

On Existence of Prime Numbers Generator

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Calculation of probability of interrelation between all prime numbers is presented.

KEYWORDS: Prime Numbers, Prime Numbers Generator, Riemann hypothesis.

Introduction

Prime Numbers Generator (PNG) is usually understood in terms of the rule which would allow one to put each of the natural numbers $1, 2, 3, \dots, n, \dots$ into one-to-one correspondence with prime numbers $p_1, p_2, p_3, \dots, p_n, \dots$. Even though many unsuccessful tries have been done to derive or suggest such a rule, no work, to the best of the author's knowledge, has proved the very existence or non-existence of the generator yet. Present treatise is attempting to fill the gap with this respect.

The route of the testing will follow the scheme:

1. It will be shown that PNG exists if and only if there is an interrelation between all prime numbers.
2. Conditional probability of taking all prime numbers except one, which is given, will be stated for a limited set of odd numbers.
3. Conditional probability of taking all prime numbers except one, which is given, will be stated and calculated for a complete set of odd numbers.

1. Lemma: PNG exists if and only if there is an interrelation between all prime numbers.

N.B.: Interrelation between all prime numbers should be perceived as an existence of the rule, which would allow one to generate all prime numbers from one arbitrary chosen prime number.

Proof: 1) If PNG exists, then there should exist a way to put one arbitrary chosen prime number p_n into correspondence with another arbitrary chosen prime number p_m by plugging known counting indices n and m into PNG. Thus, the interrelation between all prime numbers exists.

2) If interrelation between two arbitrary chosen prime numbers p_n and p_m exists, then there should exist a rule of putting natural number m into correspondence with prime number p_m . Thus, the PNG exists.

Q.E.D.

N.B.: Indices n and m are assumed to be known for (big) primes with the assist of Prime Number Theorem (PNT) [1].

Thus, the existence of PNG can be tested by testing the existence of the interrelation between all prime numbers.

2. The existence of interrelation between two events can be tested by calculating a conditional probability, defined elsewhere:

“Conditional probability is defined to be the probability of an event given that another event has occurred. If we name these events A and B , then we can talk about the probability of A given B . We can also refer to the probability of A dependent upon B .” [2].

“There is a formula for conditional probability that connects this to the probability of A and B :

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad ” [2] \quad (1)$$

where $P(A|B)$ - the conditional probability of event A given B [3];

$P(A \cap B)$ - probability of a common occurrence of events A and B
or probability of intersection of A and B [2];

$P(B)$ - probability of an event B occurring independently of any
other event (the marginal probability) [3].

Let us consider a limited set of odd numbers from 1 to x with r prime numbers within. Keeping in mind eq.1, the conditional probability of pulling out all of the prime numbers from the set except one, which is given, will be:

$$P(p_2, p_3, p_4, \dots, p_r | p_1) = \frac{P(\cap p_1, p_2, p_3, \dots, p_r)}{P(p_1)} \quad (2)$$

where $P(p_2, p_3, p_4, \dots, p_r | p_1)$ - conditional probability of taking all

primes out of the set with prime p_1 given; *i.e.* probability the interrelation between arbitrary chosen $p_2, p_3, p_4, \dots, p_r$, and p_1 exists;

$P(\cap p_1, p_2, p_3, \dots, p_r)$ - probability of taking the primes altogether;

$P(p_1)$ - probability of taking prime p_1 independently of other primes.

The probabilities can be calculated using Prime Number Theorem (PNT) [1].

According to PNT there are approximately $\frac{x}{\ln x}$ primes in the range $[1; x]$ of natural numbers [1]. As soon as we consider the Realm of odd numbers only, the number of primes r should be expressed as:

$$r \approx \frac{2x}{\ln(2x)} \quad (3)$$

N.B.: Eq.3 acts for odd numbers within the range $[1; x]$, where natural numbers from 1 to x may be considered as counting indices of the odd numbers.

Thus, the probability $P(p_1)$ of taking one arbitrary chosen prime number p_1 independently of others out of x odd numbers will be:

$$P(p_1) = \frac{r}{x} \approx \frac{\frac{2x}{\ln(2x)}}{x} \quad (4)$$

$$P(p_1) \approx \frac{2}{\ln(2x)} \quad (5)$$

By the same train of reasoning:

$$P(\cap p_1, p_2, p_3, \dots, p_r) = \frac{r}{x} \cdot \frac{(r-1)}{(x-1)} \cdot \frac{(r-2)}{(x-2)} \cdot \dots \cdot \frac{r-(r-1)}{x-(r-1)} \quad (6)$$

After the usage of:

$$a \cdot (a-1) \cdot (a-2) \cdot \dots \cdot (a-b) = \frac{a!}{[a-(b+1)]!} \quad (7)$$

eq.6 transforms into:

$$P(\cap p_1, p_2, p_3, \dots p_r) = \frac{r!(x-r)!}{x!} \quad (8)$$

Let us set eq.3 into eq.8:

$$P(\cap p_1, p_2, p_3, \dots p_r) \approx \frac{\left(\frac{2x}{\ln(2x)}\right)! \cdot \left(x - \frac{2x}{\ln(2x)}\right)!}{x!} \quad (9)$$

By setting eqs.5,9 into eq.2:

$$P(p_2, p_3, p_4, \dots p_r | p_1) \approx \frac{\ln(2x)}{2} \cdot \frac{\left(\frac{2x}{\ln(2x)}\right)! \cdot \left(x - \frac{2x}{\ln(2x)}\right)!}{x!} \quad (10)$$

It should be stressed here that eq.10 is stated for a *limited* set of x odd numbers.

3. To extend eq.10 for all odd numbers one should take a limit of eq.10 at $x \rightarrow \infty$. In this case, for big numbers, approximate character brought by PNT can be left:

$$\lim_{x \rightarrow \infty} P(p_2, p_3, p_4, \dots p_r | p_1) = \lim_{x \rightarrow \infty} \left[\frac{\frac{\ln(2x)}{2} \cdot \overbrace{\left(\frac{2x}{\ln(2x)}\right)!}^{(B)} \cdot \overbrace{\left(x - \frac{2x}{\ln(2x)}\right)!}^{(C)}}{\underbrace{x!}_{(A)}} \right] \quad (11)$$

To make eq.11 computable, one should get rid of factorial terms (A), (B), and (C), using Stirling's Approximation (SA):

$$n! \approx \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \quad (12)$$

where $\pi = 3.14\dots$ - “pi-number”;

$e = 2.71\dots$ - “Euler’s number” [4].

For the same token, approximate character brought by SA can also be left in the case of big numbers. One may find (not shown) that upon the application of SA, eq.11 converts into:

$$\begin{aligned} \lim_{x \rightarrow \infty} P(p_2, p_3, p_4, \dots, p_r | p_1) &= \\ &= \lim_{x \rightarrow \infty} \left\{ \underbrace{2\pi x}_{(A)} \cdot \underbrace{\left(\frac{2}{\ln(2x) - 2}\right)^{\left[\frac{2x}{\ln(2x)} - \frac{1}{2}\right]}}_{(C)} \cdot \underbrace{\left(1 - \frac{2}{\ln(2x)}\right)^x}_{(D)} \right\} \quad (13) \end{aligned}$$

Eq.13 can easily be generalized for the case of s primes given ($s < r$):

$$\begin{aligned} \lim_{x \rightarrow \infty} P(p_{s+1}, p_{s+2}, p_{s+3}, \dots, p_r | p_1, p_2, p_3, \dots, p_s) &= \\ &= \lim_{x \rightarrow \infty} \left\{ \underbrace{2\pi x}_{(A)} \cdot \underbrace{\left(\frac{2}{\ln(2x) - 2}\right)^{\left[\frac{2x}{\ln(2x)} - \frac{1}{2}\right]}}_{(C)} \cdot \underbrace{\left(1 - \frac{2}{\ln(2x)}\right)^x}_{(D)} \cdot \underbrace{\left(\frac{\ln(2x)}{2}\right)^{(s-1)}}_{(F)} \right\} \quad (14) \end{aligned}$$

Concluding Remarks

- (i) Based on eqs.13,14, in BASIC written code (see below) was used to calculate a conditional probability the PNG exists for sets of odd numbers of limited sizes. The results of the calculations are presented in the table (see below). As one can see, in all of the situations considered, the numeric value of conditional probability is far below a threshold value of 0.5.

Moreover, conditional probability is getting smaller and smaller with size of odd numbers' set expanding, apparently approaching zero in the infinity-limit case. The rise of number of primes given to some extent improves the situation by increasing the value of the conditional probability, though for all of the considered cases there still are *not enough* primes given to reach the crucial value of 0.5. With all of the above considerations one must sadly admit it is highly unlikely the PNG exists.

- (ii) It should be stressed here the non-existence verdict applies to *ultimate* PNG only, i.e. to such PNG, which would generate all and nothing but the prime numbers.
- (iii) Any positively constructed statement about exact distribution of prime numbers is wrong in general. Indeed, any positively constructed statement about exact distribution of prime numbers may be considered as equivalent to ultimate PNG, non-existence of which is proved above. For this reason, Riemann hypothesis [5] must be wrong in general.

References

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2. <https://www.thoughtco.com/conditional-probability-3126575>
3. <https://www.thoughtco.com/bayes-theorem-4155845>
4. <https://www.britannica.com/science/Stirlings-formula>
5. <https://www.claymath.org/millennium-problems/riemann-hypothesis>

Set of N odds	Probability the PNG exists for a set of N odd numbers																							
	Number of primes given																							
	1	37	38	66	67	93	94	119	120	144	145	169	170	193	194	216	217	240	241	262	263			
100	5.3· -10 ⁸ (-28)	9.0· -10 ⁸ (-13)	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA		
200	2.4· -10 ⁸ (-54)	3.5· -10 ⁸ (-37)	1.0· -10 ⁸ (-36)	3.3· -10 ⁸ (-23)	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA		
300	7.5· -10 ⁸ (-80)	1.1· -10 ⁸ (-61)	3.6· -10 ⁸ (-61)	4.9· -10 ⁸ (-47)	1.6· -10 ⁸ (-46)	2.1· -10 ⁸ (-33)	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA		
400	7.7· -10 ⁸ (-105)	5.6· -10 ⁸ (-86)	1.9· -10 ⁸ (-85)	8.9· -10 ⁸ (-71)	3.0· -10 ⁸ (-70)	1.3· -10 ⁸ (-56)	4.2· -10 ⁸ (-56)	5.3· -10 ⁸ (-43)	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	
500	1.9· -10 ⁸ (-129)	4.6· -10 ⁸ (-110)	1.1· -10 ⁸ (-111)	1.9· -10 ⁸ (-94)	6.4· -10 ⁸ (-94)	6.4· -10 ⁸ (-80)	2.2· -10 ⁸ (-79)	6.3· -10 ⁸ (-66)	2.2· -10 ⁸ (-65)	1.8· -10 ⁸ (-52)	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	
600	9.4· -10 ⁸ (-154)	5.7· -10 ⁸ (-134)	2.0· -10 ⁸ (-133)	5.0· -10 ⁸ (-117)	1.8· -10 ⁸ (-117)	3.4· -10 ⁸ (-103)	1.2· -10 ⁸ (-102)	6.7· -10 ⁸ (-89)	2.4· -10 ⁸ (-88)	3.7· -10 ⁸ (-75)	1.3· -10 ⁸ (-74)	2.0· -10 ⁸ (-61)	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA	
700	8.1· -10 ⁸ (-178)	1.1· -10 ⁸ (-157)	3.9· -10 ⁸ (-157)	1.7· -10 ⁸ (-141)	6.3· -10 ⁸ (-141)	2.1· -10 ⁸ (-126)	7.8· -10 ⁸ (-126)	7.3· -10 ⁸ (-112)	2.6· -10 ⁸ (-111)	6.9· -10 ⁸ (-98)	2.5· -10 ⁸ (-97)	6.5· -10 ⁸ (-84)	2.3· -10 ⁸ (-83)	1.7· -10 ⁸ (-70)	NA	NA	NA	NA	NA	NA	NA	NA	NA	NA
800	1.1· -10 ⁸ (-201)	2.8· -10 ⁸ (-181)	1.1· -10 ⁸ (-180)	7.8· -10 ⁸ (-165)	2.9· -10 ⁸ (-164)	1.6· -10 ⁸ (-149)	5.9· -10 ⁸ (-149)	8.7· -10 ⁸ (-135)	3.2· -10 ⁸ (-134)	1.3· -10 ⁸ (-120)	4.8· -10 ⁸ (-120)	1.9· -10 ⁸ (-106)	7.1· -10 ⁸ (-106)	7.8· -10 ⁸ (-93)	2.9· -10 ⁸ (-92)	8.5· -10 ⁸ (-80)	NA	NA	NA	NA	NA	NA	NA	NA
900	2.3· -10 ⁸ (-225)	1.0· -10 ⁸ (-204)	3.9· -10 ⁸ (-204)	4.6· -10 ⁸ (-188)	1.7· -10 ⁸ (-187)	1.4· -10 ⁸ (-172)	5.3· -10 ⁸ (-172)	1.2· -10 ⁸ (-157)	4.4· -10 ⁸ (-157)	2.6· -10 ⁸ (-143)	9.7· -10 ⁸ (-143)	5.7· -10 ⁸ (-129)	2.1· -10 ⁸ (-128)	3.4· -10 ⁸ (-115)	1.3· -10 ⁸ (-114)	5.3· -10 ⁸ (-102)	2.0· -10 ⁸ (-101)	3.1· -10 ⁸ (-88)	NA	NA	NA	NA	NA	
1000	6.9· -10 ⁸ (-249)	5.2· -10 ⁸ (-228)	2.0· -10 ⁸ (-227)	3.4· -10 ⁸ (-211)	1.3· -10 ⁸ (-210)	1.5· -10 ⁸ (-195)	5.8· -10 ⁸ (-195)	1.8· -10 ⁸ (-180)	6.9· -10 ⁸ (-180)	5.7· -10 ⁸ (-166)	2.2· -10 ⁸ (-165)	1.8· -10 ⁸ (-151)	6.8· -10 ⁸ (-151)	1.5· -10 ⁸ (-137)	5.6· -10 ⁸ (-137)	3.2· -10 ⁸ (-124)	1.2· -10 ⁸ (-123)	2.6· -10 ⁸ (-110)	1.0· -10 ⁸ (-109)	1.5· -10 ⁸ (-97)	NA	NA	NA	

NA = Not Applicable because number of primes given exceeds the total number of primes in the set

'The program calculates probability the Prime Number Generator exists for the sets of odd numbers up to 1000.

```
[start]
  input "How many primes are given?: "; s
  let i=0
  let A=0
  let B=0
  let C=0
  let D=0
  let F=0
  let startodds=4
  CLS
  print "NUMBER OF PRIMES GIVEN: "; s
  'setting the odd numbers' cutoff to avoid LOG(negative)
  if s<=0 goto [foolproof]
  if s=1 goto [startcalc]
  if s=2 goto [startcalc]
  if s=3 goto [startcalc]
  'check for consistency between one thousand of odds and the number of primes given
  if INT (2*1000/LOG(2*1000))>s then goto [startoddscale]
  print "Number of primes given exceeds or equal to total number of primes
    among 1000 of odd numbers."
  goto [start]
[foolproof]
  print "Number of primes given cannot be less than 1"
  goto [start]
[startoddscale]
  'calculating the number of odds to start from
  let allprimes = INT(2*startodds/LOG(2*startodds))
  if allprimes = s then goto [startcalc]
  let startodds = startodds+1
  goto [startoddscale]
[startcalc]
  for i = startodds to 1000
  'calculating term A
  let pi = 4*atn(1)
  let A = SQR(2*pi*i)
  'calculating term B
  let B = 2*i/LOG(2*i)-0.5
  'calculating term C
  let C = 2/(LOG(2*i)-2)
  'calculating term D
  let D = 1-2/LOG(2*i)
  'calculating term F
  let F = (LOG(2*i)/2)
  'calculating conditional probability
  Let condprob= A*(C^B)*(D^i)*(F^(s-1))
  'output of the results
  print
  print "set of odd numbers the probability is calculated for: ";i
  print "conditional probability the PNG exists for the set: "; condprob
  print
  next
  goto [start]
```