An Elementary Proof of Goldbach's Conjecture

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Abstract

Goldbach's conjecture is proven using the Chinese Remainder Theorem. It is shown that an even number $2N$ greater than four cannot exist if it is congruent to every prime p less than N (mod a different prime number).

1 Introduction

Goldbach's Conjecture states that every even number greater than two is the sum of two primes. In 2013, Helfgott showed that Goldbach's weak conjecture was true. [1] The strong conjecture has been empirically verified to $4\text{x}10^{18}$ [2] but remains unproven. The following is a proof of the strong conjecture.

2 Proof

Every even number greater than four can be written as the sum of two odd numbers. Let $\mathbb{P}$ be the set of odd prime numbers. Then for $p \in \mathbb{P}$ and $q \in \mathbb{P}$, assume the following theorem is true

Theorem 1 *There exists an even number $2N \in \mathbb{Z}$, $N > 2$, such that for all p_i where $i \leq \pi(N)$*

$$2N = p_i + a_i q_i$$

where $a_i \in \mathbb{Z}$, $a_i > 1$, and $\pi(N)$ is the prime-counting function.

Theorem 1 requires a solution to the following system of congruences

$$\begin{aligned} 2N &\equiv p_1 \pmod{q_1} \\ 2N &\equiv p_2 \pmod{q_2} \\ &\vdots \\ 2N &\equiv p_{\pi(N)} \pmod{q_{\pi(N)}} \end{aligned}$$

For any number m, the modular multiplicative inverse of $2 \pmod{m}$ is $(m-1)/2+1$ since

$$2\left(\frac{m-1}{2}+1\right) \equiv 1 \pmod{m}$$

Then the system of congruences becomes

$$\begin{aligned} N &\equiv p_1\left(\frac{q_1-1}{2}+1\right) \pmod{q_1} \\ N &\equiv p_2\left(\frac{q_2-1}{2}+1\right) \pmod{q_2} \\ &\vdots \\ N &\equiv p_{\pi(N)}\left(\frac{q_{\pi(N)}-1}{2}+1\right) \pmod{q_{\pi(N)}} \end{aligned}$$

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From the Chinese Remainder Theorem, one solution to this system of congruences is

$$N = \sum_{i=1}^{\pi(N)} p_i \left(\frac{q_i - 1}{2} + 1 \right) \frac{Q}{q_i} z_i$$

where $Q = \prod_{i=1}^{\pi(N)} q_i$ and z_i is the modular multiplicative inverse of Q/q_i. Q is independent of the sum so it can be factored out

$$N = Q \sum_{i=1}^{\pi(N)} \frac{p_i \left(\frac{q_i-1}{2} + 1 \right) z_i}{q_i} \tag{1}$$

$Q \geq 3^{\pi(N)} \approx 3^{\frac{N}{ln(N)}}$ so $N < Q$. Then the summation in the right side of (1) must be less than one for a valid solution to exist.

$$\sum_{i=1}^{\pi(N)} \frac{p_i \left(\frac{q_i-1}{2} + 1 \right) z_i}{q_i} < 1$$

$$\sum_{i=1}^{\pi(N)} \left(\frac{p_i z_i}{2} + \frac{p_i z_i}{2q_i} \right) < 1$$

$$\sum_{i=1}^{\pi(N)} \frac{p_i z_i}{2} + \sum_{i=1}^{\pi(N)} \frac{p_i z_i}{2q_i} < 1$$

For the left side to be less than 1, $\sum_{i=1}^{\pi(N)} p_i z_i$ must be less than 2. But $p_i \geq 1$ and $z_i \geq 1$, so $\sum_{i=1}^{\pi(N)} p_i z_i \geq 2$ since $\pi(N)$ is at least 2.

Therefore, no solution exists and Theorem 1 is false. Together with $4 = 2 + 2$, every even number greater than two can be written as the sum of two primes.

References

[1] Helfgott, Harald A. The ternary Goldbach conjecture is true. Available at arXiv:1312.7748

[2] Toms Oliveira e Silva, Siegfried Herzog, and Silvio Pardi, Empirical verification of the even Goldbach conjecture and computation of prime gaps up to 4x10^{18}, Mathematics of Computation, vol. 83, no. 288, pp. 2033-2060, July 2014 (published electronically on November 18, 2013).