ζ(z) = X(z) – Y(z) A decomposition of the Riemann Zeta function for $\boldsymbol{Re(z) > 0, z \neq 1}$

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Abstract:

In this paper, we define the C-transformation as:

[1] $C_n\{f\} = \sum_{k=1}^{n} f(k) - \int f(n)\ dn$

And the C-values as:

[2] $C\{f\} = \lim_{n\to\infty} C_n\{f\}$

And we obtain a new representation for ζ(z) in the form ζ(z) = X(z) – Y(z) applying the C-transformation to the function $f(x) = \frac{1}{x^z}$ for $z \in C, Re(z) \geq 0, z \neq 1$.

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Nomenclature and conventions

a. ζ(z)= $\sum_{k=1}^{\infty} k^{-z}$ is the Riemann Zeta function
b. α=a=Re(z) is the real part of a complex number z
c. ß=b=Im(z) is the imaginary part of a complex number z

1. C-Transformation of f(x)

The C-transformation of an integrable function f(x) is defined by:

[3] $C_n\{f(x)\} = \sum_{k=1}^{n} f(k) - \int f(n)\ dn$

And the C-values iss the limit, if it exists, of the C-transformation when $n \to \infty$:

[4] $C\{f(x)\} = \lim_{n\to\infty} C_n\{f(x)\}$

1.1. C-Transformation of $f(x) = \frac{1}{x}$ for $x \in R$:

[5] $C_n\left\{\frac{1}{x}\right\} = \sum_{k=1}^{n} \frac{1}{k} - \int \frac{dn}{n}$

and

[6] $C\left\{\frac{1}{x}\right\} = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln(n)\right) = \gamma$

(γ = Euler-Mascheroni constant = 0.5772…)

1.2. C-Transformation of $f(x) = m$, for $m \in R$ constant:

[7] $C_n\{m\} = \sum_{k=1}^{n} m - \int m\ dn$

[8] $C_n\{m\} = m * n - m * n = 0$

and the C-values of f(x) =m constant is:

[9] $C\{m\} = 0$

1.3. C-Transformation of $f(x) = \sin(x)$ for $x \in R$:

[10] $C_n\{\sin(x)\} = \sum_{k=1}^{n} \sin(k) - \int \sin(n)\, dn$

[11] $C\_n\ \{\sin(x)\} = 1/2(\sin(n) - \cot\left(\frac{1}{2}\right)\cos(n) + \cot\left(\frac{1}{2}\right) + \cos(n))$

And the C-values of $f(x) = \sin(x)$ are in the interval:

[12] $C\{\sin(x)\} \in \left[\frac{1}{2}\left(2\cot\left(\frac{1}{2}\right) - 3\right),\ \frac{3}{2}\right]$

One can also calculate that:

[13] $C\{\cos(x)\} \in \left[\frac{1}{2}\left(\cot\left(\frac{1}{2}\right) - 4\right), \frac{1}{2}\left(2 - \cot\left(\frac{1}{2}\right)\right)\right]$

1.4. C-Transformation of $f(x) = \mathrm{e}^{-\mathrm{x}}$ for $x \in R$:

[14] $$C_n\{\mathrm{e}^{-\mathrm{x}}\} = \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{e}^{-\mathrm{k}} - \int e^{-n} dn$$

[15] $$C_n\{\sin(x)\} = \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{e}^{-\mathrm{k}} + \frac{e^{-n}}{n}$$

And the C-values of $f(x) = \mathrm{e}^{-\mathrm{x}}$ are:

[16] $$\mathrm{C}\{\mathrm{e}^{-\mathrm{x}}\} = \frac{1}{\mathrm{e}-1}$$

1.5. C-Transformation of $f(x) = x^{-s}$ for $x, s \in R$, s>1:

[17] $$C_n\left\{\frac{1}{x^s}\right\} = \sum_{\mathrm{k}=1}^{\mathrm{n}} \frac{1}{\mathrm{k}^{\mathrm{s}}} - \int \frac{dn}{n^s}$$

[18] $$C_n\left\{\frac{1}{x^s}\right\} = \sum_{\mathrm{k}=1}^{\mathrm{n}} \frac{1}{\mathrm{k}^{\mathrm{s}}} - \frac{n^{1-s}}{1-s}$$

and the C-value of $f(x) = \frac{1}{x^s}$ is the Riemann Zeta function for s>1:

[19] $$\mathrm{C}\left\{\frac{1}{x^s}\right\} = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{\mathrm{k}^{\mathrm{s}}} - \frac{\mathrm{n}^{1-\mathrm{s}}}{1-\mathrm{s}}\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{\mathrm{k}^{\mathrm{s}}}\right) - \lim_{n\to\infty}\left(\frac{\mathrm{n}^{1-\mathrm{s}}}{1-\mathrm{s}}\right) = \zeta(\mathrm{s}) - 0 = \zeta(\mathrm{s})$$

1.6. C-Transformation of $f(z) = \frac{1}{x^z}$ for $z \in C, Re(z) \geq 0, z \neq 1$

[20] $$C_n\left\{\frac{1}{x^z}\right\} = \sum_{\mathrm{k}=1}^{\mathrm{n}} \frac{1}{\mathrm{k}^{\mathrm{z}}} - \int \frac{dn}{n^z}$$

We will use Euler's identity:

[21] $$e^x = \cos(x) + i * \sin(x)$$

To calculate [20] for z=α+ßi:

[22] $$k^{-z} = k^{-\alpha}\ [\cos(ß * \ln k) - i\ (\sin\ (ß * \ln \mathrm{k})]$$

And:

[23] $$\int dn/n^z = n^{(1-\alpha)}\ [\cos(ß * ln(n) - i\, sin(ß * ln(n)] * \frac{[(1-\alpha)+iß]}{[(1-\alpha)^2+ß^2]}$$

One can now express the real and imaginary components of $C_n\{f\}$ as:

[24] $Re(C_n\{f\}) = \sum_{k=1}^{n} k^{-\alpha}\,(\cos(ß * \ln(k)) +$
$+ \frac{1}{[(1-\alpha)^2+ß^2]}\,(n^{(1-\alpha)}$ [(1-α)*cos(ß*ln(n))+ß* sin(ß*ln(n))])

[25] $Im(C_n\{f\}) = -\sum_{k=1}^{n} k^{-\alpha}\,(\sin(ß * \ln(k))+$
$+ \frac{1}{[(1-\alpha)^2+ß^2]}\,(n^{(1-\alpha)}$ [ß*cos(ß*ln(n)) - (1-α)*sin(ß*ln(n))])

One can calculate that, for *α=Re(z)>2*, and for any ϵ arbitrarily small, there is a value of n=N such that for n>N, $C_N\{f\}$ - ζ(z)< ϵ, as the following table shows:

α	*ß*	$C_N\{f\}$ *for N=500*	$\zeta(z)$	$\lvert C\_N\,\{f\} - \zeta(z)\rvert$
2	0	1.644934068	1.654934067	$< 10^{-8}$
2	1	1.150355702 + 0.437530865 i	1.150355703+0.437530866 i	$< 10^{-8}$
3	0	1.202056903	1.202056903	$< 10^{-9}$

Table 1. Values of $C_n\{f(n) = k^{-z}\}$ *for α=Re(z)>1 for N=500*

The error $C_n\{f\}$ - $\zeta(z)$ grows significantly in the critical strip for 0≤α<1 as we can see in the following table:

A	*ß*	$C_n\{f\}$	$\zeta(z)$	$\lvert C\_n\,\{f\} - \zeta(z)\rvert$
0.0	0	$C_N\{f\}$ *for N=500*	-0.5	0.5
0.2	2	0.399824505+0.322650799 i	0.360103 + 0.266246 i	> 0.05
0.7	0	-2.777900606	-2.7783884455	$> 10^{-4}$

Table 2. Values of $C_n\{f(n) = k^{-z}\}$ *for 0≤Re(z)<1 for N=500*

To understand better the value of the difference $C_n\left\{\frac{1}{k^z}\right\}$ - ζ(z), one can plot the difference for $\alpha \in [0,1)$ and ß = 0: (Similar exponential charts occur for all values of $\alpha \in [0,1)$ for any given value of ß)

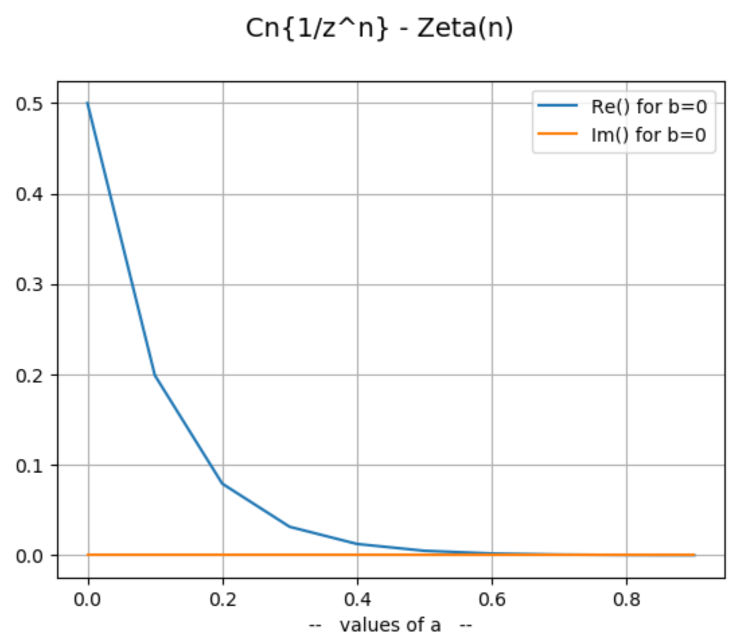

Figure 1 where a=α=Re(z) and b=ß=Im(z)

And plot the difference for variable values of ß ∈ [0,1) and $\alpha = 0$: (Similar sine charts occur for all values of ß ∈ [0,1) for any given value of α)

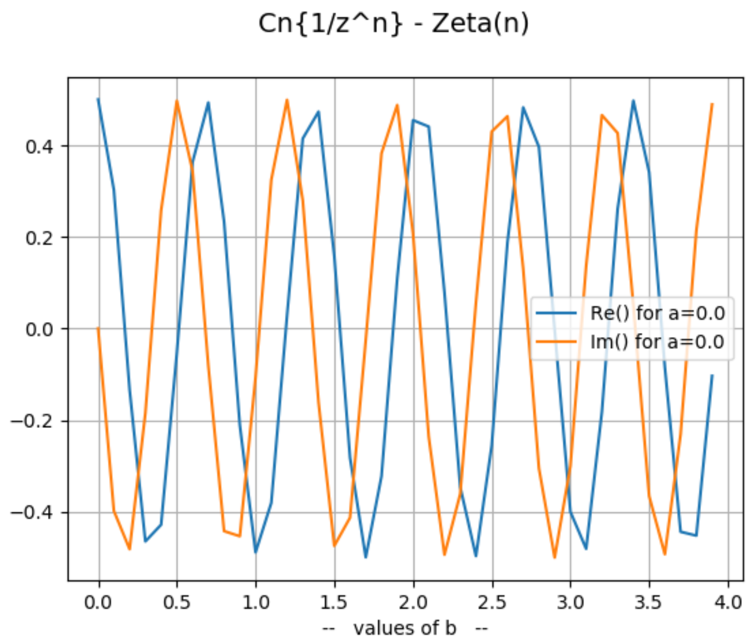

Figure 2 where a=α=Re(z) and b=ß=Im(z)

These charts lead to the following calculation of the difference $C_n\left\{\frac{1}{k^z}\right\} - \zeta(z)$:

[26] $\mathrm{Re}[C_n\left\{\frac{1}{k^z}\right\} - \zeta(z)] = \frac{1}{2}\, n^{-a} * \cos\,(ß * \ln\,(n)) + O(\frac{1}{n})$

[27] $\mathrm{Im}[C_n\left\{\frac{1}{k^z}\right\} - \zeta(z)] = \frac{1}{2}\, n^{-a} * \sin\,(ß * \ln\,(n)) + O(\frac{1}{n})$

With *O(1/n)->0* when n->∞.

And one can finally write:

[28] $Re(C_n\{f\}) = \sum_{k=1}^{n} k^{-\alpha}\,(\cos\,(ß * \ln(k)) +$

$+ \frac{1}{[(1-\alpha)^2+ß^2]}\,(n^{(1-\alpha)}$ [(1-α)*cos(ß*ln(n))+ß* sin(ß*ln(n))])

$+\frac{1}{2}\, n^{-a} * \cos(ß * \ln(n))$

[29] $Im(C_n\{f\}) = -\sum_{k=1}^{n} k^{-\alpha}\,(\sin\,(ß * \ln(k)) +$

$+ \frac{1}{[(1-\alpha)^2+ß^2]}\,(n^{(1-\alpha)}$ [ß*cos(ß*ln(n)) - (1-α)*sin(ß*ln(n))])

$+\frac{1}{2}\, n^{-a} * \sin(ß * \ln(n))$

and the C-value of $f(x) = \frac{1}{x^z}$ for $z \in C, Re(z) \geq 0, z \neq 1$ is the Riemann Zeta function ζ(z).

1.7. A decomposition of ζ(z) based on the C-transformation of $f(x) = \frac{1}{x^z}$ for $z \in C, Re(z) \geq 0, z \neq 1$

One can rewrite [28] and [29] creating the X(z,n) and Y(z,n) functions:

[30] $$\zeta(z) = \lim_{n\to\infty} [X(z,n) - Y(z,n)], \text{ where:}$$

[31] $$X(z,n) = (\textstyle\sum_{k=1}^{n} k^{-\alpha} (\cos(ß * \ln(k)) + \frac{1}{2} n^{-\alpha} \cos(ß \ln(n)) + $$
$$+ i * (\textstyle\sum_{k=1}^{n} k^{-\alpha} (\sin(ß * \ln(k)) + \frac{1}{2} n^{-\alpha} \sin(ß \ln(n)))$$

[32] $$Y(z,n) = n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2 + ß^2]} [((1-\alpha) * \cos(ß \ln(n)) + ß * \sin(ß \ln(n))) + $$
$$+ i \; (ß * \cos(ß \ln(n)) - (1-\alpha) * \sin(ß \ln(n)))]$$

If :

$X(z) = \lim_{n\to\infty} X(z,n)$ and

$Y(z) = \lim_{n\to\infty} Y(z,n)$

Then, one can write:

$\zeta(z) = X(z) - Y(z)$

The following table shows values for [30]:

z= 0 +j* 0 and n=500
Zeta(z) = -0.5 + i* 0.0 X(z)-Y(z) = -0.5 +i* 0.0 ---> Error = 0.0 +i* 0.0
z= 0.2 +j* 2 and n=500
Zeta(z) = 0.360102590022591 + i* -0.266246199765574 X(z)-Y(z) = 0.360102741838091 +i* -0.266246128959438 ---> Error= -1.5181550 e-7 +i* -7.080613 e-8
z= 0.4 +j* 0 and n=500
Zeta(z) = -1.13479778386698 + i* 0.0 X(z)-Y(z) = -1.1347977871726 +i* 0.0 ---> Error= 3.305619 e-9 +i* 0.0

Table 3. ζ(z) compared to X(z) - Y(z)

The highest error for α∈ [0,1), ß ∈ [0,100], n=1000 is $8x10^{-6}$.

2. Representation of the function X(z,n)

The following chart represents X(z,n) for a=1/2 and b∈ [1,6] and n=250

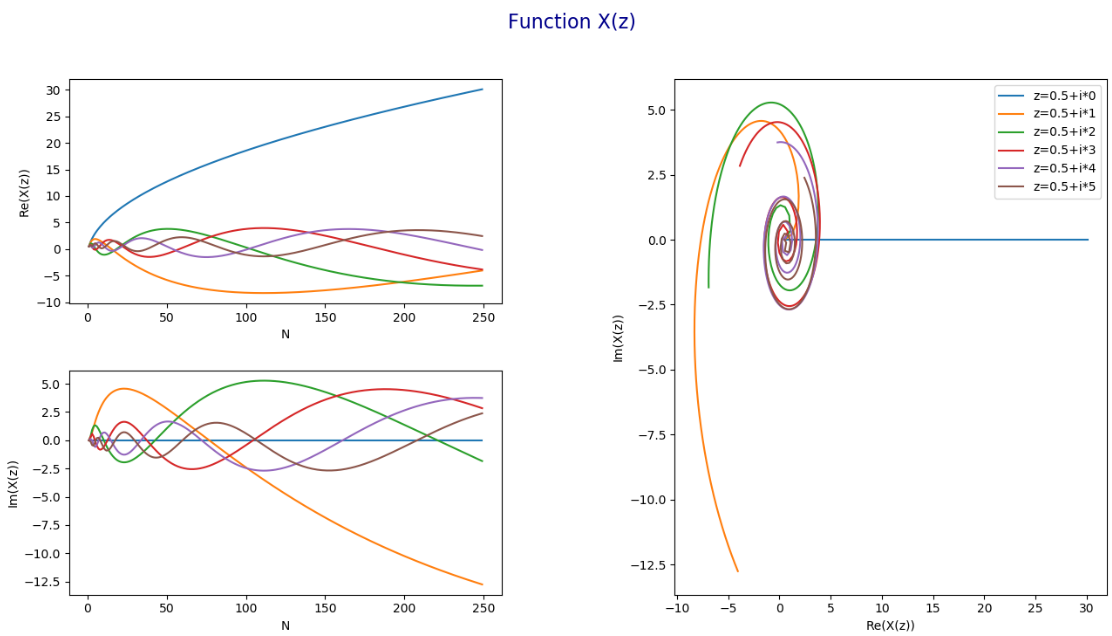

Fig. 3: X(z,n)

The following chart represents X(z,n) for a∈[1,6] and b= 1 and n=250

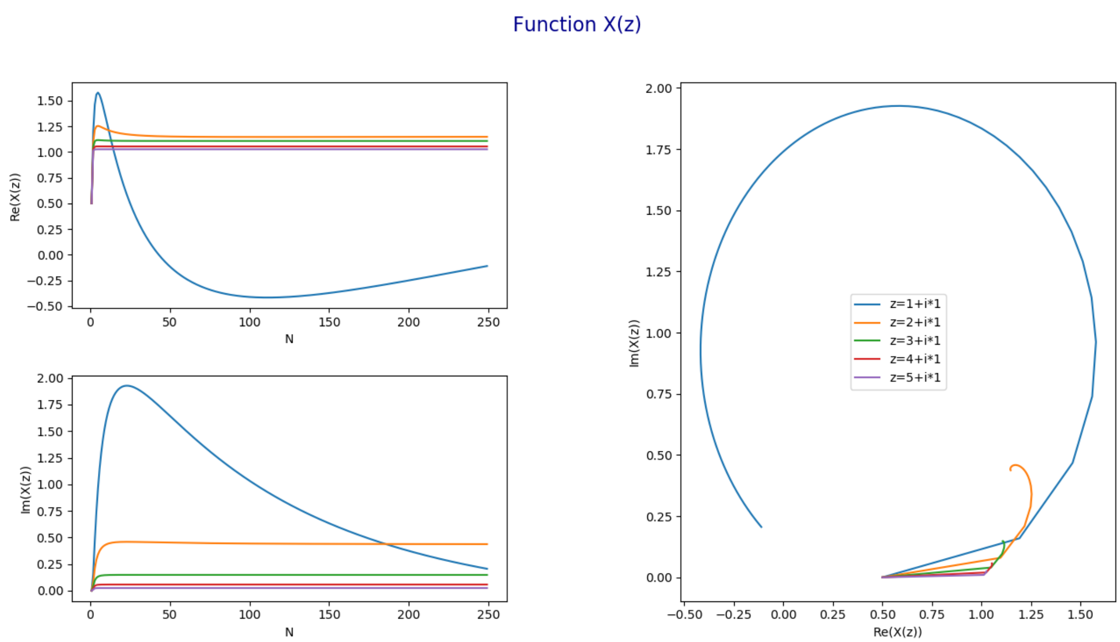

Fig. 4: X(z,n)

3. Representation of the function Y(z,n)

The following chart represents Y(z,n) for a=1/2 and b∈ [1,6] and n=250

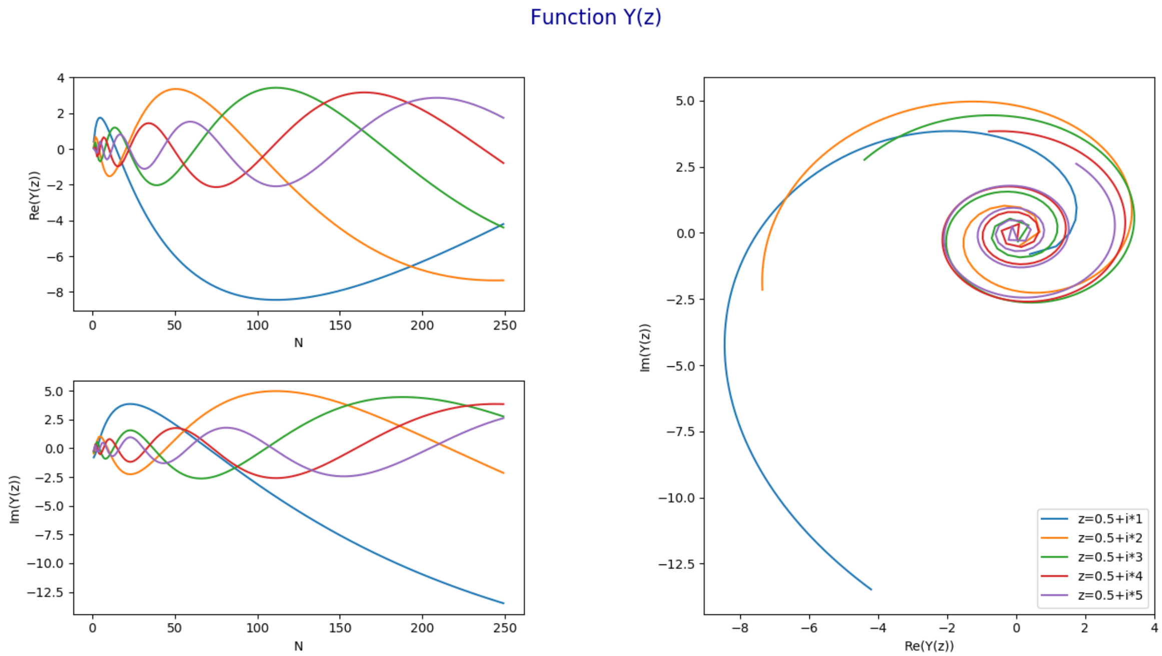

Fig. 5: Y(z,n)

The following chart represents Y(z,n) for a∈[1,6] and b= 1 and n=250

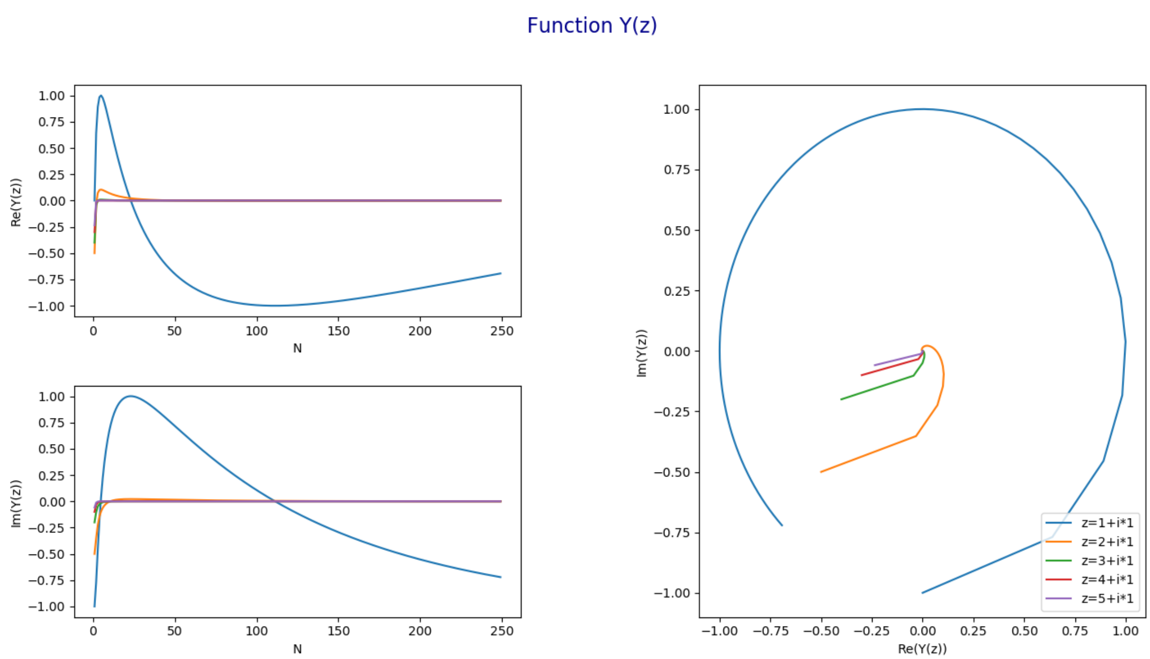

Fig. 6: Y(z,n)

4. Graphical representations of Square Modulus of ζ(z), X(z), Y(z), and X(z) - Y(z)

Representation of $|\zeta(z)|^2$ and $|X(z)\text{-}Y(z)|^2$. These two charts are the same as expected.

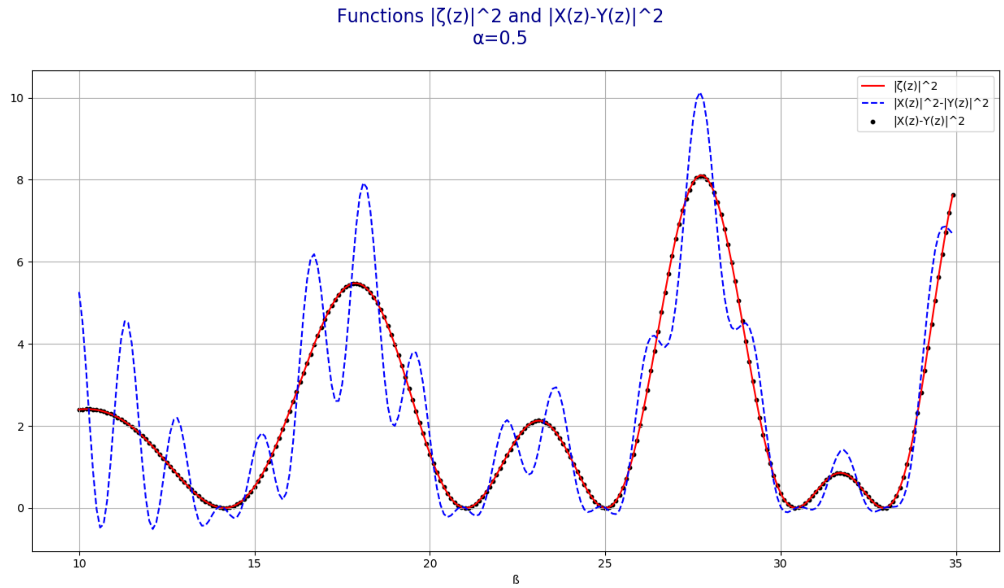

Fig. 7 : $|\zeta(z)|^2 = |X(z)\text{-}Y(z)|^2$

Representation of $|\zeta(z)|^2$, $|X(z)|^2$, $|Y(z)|^2$, and $|X(z)|^2 - |Y(z)|^2$

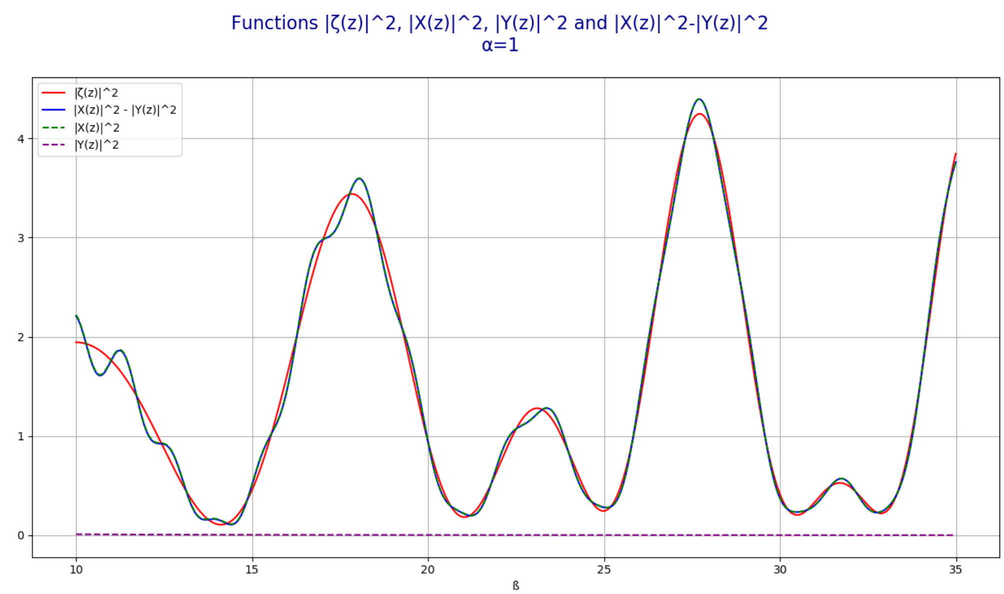

Fig. 8

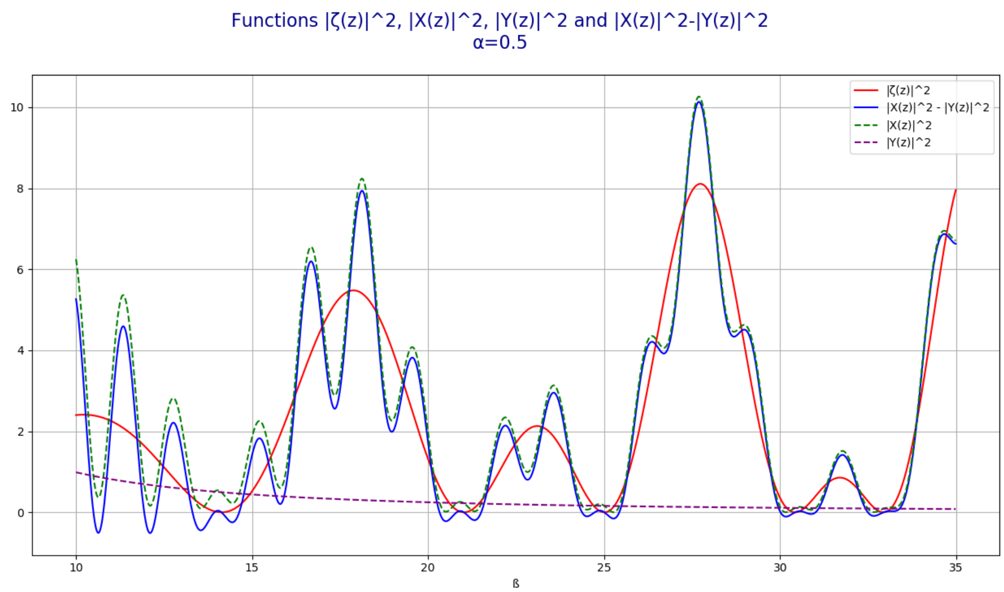

Fig. 9: if $|\zeta(z)|^2 = 0$ then $|X(z)|^2 = |Y(z)|^2$ when $Re(z)=1/2$

One can observe in the following chart that the zeros of $|X(z)|^2 - |Y(z)|^2$ happen on $z=1/2+i\beta_n$ with β_n the imaginary part of the nontrivial zeros of ζ(z). This shows that the for z a nontrivial zero of ζ(z):

$$|X(z)|^2 = |Y(z)|^2$$

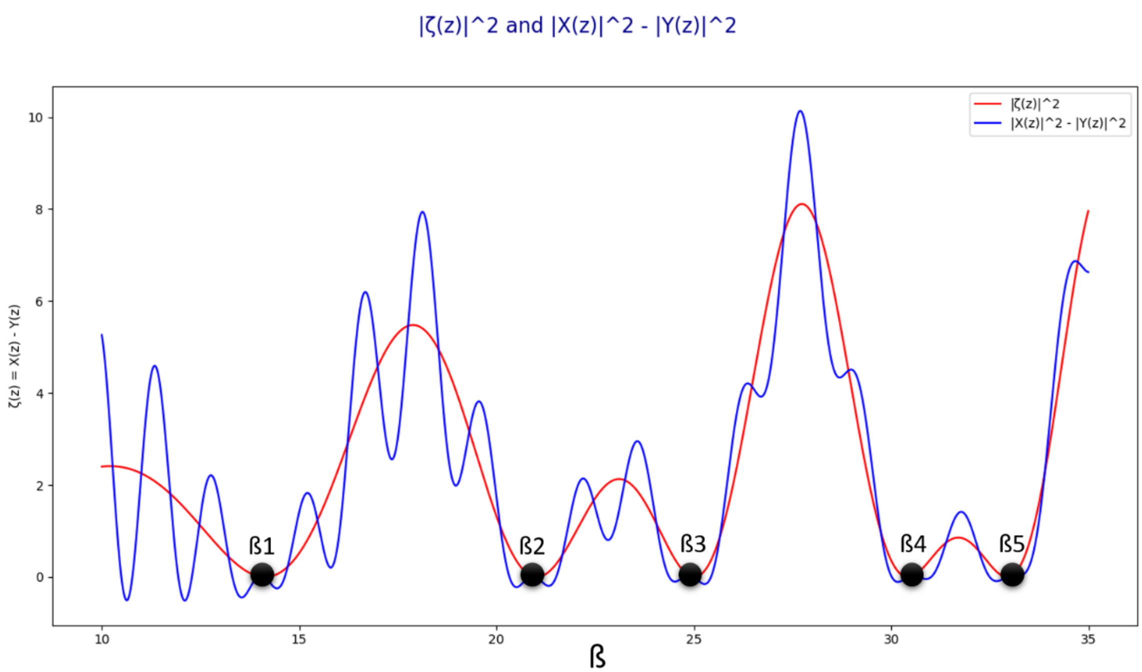

Fig. 10: if $|\zeta(z)|^2 = 0$ then $|X(z)|^2 = |Y(z)|^2$ when $Re(z)=1/2$

5. Conclusion

Using the defined C-transformation, one can write the Riemann Zeta function as the difference of two functions X(z) and Y(z) which will provide a new way of analyzing the zeros of the Zeta function.

$\zeta(z) = X(z) - Y(z)$, where:

$$X(z, n) = \left(\sum_{k=1}^{n} k^{-\alpha} \left(\cos\left(ß * \ln(k)\right) + \frac{1}{2}\, n^{-\alpha} \cos\left(ß \ln(n)\right) + \right.\right.$$
$$\left. + i * \left(\sum_{k=1}^{n} k^{-\alpha} \left(\sin\left(ß * \ln(k)\right) + \frac{1}{2}\, n^{-\alpha} \sin\left(ß \ln(n)\right)\right)\right)$$

and: $X(z) = \lim_{n\to\infty} X(z, n)$

$$Y(z, n) = n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2 + ß^2]} \left[\left((1-\alpha) * \cos(ß \ln(n)) + ß * \sin(ß \ln(n))\right) + \right.$$
$$\left. + i\ \left(ß * \cos(ß \ln(n)) - (1-\alpha) * \sin(ß \ln(n))\right)\right]$$

and: $Y(z) = \lim_{n\to\infty} Y(z, n)$

The following chart (Fig. 1) shows the equivalence of $\zeta(z) = X(z) - Y(z)$:

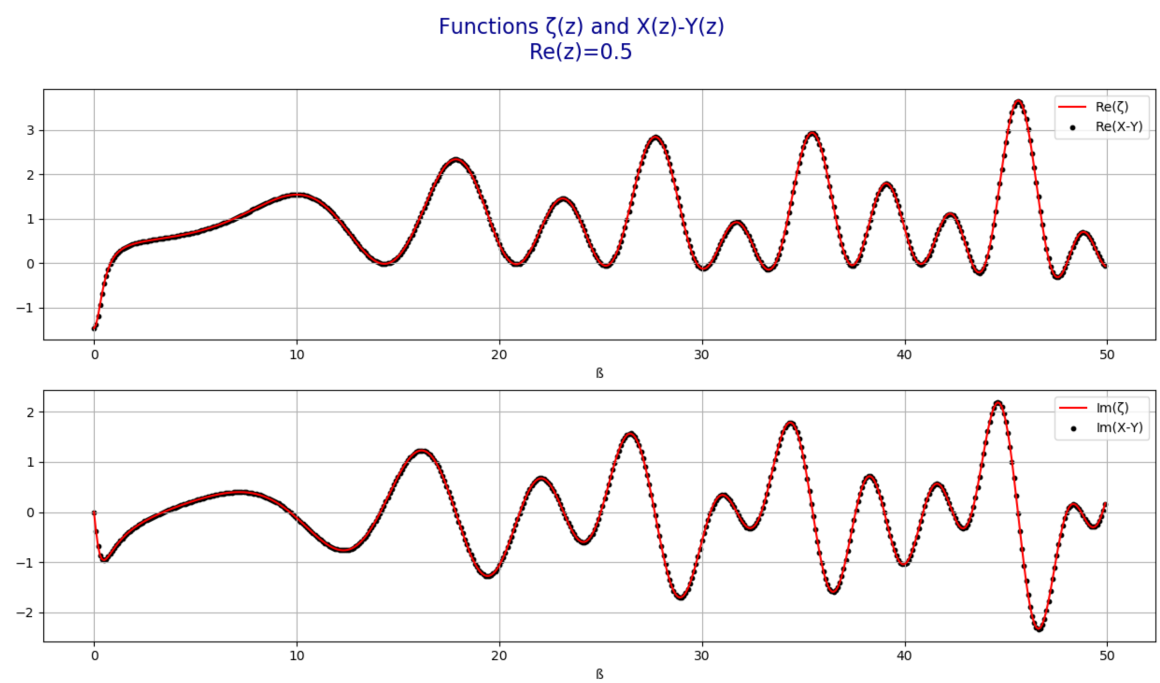

Fig. 11: ζ(z) = X(z) - Y(z)

The following chart (Fig. 2) shows that |ζ(z)| = |X(z) - Y(z)|:

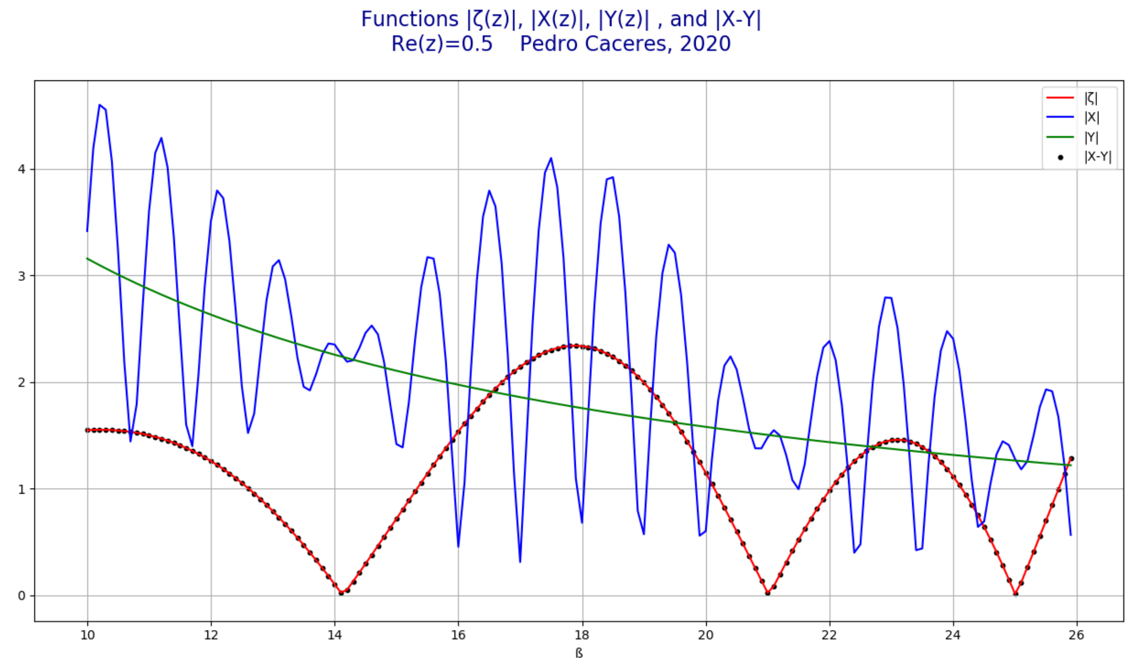

Fig. 12: $|\zeta(z)|^2 = |X(z) - Y(z)|^2$

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