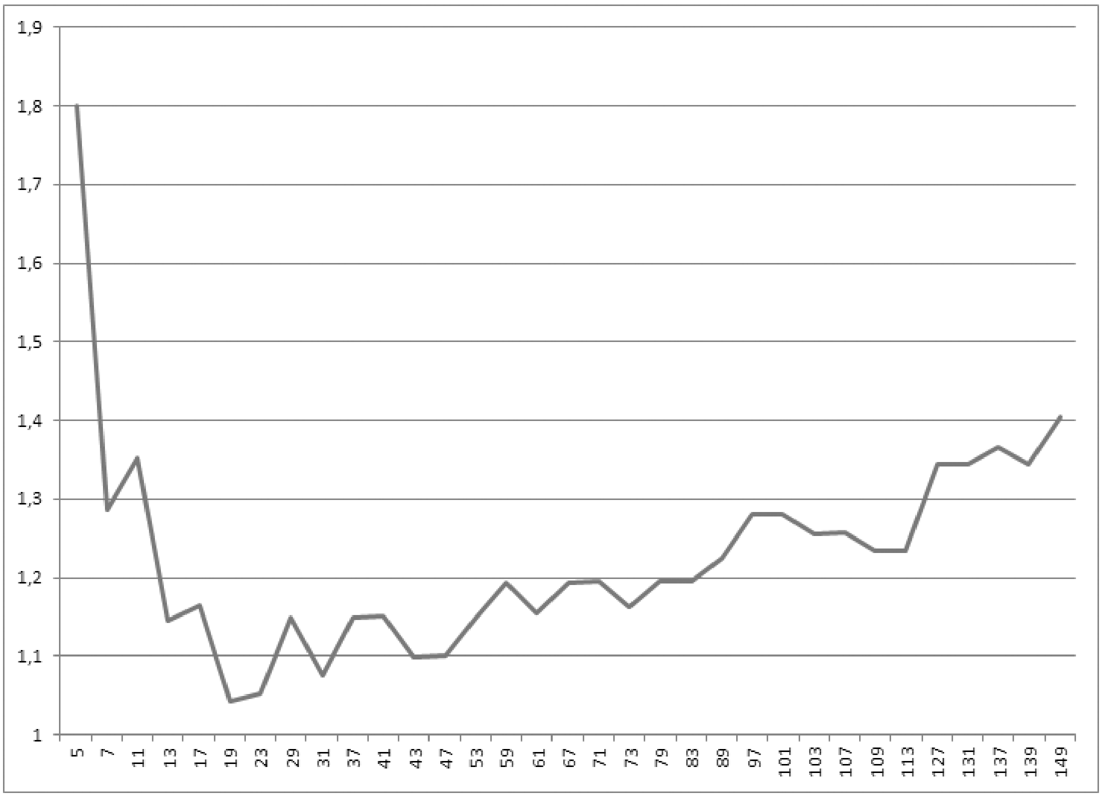

Figure 1. $p\eta(p)^2$ for $p = 5 \ldots 149$

Proof. We consider the properties of $p\eta(p)^2$ for two cases:

A) $p' \geq p + 4$:

$$\begin{aligned} p'\eta(p')^2 - p\eta(p)^2 &= \eta(p)^2\left(p'\frac{(p'-2)^2}{p'^2} - p\right) \\ &= \eta(p)^2\left(\frac{p'(p'-4)+4}{p'} - p\right) \\ &= \eta(p)^2\left(p' - 4 - p + \frac{4}{p'}\right) \\ &> \frac{4\eta(p)^2}{p'} > 0. \end{aligned}$$

Hence it holds in this case $p'\eta(p')^2 > p\eta(p)^2$.

B) $p' = p + 2$:

$$
\begin{aligned}
p'\eta(p')^2 - p\eta(p)^2 &= \eta(p)^2 \left(p' \frac{(p'-2)^2}{p'^2} - p \right) \\
&= p\eta(p)^2 \left(\frac{p}{p'} - 1 \right) \\
&= p\eta(p)^2 \cdot \left(\frac{p-p'}{p'} \right) \\
&= -\frac{2p}{p+2}\eta(p)^2 < 0.
\end{aligned}
$$

Therefore is $p'\eta(p')^2 = p\eta(p)^2 - v(p)$ with

$$
v(p) := \frac{2p}{p+2}\eta(p)^2.
$$

The "loss function" $v(p)$ is monotonously decreasing, because it is for two subsequent twin primes $p, p+2$ und $p+6, p+8$:

$$
\begin{aligned}
v(p+6) &= \frac{2(p+6)}{p+8}\eta(p+6)^2 \\
&= \frac{2}{p+8} \cdot \frac{(p+4)^2}{p+6}\eta(p+2)^2 \\
&= \frac{2}{p+8} \cdot \frac{(p+4)^2}{p+6} \cdot \frac{p^2}{(p+2)^2}\eta(p)^2 \\
&= v(p) \cdot \frac{p(p+4)^2}{(p+2)(p+6)(p+8)} < v(p).
\end{aligned}
$$

Because this is valid for the least distance of two twin primes, it holds also for arbitrary distances. How we can see in Figure 1, the minimum of $p'\eta(p')^2$ lies by $p' = 19$ and $p = 17$:

$$
17 \cdot \eta(17)^2 > 1,165 \text{ and } v(17) < 0,123 \longrightarrow 19 \cdot \eta(19)^2 > 1.
$$

Because in case A) $p\eta(p)^2$ increases and in case B) always holds $v(p) < 0,123$, it holds for $p \geq 19$

$$
p\eta(p)^2 > 1 \longrightarrow \frac{1}{\eta(p)^2} < p.
$$

□

Through a deeper view we see the following situation. The greatest twin prime < 200 is $(197, 199)$. The next prime number 211 belongs to $\mathbb{P}_+$ and it is $211 \cdot \eta(211)^2 > 1,5159$. On the other hand is $v(197) < 0,0148$ and all other $v(p)$ are still less because $v(p)$ is monotonously decreasing. Therefore we get for $p > 200$

$$
p\eta(p)^2 > \frac{3}{2} \longrightarrow \frac{1}{\eta(p)^2} < \frac{2}{3}p < \frac{2}{3}(p+1) \leq d_p.
$$

Hence for $p > 200$ is the square of the average distance between the ω_p–numbers always less than d_p:

$$\bar{\delta}(p)^2 < d_p \text{ for } p > 200. \tag{5.2}$$

Now we will look for more properties of the in (3.9) defined function $\eta(p)$.

Theorem 8. *Let be $\pi(n)$ the number of primes in the interval $[2, n]$. It holds for $p \geq 31$*

$$\eta(p) > \frac{2}{\pi(p)}.$$

Proof. We prove at first

$$\frac{\pi(p)}{\pi(p)+1} \cdot \frac{p'}{p'-2} < 1.$$

We consider the difference of numerator and divisor

$$\begin{aligned} & \pi(p)p' - (\pi(p)+1)(p'-2) \\ = \quad & 2 - p' < 0 \end{aligned}$$

From this it deduces the claim. The theorem we will prove by mathematical induction. We assume that the asserted inequation for p is valid. We transform it and multiply both sides with $\frac{\pi(p)}{\pi(p)+1} \cdot \frac{p'}{p'-2}$

$$\frac{2}{\pi(p) \cdot \eta(p)} < 1 \mid \cdot \frac{\pi(p)}{\pi(p)+1} \cdot \frac{p'}{p'-2}$$

For the left side we obtain

$$\begin{aligned} & \frac{2}{\pi(p) \cdot \eta(p)} \cdot \frac{\pi(p)}{\pi(p)+1} \cdot \frac{p'}{p'-2} \\ & \text{because } \pi(p)+1 = \pi(p') \text{ and } \eta(p) \cdot \frac{p'-2}{p'} = \eta(p') \\ = \quad & \frac{2}{\pi(p') \cdot \eta(p')}. \end{aligned}$$

The right side is like proved above

$$\frac{\pi(p)}{\pi(p)+1} \cdot \frac{p'}{p'-2} < 1.$$

Therefore holds

$$\frac{2}{\pi(p') \cdot \eta(p')} < 1.$$

The proof will be completed by the start of the induction

$$\eta(31) = \frac{3}{7} \cdot \frac{9}{13} \cdot \frac{15}{19} \cdot \frac{21}{23} \cdot \frac{27}{31} > \frac{2}{11} = \frac{2}{\pi(31)}.$$

□

Theorem 9. *The function*

$$\theta(p) := O_p \cdot \eta(p)$$

is a strong monotonously increasing function.

Proof.

$$\begin{aligned}
\theta(p') - \theta(p) &= O_{p'} \cdot \eta(p') - O_p \cdot \eta(p) \\
&= \eta(p) \left(O_{p'} \left(1 - \frac{2}{p'} \right) - O_p \right).
\end{aligned}$$

$$\begin{aligned}
O_{p'} \left(1 - \frac{2}{p'} \right) - O_p &= O_{p'} - O_p - \frac{2O_{p'}}{p'} \\
&= O_{p'} - O_p - \frac{p'^2 - 1}{3p'} \\
&> O_{p'} - O_p - \frac{p'}{3} \\
&= \frac{(p'^2 - p^2)}{6} - \frac{p'}{3} \text{ and because } p' - 2 \geq p \\
&\geq \frac{(p'^2 - (p' - 2)^2)}{6} - \frac{p'}{3} \\
&= \frac{2(p' - 1)}{3} - \frac{p'}{3} \\
&= \frac{p' - 2}{3} > 0.
\end{aligned}$$

Hence we get $\theta(p') > \theta(p)$. □

Corollary 5.1. For all $x \in \mathcal{A}_p$ is also $x \cdot \eta(p)$ a strong monotonously increasing function because here $\eta(p)$ remains constant, while x strong monotonously increases. The monotony get lost by the transition from $\mathcal{A}_p$ to $\mathcal{A}_{p'}$ because

$$\begin{aligned}
O_{p'} \cdot \eta(p') - \big(O_{p'} - 1\big) \cdot \eta(p) &= O_{p'} \cdot \eta(p) \left(1 - \frac{2}{p'} \right) - \big(O_{p'} - 1\big) \cdot \eta(p) \\
&= \eta(p) \left(1 - \frac{2O_{p'}}{p'} \right) \\
&= \frac{\eta(p)}{p'} \left(p' - \frac{p'^2 - 1}{3} \right) \\
&< 0 \text{ für } p' > 3.
\end{aligned}$$

Theorem 10. *Also the function*

$$\hat{\theta}(p) := O_p \cdot \eta(p)^2$$

is a strong monotonously increasing function.

Proof.

$$\begin{aligned}
\hat{\theta}(p') - \hat{\theta}(p) &= \eta(p)^2 \left(O_{p'} \cdot \left(\frac{p'-2}{p'} \right)^2 - O_p \right) \\
&= \eta(p)^2 \left(O_{p'} \cdot \frac{p'-4}{p'} - O_p + \frac{4O_{p'}}{p'^2} \right) \\
O_{p'} \cdot \frac{p'-4}{p'} - O_p + \frac{4O_{p'}}{p'^2} &= O_{p'} - O_p + \frac{4O_{p'}}{p'^2} - \frac{4O_{p'}}{p'} \\
&= \frac{p'^2 - p^2}{6} - 4\frac{p'^2-1}{6p'} \left(1 - \frac{1}{p'} \right) \\
&> \frac{p'^2 - p^2}{6} - 4\frac{p'^2}{6p'} \left(1 - \frac{1}{p'} \right) \\
&= \frac{p'^2 - p^2}{6} - \frac{4p'}{6} \left(1 - \frac{1}{p'} \right) \\
&= \frac{1}{6} \left(p'^2 - p^2 - 4p' + 4 \right) \\
&= \frac{1}{6} \left((p'-2)^2 - p^2 \right) \\
&\geq 0, \text{ because } p' - 2 \geq p.
\end{aligned}$$

Hence it holds $\hat{\theta}(p') > \hat{\theta}(p)$. □

If we expand the product on the right side of (3.9) then we obtain

$$\eta(p) = \sum_{d | p\sharp_5} \mu(d) \frac{2^{\nu(d)}}{d}, \tag{5.3}$$

whereupon $\nu(d)$ means the number of the prime factors of d with $\nu(1) = 0$, and $\mu(d)$ is the Möbius-function

$$\mu(d) = \begin{cases} 1 & \text{if } d = 1 \\ 0 & \text{if } d \text{ is not sqarefree} \\ (-1)^{\nu(d)} & \text{else.} \end{cases}$$

If we multiply (5.3) with $p\sharp_5$ then we obtain

$$\begin{aligned}
\omega(p) &= p\sharp_5 \cdot \eta(p) \\
&= \sum_{d | p\sharp_5} \mu(d) \frac{2^{\nu(d)} p\sharp_5}{d} \\
&\quad \text{and because all } d \text{ are divisors of } p\sharp_5, \\
&= \sum_{d | p\sharp_5} \mu(d) \left[\frac{2^{\nu(d)} p\sharp_5}{d} \right].
\end{aligned}$$

6. Symmetry of the Positions of the ω_p–Numbers

In the interval $\mathcal{P}_p$ the element $x_p^{(0)} := p\sharp_5$ has a particular importance. Because $p\sharp_5$ is divisible by all primes between 5 and p it holds

$$p\sharp_5 \equiv 0 (\operatorname{mod} q) \mid q \leq p, q \in \mathbb{P}^*$$

and hence

$$x_p^{(0)} \not\equiv \pm\kappa(q) (\operatorname{mod} q) \mid q \leq p, q \in \mathbb{P}^*.$$

Therefore $x_p^{(0)}$ is an ω_p–number and thus a member of $\mathbb{K}_p$. Because of $O_p < p\sharp_5 < O_p + p\sharp_5$ the number $p\sharp_5$ is in the inner of $\mathcal{P}_p$ but near to the end. Furthermore for all primes q between 5 and p [7] is

$$\begin{aligned} x_p^{(0)} + \kappa(q) &\equiv \kappa(q) (\operatorname{mod} q) \text{ and therefore} \\ \tau(x_p^{(0)}, q) &= (p\sharp_5 + \kappa(q)) \operatorname{Mod} q \text{ and because of } q \mid p\sharp_5 \\ &= \kappa(q). \end{aligned}$$

$x_p^{(0)}$ is marked by the fact that all sieves $\boldsymbol{S}_5 \times \ldots \times \boldsymbol{S}_p$ on this position have a τ–value of $\kappa(q)$. This means that for all $q \leq p$ is

$$\begin{aligned} &\tau(x_p^{(0)} - \kappa(q), q) = 0 \text{ and} \\ &\tau(x_p^{(0)} + \kappa(q), q) = 2\kappa(q), \text{ or with the } \psi - \text{function} \\ &\psi(x_p^{(0)} \pm \kappa(q), q) = 0. \end{aligned} \tag{6.1}$$

Therefore bars are in the sieves $\boldsymbol{S}_5 \times \ldots \times \boldsymbol{S}_p$ at the positions $x_p^{(0)} \pm \kappa(q)$. Evidently there is symmetry around $p\sharp_5$ with respect to the positions of the bars.

Theorem 11. *In every sieve $\boldsymbol{S}_q$ it's ψ–symmetry around on the position $p\sharp_5$:*

$$\psi(p\sharp_5 - a, q) = \psi(p\sharp_5 + a, q).$$

Proof. Corresponding with (2.8) we have

$$\psi(x, q) = \tau(x, q) \cdot (\tau(x, q) - 2\kappa(q)) \operatorname{Mod} q.$$

We put $p\sharp_5 - a$ for x and have

$$\begin{aligned} \psi(p\sharp_5 - a, q) &= \tau(p\sharp_5 - a, q) \cdot (\tau(p\sharp_5 - a, q) - 2\kappa(q)) \operatorname{Mod} q \\ &= (\kappa(q) - a) \cdot ((\kappa(q) - a) - 2\kappa(q)) \operatorname{Mod} q \\ &= -\left(\kappa(q)^2 - a^2\right) \operatorname{Mod} q. \end{aligned}$$

For $p\sharp_5 + a$ it results

$$\begin{aligned} \psi(p\sharp_5 + a, q) &= \tau(p\sharp_5 + a, q) \cdot (\tau(p\sharp_5 + a, q) - 2\kappa(q)) \operatorname{Mod} q \\ &= (\kappa(q) + a) \cdot ((\kappa(q) + a) - 2\kappa(q)) \operatorname{Mod} q \\ &= -\left(\kappa(q)^2 - a^2\right) \operatorname{Mod} q \\ &= \psi(p\sharp_5 - a, q). \end{aligned}$$

□

[7] Unless otherwise specified the use of the variable q means below $q \in \mathbb{P}^* \mid q \leq p$.

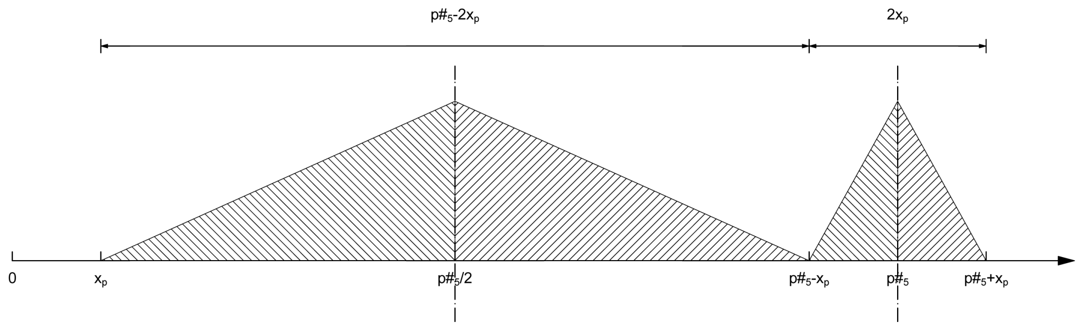

Figure 2. Sections of symmetry in $\mathcal{P}_p$

Hence there is a section of symmetry with a length of $2O_p$ at the end of $\mathcal{P}_p$ around on $x_p^{(0)} = p\sharp_5$. But in the rest of the interval there is symmetry too.

Theorem 12. *In every sieve* $\boldsymbol{S}_q$ *there is* ψ*–symmetry for all primes between* 5 [8] *and* p *around the middle between the positions* $\frac{p\sharp_5 - 1}{2}$ *and* $\frac{p\sharp_5 + 1}{2}$*:*

$$\psi\left(\frac{p\sharp_5 - 1}{2} - a, q\right) = \psi\left(\frac{p\sharp_5 + 1}{2} + a, q\right).$$

Proof. Because of the periodicity of the sieve function we have

$$\begin{aligned}\psi(O_p + a) &= \psi(O_p + p\sharp_5 + a) \text{ and because of Theorem 11}\\ &= \psi(p\sharp_5 - O_p - a).\end{aligned}$$

Therefore there is symmetry around

$$x_p^{(1)} := \frac{O_p + a + p\sharp_5 - O_p - a}{2} = \frac{p\sharp_5}{2} = \frac{1}{2}\left(\frac{p\sharp_5 - 1}{2} + \frac{p\sharp_5 + 1}{2}\right).$$

We put $x = \frac{p\sharp_5 + 1}{2} + a$ and get

$$\begin{aligned}\psi\left(\frac{p\sharp_5 + 1}{2} + a, q\right) &= \psi\left(p\sharp_5 - \frac{p\sharp_5 + 1}{2} - a, q\right)\\ &= \psi\left(\frac{p\sharp_5 - 1}{2} - a, q\right).\end{aligned}$$

□

Figure 2 shows schematically (and not in scale) the sections of symmetry in $\mathcal{P}_p$. The section around on $x_p^{(1)}$ has a length of $p\sharp_5 - 2O_p$. It raises more than the length of the section around on $x_p^{(0)}$ because of $p\sharp_5$ raises [9] more quickly than O_p.

---

[8] For the case $p = 5$ we use $p\sharp_5 + \frac{p\sharp_5 - 1}{2}$, hence $\frac{p\sharp_5 - 1}{2} = 2 < O_5 = 4$.

[9] Already for $p = 29$ the relation $\frac{p\sharp_5 - 2O_p}{p\sharp_5}$ has a value of more than $0,99999974$.

Corollary 6.1. Since all sieves $\boldsymbol{S}_q$ have the same symmetry qualities, the cooperation of the sieves $\boldsymbol{S}_5 \times \ldots \times \boldsymbol{S}_p$ indicates these qualities too

$$\Psi\left(p\sharp_5 - a, p\right) = \Psi\left(p\sharp_5 + a, p\right) \text{ for } 1 \le a \le O_p$$
and
$$\Psi\left(\frac{p\sharp_5 - 1}{2} - a, p\right) = \Psi\left(\frac{p\sharp_5 + 1}{2} + a, p\right) \text{ for } 1 \le a \le \frac{p\sharp_5 - 1}{2} - O_p.$$

Therefore the ω_p–numbers are symmetrically distributed around the axes $x_p^{(0)}$ und $x_p^{(1)}$.

Corollary 6.2. The $2\varphi(p)$ excludings of the sieve $\boldsymbol{S}_{p'}$ (see (3.12)) have the same symmetry around $x_{p'}^{(0)}$ and $x_{p'}^{(1)}$. This means that also the beating bars are symmetrically distributed.

Let's denote the symmetry half sections as

$$S_p^{(1)} := \left[O_p, \frac{p\sharp_5 - 1}{2}\right] \text{ and } S_p^{(0)} := \left(p\sharp_5 - O_p, p\sharp_5\right).$$

Theorem 13. *A positive integer x with $\kappa(p) < x < O_p$ is a Twin Prime Generator ($x \in \mathbb{E}$) if and only if $x + p\sharp_5$ is an ω_p–Zahl ($x + p\sharp_5 \in \mathbb{K}_p$).*

Proof.

a) For all prime numbers q with $5 \le q \le p$ holds

$$\begin{aligned}(6(x + p\sharp_5) \pm 1) \operatorname{Mod} q &= (6x \pm 1 + 6p\sharp_5) \operatorname{Mod} q \\ &= (6x \pm 1) \operatorname{Mod} q, \text{ because } q \mid p\sharp_5 \\ &> 0, \text{ because } 6x \pm 1 \in \mathbb{P}.\end{aligned}$$

Hence $x + p\sharp_5$ is an ω–number.

b) If $x + p\sharp_5$ is an ω_p–number and it is $x > \kappa(p)$ then holds

$$\begin{aligned}\Psi(x + p\sharp_5, p) &> 0 \text{ and because of the periodicity of } \Psi(x,p) \\ &= \Psi(x,p) > 0.\end{aligned}$$

And because all ω_p–numbers $< O_p < O_{p'}$ are twin prime generators therefore $x < O_p$ is a twin prime generator.

The case $x \le \kappa(p)$ must be excluded because of (6.1). □

Corollary 6.3. The number of twin prime generators in the interval $(\kappa(p), O_p)$ is equal to the number of ω_p–numbers in $S_p^{(0)}$.

$$\pi_2\left(p^2\right) - \pi_2\left(p\right) = \sharp\left(\mathbb{K}_p \cap S_p^{(0)}\right),$$

because it is $6O_p + 1 = p^2$ and $6\kappa(p) \pm 1 = p$.

Theorem 14. *In the symmetry half $S_p^{(0)}$ there is only at position $p\sharp_5 - \kappa(p)$ a beating bar of the sieve $\boldsymbol{S}_p$.*

Proof.

a. We prove at first: At position $p\sharp_5 - \kappa(p)$ it is possible to have a beating bar. Let be t_p the p immediately preliminary prime. With $p \in \mathbb{P}_-$ it holds $\kappa(t_p) < \kappa(p)$. Because of (6.1) holds furthermore for $q \leq t_p$ $\psi\left(p\sharp_5 - \kappa(q), q\right) = 0$. Then is $\Psi\left(p\sharp_5 - \kappa(p), t_p\right) > 0$ if

$$p \in \mathbb{P}_- \text{ and } \psi\left(p\sharp_5 - \kappa(p), q\right) > 0 \mid \forall q \leq t_p.$$

Due to (6.1) is $\psi\left(p\sharp_5 - \kappa(p), p\right) = 0$. Then we have at $p\sharp_5 - \kappa(p)$ a **beating** bar in the sieve $\boldsymbol{S}_p$. Subsequent up to $p\sharp_5$ because of (6.1) it is not possible to have a beating bar.

b. Because of the periodicity of $\Psi(x, t_p)$ is

$$\begin{aligned}
\Psi(x, t_p) &= \Psi(x + t_p\sharp_5, t_p) \text{ and because of the symmetry} \\
&= \Psi(t_p\sharp_5 - x, t_p) \\
&= \Psi(a \cdot t_p\sharp_5 - x, t_p) \text{ and with } a = p \\
&= \Psi(p\sharp_5 - x, t_p).
\end{aligned}$$

Would be at position $p\sharp_5 - x$ on $x < O_p$ a beating bar of the sieve $\boldsymbol{S}_p$, it would be

$$\Psi(p\sharp_5 - x, t_p) > 0 \text{ and } \psi(p\sharp_5 - x, p) = 0. \tag{6.2}$$

However then is also $\Psi(x, t_p) > 0$ and x is therefore a twin prime generator. And in contradiction to (6.2) is $\psi(x, p) > 0$ because $6x \pm 1$ as prime numbern cannot be divisible by p. Therefore there is in the interval $(p\sharp_5 - O_p, p\sharp_5 - \kappa(p))$ no beating bars of the sieve $\boldsymbol{S}_p$.

□

Accordingly with (3.12) there are in $\mathcal{P}_p$ just $2\varphi(t_p)$ beating bars. With it this theorem means that in the symmetry half $S_p^{(1)}$ are all $\varphi(t_p)$, at least however $\varphi(t_p) - 1$ beating bars. Let be

$$\iota(p) := p\sharp_5 \operatorname{Mod} 6 \text{ or } p\sharp_5 \equiv \iota(p) (\operatorname{mod} 6).$$

The range of $\iota(p)$ is obviously $\{-1, +1\}$ and it counts with $\iota(5) = -1$

$$\iota(p') = \begin{cases} -\iota(p) & \text{if } p' \in \mathbb{P}_- \\ \iota(p) & \text{if } p' \in \mathbb{P}_+. \end{cases}$$

From the definition of $\iota(p)$ we see that $p\sharp_5 - \iota(p)$ is divisible by 6. Hence $\frac{p\sharp_5 - \iota(p)}{3}$ is an even integer. On the other hand $p\sharp_5 + \iota(p)$ is not divisible by 3, because only one of three numbers can be divisible by 3.

Theorem 15. *At the positions* $\dfrac{p\sharp_5 - \iota(p)}{3}$ *and* $\dfrac{p\sharp_5 + 5\iota(p)}{6}$ *there are ω_p–numbers:*

$$\Psi\left(\frac{p\sharp_5 - \iota(p)}{3}, p\right) > 0.$$

Proof. At first we prove $\Psi\left(\frac{p\sharp_5-\iota(p)}{3},p\right)>0$. This is equivalent to $\psi\left(\frac{p\sharp_5-\iota(p)}{3},q\right)>0$ for all primes $5\le q\le p$. Therefore below q holds for all primes between 5 and p.

With (2.2) is

$$\psi\left(\frac{p\sharp_5-\iota(p)}{3},q\right)=\left(\left(\frac{p\sharp_5-\iota(p)}{3}\right)^2-\kappa(q)^2\right)\operatorname{Mod}q.$$

Because $q\ge 5$ is prime to 36 it holds also

$$36\psi\left(\frac{p\sharp_5-\iota(p)}{3},q\right)\operatorname{Mod}q=36\left(\left(\frac{p\sharp_5-\iota(p)}{3}\right)^2-\kappa(q)^2\right)\operatorname{Mod}q$$

and hence $36\kappa(q)^2\operatorname{Mod}q=1$ and $\iota(p)^2=1$ we get

$$\begin{aligned}36\psi\left(\frac{p\sharp_5-\iota(p)}{3},q\right)\operatorname{Mod}q &= \left(\left(6\frac{p\sharp_5-\iota(p)}{3}\right)^2-1\right)\operatorname{Mod}q\\ &= ((4p\sharp_5(p\sharp_5-2\iota(p)))+4-1)\operatorname{Mod}q\\ &= 3\ne 0.\end{aligned}$$

For $\Psi\left(\frac{p\sharp_5+5\iota(p)}{6},p\right)>0$ we do analogously. We multiply $\psi\left(\frac{p\sharp_5+5\iota(p)}{6},q\right)$ by 36 and obtain

$$\begin{aligned}36\psi\left(\frac{p\sharp_5+5\iota(p)}{6},q\right)\operatorname{Mod}q &= \left(25-(6\kappa(q))^2\right)\operatorname{Mod}q\\ &= (25-1)\operatorname{Mod}q=2^3\cdot 3\operatorname{Mod}q\\ &\ne 0.\end{aligned}$$

□

Because of Theorem 12 there are ω_p–numbers at the positions

$$p\sharp_5-\frac{p\sharp_5-\iota(p)}{3}\text{ and }p\sharp_5-\frac{p\sharp_5+5\iota(p)}{6}.$$

too.

Corollary 6.4. With an analogous way of proof we can show that also

$$\frac{p\sharp_5+5\iota(p)}{6}\pm 2\iota(p)$$

and its mirror images by $\frac{p\sharp_5}{2}$ for all primes $5\le q\le p$ are ω_p–numbers. The proof results finally the congruences

$$\psi\left(\frac{p\sharp_5+17\iota(p)}{6},q\right)\operatorname{Mod}q = 8$$

und

$$3\psi\left(\frac{p\sharp_5-7\iota(p)}{6},q\right)\operatorname{Mod}q = 4,$$

which with $\psi\left(\frac{p\sharp_5+5\iota(p)}{6}\pm 2\iota(p),q\right)=0$ is never accomplishable.

Theorem 16. *At the positions* $\dfrac{p\sharp_5-1}{2}$ *and* $\dfrac{p\sharp_5+1}{2}$ *are* ω_p*–numbers:*

$$\Psi\left(\frac{p\sharp_5-1}{2},p\right)>0 \text{ and } \Psi\left(\frac{p\sharp_5+1}{2},p\right)>0.$$

Proof. For the position function according to (2.7) with $n\in\mathbb{N}$ we have

$$\begin{aligned} n\tau(x+c,q) &= (n\tau(x,q)+nc)\operatorname{Mod}q \\ &= (nx+n\kappa(q)+nc)\operatorname{Mod}q. \end{aligned}$$

Hence with $x=\dfrac{p\sharp_5+1}{2}$ holds

$$\begin{aligned} \tau\left(\frac{p\sharp_5+1}{2},q\right) &= \left(\frac{p\sharp_5+1}{2}+\kappa(q)\right)\operatorname{Mod}q \text{ and therefore} \\ 2\tau\left(\frac{p\sharp_5+1}{2},q\right)\operatorname{Mod}q &= (p\sharp_5+1+2\kappa(q))\operatorname{Mod}q \\ &= 2\kappa(q)+1. \end{aligned}$$

This equation is never accomplishable with τ–values 0 or $2\kappa(q)$. Therefore it holds

$$\psi\left(\frac{p\sharp_5+1}{2},q\right)\neq 0 \mid 5\le q\le p \Rightarrow \frac{p\sharp_5+1}{2}\in\mathbb{K}_p.$$

Because of the symmetry around $x_p^{(1)}$ also is

$$\psi\left(\frac{p\sharp_5-1}{2},q\right)\neq 0 \mid 5\le q\le p \Rightarrow \frac{p\sharp_5-1}{2}\in\mathbb{K}_p.$$

□

Because of $\tau(x+c,q)=(\tau(x,q)+c)\operatorname{Mod}q$ and with

$$c_q := 2\kappa(q)-\begin{cases} 1 & \text{für } q\in\mathbb{P}_- \\ 0 & \text{für } q\in\mathbb{P}_+ \end{cases}$$

we get

A. for $q\in\mathbb{P}_-$:

$$\begin{aligned} 2\tau\left(\frac{p\sharp_5+1}{2}+c_q,q\right)\operatorname{Mod}q &= (p\sharp_5+1+2\kappa(q)+2c_q)\operatorname{Mod}q \\ &= (2\kappa(q)+1+4\kappa(q)-2)\operatorname{Mod}q \\ &= (6\kappa(q)-1)\operatorname{Mod}q \\ &= q\operatorname{Mod}q=0. \end{aligned}$$

B. for $q \in \mathbb{P}_+$:

$$\begin{aligned} 2\tau\left(\frac{p\sharp_5+1}{2}+c_q, q\right) \operatorname{Mod} q &= (p\sharp_5+1+2\kappa(q)+2c_q) \operatorname{Mod} q \\ &= (2\kappa(q)+1+4\kappa(q)) \operatorname{Mod} q \\ &= (6\kappa(q)+1) \operatorname{Mod} q \\ &= q \operatorname{Mod} q = 0. \end{aligned}$$

We see in both cases that

$$\tau\left(\frac{p\sharp_5+1}{2}+c_q, q\right) = 0 \mid 5 \le q \le p$$

and therefore is

$$\psi\left(\frac{p\sharp_5+1}{2}+c_q, q\right) = 0 \mid 5 \le q \le p.$$

Because of the symmetry this is for $\frac{p\sharp_5-1}{2} - c_q$ valid too. At these positions around $\frac{p\sharp_5-1}{2}$ and $\frac{p\sharp_5+1}{2}$ there are b– or rather a–bars of the sieves $\boldsymbol{S}_5 \times \ldots \times \boldsymbol{S}_p$. For positions x for which $6x-1$ and $6x+1$ are not primes, there is always a prime divisor $q < \hat{p}(x)$ of $6x \pm 1$, whose sieve $\boldsymbol{S}_q$ has a bar at this position. Therefore the sieves $\boldsymbol{S}_5 \times \ldots \times \boldsymbol{S}_p$ exclude the positions $\frac{p\sharp_5-1}{2} - x$ and $\frac{p\sharp_5+1}{2} + x$ for $x = 1, 2, 3, ..., c_p$ comprehensively. This means that in the intervals $\left[\frac{p\sharp_5-1}{2} - c_p, \frac{p\sharp_5-1}{2} - 1\right]$ and $\left[\frac{p\sharp_5+1}{2} + 1, \frac{p\sharp_5+1}{2} + c_p\right]$ there are no ω_p–numbers.

Definition 6.1. Let be a and b ω_p–numbers with $a < b$. An interval $[a+1, b-1]$ will be called ω_p**–gap** with a length of $b-a-1$ if in this interval there are **no** ω_p–numbers.

With this we have around $\frac{p\sharp_5-1}{2}$ and $\frac{p\sharp_5+1}{2}$ ω_p–gaps with a length $\ge c_p$. On the other hand we see, that an ω_p–gap never can reach over the symmetry axis $\frac{p\sharp_5}{2}$. This means that every ω_p–gap occurs twice in $\mathcal{P}_p$ and has a length of c_p at least. Because for all sieves $\boldsymbol{S}_5 \times \ldots \times \boldsymbol{S}_p$ corresponding with (6.1) there are bars around on $p\sharp_5$ too, here we have also two ω_p–gaps but only with a length $\ge \kappa(p)$. This result is in accordance with the fact that in the interval $[p\sharp + 2, p\sharp + p]$ no primes can exist.

In addition to Corollary 6.1 the symmetries around $x_p^{(0)}$ and $x_p^{(1)}$ are valid for ω_p–gaps too.

7. The Statistical Distribution of the ω_p–Numbers

The intervals $\mathcal{A}_p, p \ge 5$ defined by (3.1) cover the positive integers ≥ 4 gapless and closely. It is

$$\mathbb{N} = \{1, 2, 3\} \cup \bigcup_{p \in \mathbb{P}^*} \mathcal{A}_p \text{ and } \bigcap_{p \in \mathbb{P}^*} \mathcal{A}_p = \emptyset.$$

They are the beginnings of the period sections $\mathcal{P}_p$ of the ω_p–numbers. Hereafter let's say **A-sections** to them. Every ω_p–number which is in an A-section is a twin prime generator. In contrast to the A-sections the period sections $\mathcal{P}_p$ overlap very strong. So the period section $\mathcal{P}_{23}$ reachs over 1740 A-sections up to the period section $\mathcal{P}_{14929}$ and the next $\mathcal{P}_{29}$ over 7864 A-sections up to $\mathcal{P}_{80429}$.

Obviously the distribution of the primes on the number line is irregular. Although it is possible by the formula of **Gandhi** (see [26]) from the knowlegde of all primes up to p to predict the subsequent prime p'. But this results no order of the prime distances. It follow less distances after greater ones and reversed in seeming irregular sequence. Because of the *Prime Number Theorem* the distances between primes rise in tendency, the distances cannot be cyclic. Similar to the decimal digits of irrational numbers we will call such an arrangement as an **irrational** distribution.

Corresponding with (2.5) the OPs O_p are proportional to the squares of the primes. Therefore they are irrationally distributed on the number line too with a decreasing density like the primes. And according to Lemma 3.1 and (3.7) every sieve $\boldsymbol{S}_{p'}$ starts always in the inner of the symmetry section $S_p^{(1)}$ of the previous sieves $\boldsymbol{S}_5 \times \ldots \times \boldsymbol{S}_p$.

On the other hand the sieves have a strict regular inner structure. Each sieve $\boldsymbol{S}_p$ starts with a bar on position O_p. The further bars change with a distance of $2\kappa(p)$ or rather $4\kappa(p) \pm 1$ and are consequently dependent on the prime p. Therefore they follow with their positions on the number line to the irrationality of the distribution of the primes.

With the transition $\boldsymbol{S}_p \to \boldsymbol{S}_{p'}$ the symmetry of the sieves $\boldsymbol{S}_5 \times \ldots \times \boldsymbol{S}_p$ in their period sections $\mathcal{P}_p$ repeats p'–times oneself in $\mathcal{P}_{p'}$, disturbed by the $2\varphi(p)$ (see (3.12)) excludings of the sieve $\boldsymbol{S}_{p'}$. Which of the $2p\sharp_5$ bars meet the ω_p–numbers is uncertain. Nevertheless in $\mathcal{P}_{p'}$ is symmetry again (see corollaries 6.1 and 6.2), but only on the whole. In the separate symmetry half sections, namely in $S_{p'}^{(1)}$ the order is disturbed. Because of the irregular positions of the OPs O_p und therefore of the period sections $\mathcal{P}_p$ too, the positions of the ω_p–gaps relativ to O_p change from sieve to sieve **unsystematically**.

From all these it concludes that the distribution of the ω_p–gaps in their symmetry half section $S_{p'}^{(1)}$ is irrational and change from sieve to sieve unsystematically. It follow less gaps to greater ones and reversed in an irrational sequence with a mean of $\bar{\delta}(p)$. A lot of empirical examinations from the author with frequency distributions of ω_p–gaps for several values of p verify a very good appoximation to a **geometrical distribution** with the expected value $\bar{\delta}(p)$ and the variance $\bar{\delta}(p)^2 - \bar{\delta}(p)$:

$$\begin{aligned} G(d) = \quad & P(X \leq d) = 1 - (1 - \eta(p))^d \text{ and thus} \\ & P(X > d) = (1 - \eta(p))^d. \end{aligned}$$

Figure 3 shows the cumulative frequencies of the ω_p–gaps for $p = 1511$ in the intervals from $O_p = 380520$ to $7O_p = 2663640$ in comparison with the geometrical distribution. In this intervall there are 95012 ω_{1511}–numbers, 217 of these in the A-section. The χ^2–approximation test with 50 classes for the gap lengths with a step of $0.2 \cdot \bar{\delta}_p$ produces a test value of less than 0.0582. So we have a very good conformity. Additionally we get a very small theoretical frequency of ω_p–gaps with a length greater than $\bar{\delta}(p)^2$

$$P(X > \bar{\delta}(1511)^2) = (1 - \eta(1511))^{\bar{\delta}(1511)^2} < 2.40 \cdot 10^{-10}.$$

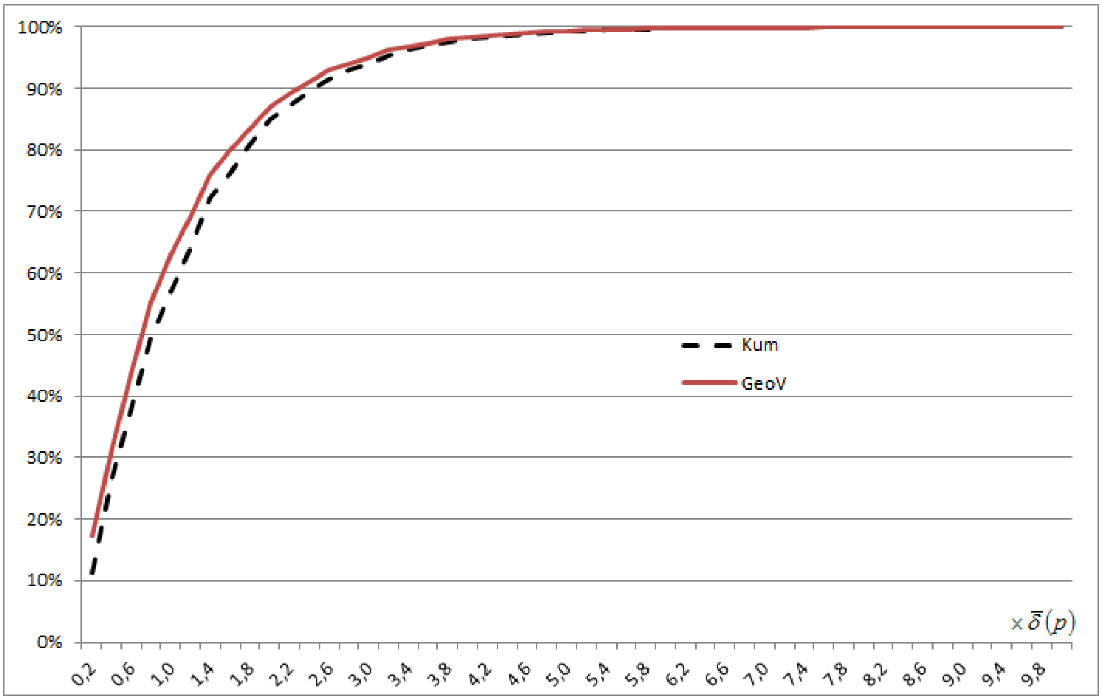

Figure 3. Cumulative frequencies of the ω_p–gaps for $p = 1511$ in the intervals $[O_p, 7O_p]$ in comparison with the geometrical distribution

And the frequency for a gap greater than d_p is much smaller:

$$P(X > d_{1511}) < 1.88 \cdot 10^{-125}.$$

It is easy to show generally that

$$\lim_{p\to\infty} (1 - \eta(p))^{\bar{\delta}(p)} = e^{-1} < 0,4 \text{ and}$$

$$\lim_{p\to\infty} (1 - \eta(p))^{\bar{\delta}(p)^2} = 0.$$

Corollary 7.1. More than 60% of all ω_p–gaps are less than $\bar{\delta}(p)$ and **nearly all** are less than $\bar{\delta}(p)^2$.

8. Another Proof of the Twin Prime Conjecture

Although the Twin Prime Conjecture is already proved in [25] and [27] we will show here another proof on the basis of the statistical properties of the Twin Prime Generators.

Proof. The proof will be done indirectly. We assume that there is only a finite number of twin primes and therefore there is only a finite number of generators of twin primes too. Let be x_o the greatest one. It is in the section $\mathcal{A}_{p_o}$ with $p_o = \hat{p}(x_o)$, the beginning of the period section $\mathcal{P}_{p_o}$. In the subsequent sections $\mathcal{A}_q$ with $q > p_o$ consequently there cannot be any twin prime generators and therefore no ω_q–numbers. But then we have ω_q–gaps with lengths $> d_q$ in **all** (infinitely many) sections $\mathcal{P}_q$ for $q > p_o$. And with (5.2) the gaps are greater than $\bar{\delta}(q)^2$ if $p_o > 200$.

Because

1. the ω_p–gaps are irrationally distributed in their period sections,
2. the period sections itself are disordered distributed on the number line and
3. nearly all ω_q–gaps are smaller than $\bar{\delta}(q)^2$

therefore it is not possible to have for **all** $q > p_o$ period sections $\mathcal{P}_q$ with ω_q–gaps at their beginnings which are all greater than $\bar{\delta}(q)^2$.

Therefore the proof assumtion cannot be valid and thus the Twin Prime Conjecture is true.

$\square$

References

[1] *H. Halberstam, H.-E. Richert*: Sieve methods. Academic Press 1974.
[2] *P. Bundschuh*: Einführung in die Zahlentheorie , Springer, 2008.
[3] *P. Ribenboim*: Die Welt der Primzahlen , Springer, 2011.
[4] *V. Brun*: Le crible d'Eratosthene et le theoreme de Goldbach. Videnskapsselskapets Skrifter Kristiana, Mat.-nat. Kl. 1920, No. 3
[5] *G.H. Hardy, J.E. Littlewood*: Some problems of 'Partitio numerorum'; III: On the expression of a number as a sum of primes. Acta Mathematica, 44, 1923, 1–70
[6] *P. Erdös*: The difference of consecutive primes. Duke Mathematical Journal, 6, 1940, 438–441
[7] *P.A. Clement*: Congruences for sets of primes. Amer. Math. Monthly 56 (1949), 23-25
[8] *A. Selberg*: An elementary proof of the prime number theorem. Annals of Math. 50 (1949), 305-313
[9] *P. Erdös*: On almost primes. Amer. Math. Monthly 57 (1959), 404-407
[10] *E. Bombieri, H. Davenport*: Small differences between prime numbers. Proc. Roy. Soc. (A) 293 (1966), 1-18
[11] *R.P. Brent*: Irregularities in the distribution of primes and twin primes. Math. Comp. 29 (1975), 43-56.
[12] *C.J. Mozzochi*: On the difference between consecutive primes. J. Number Theory 24 (1986), 181-187
[13] *E. Bombieri, J.B. Friedlander und H. Iwaniec*: Primes in arithmetic progressions to large moduli. Acta Mathematica 156.1 (1986), 203–251.
[14] *D. Wolke*: Über das Primzahl-Zwillingsproblem, Mathematische Annalen (1989), 529-537
[15] *H. Mikawa*: On prime twins, Tsukuba Journal of Mathematics (1991), 19-29
[16] *H. Mikawa*: On prime twins in arithmetical progressions, Tsukuba Journal of Mathematics (1992), 377-387
[17] *St. Baier*: Primzahlzwillinge, Diplomarbeit, Freie Universität Berlin, 1997
[18] *M. Madritsch*: Siebmethoden und das Goldbachsche Problem, Diplomarbeit, Technische Universität Graz, 2005
[19] *D.A. Goldston, Y. Motohashi, J. Pintz, C.Y. Yıldırım*: Small gaps between primes exist. In: Proceedings of the Japan Academy, Series A 82, 2006, 61–65
[20] *B. Green, T. Tao*: The primes contain arbitrarily long arithmetic progressions. Annals of Math. (2) 167 (2008), 481-547
[21] *D.A. Goldston, Y. Motohashi, J. Pinz, C.Y. Yıldırım*: Prime in tuples I. Annals of Math. 170 (2009), 819-862
[22] *J. Maynard*: Small gaps between primes. arxiv:1311.4600 Preprint 2013
[23] *Y. Zhang*: Bounded gaps between primes. Annals of Math. 179 (2014), 1121-1174
[24] *M. Köck*: Abstände zwischen Primzahlen, Diplomarbeit, Technische Universität Wien, 2016
[25] *T. Agama*: A lower bound for the two-point correlation of the master function and applications. arXiv:1903.09257v3, Preprint 2019
[26] *B. Gensel*: The Prime Number Formula of Gandhi, arXiv:1910.08362, 2019.
[27] *B. Gensel*: An Elementary Proof of the Twin Prime Conjecture, arXiv:1909.07975, 2020.

Berndt Gensel, Carinthia University of Applied Sciences, Villacher Str. 1, A-9800, Spittal an der Drau, Austria
e-mail: b.gensel@fh-kaernten.at