$\xi(s)$ i.e the Riemann's Xi function satisfies the functional equation ②

$$\xi(1-s) = \xi(s)$$

Let $\xi(s) = 0$

Claim :- $Re(s) = \frac{1}{2}$.

$\because \xi(s) = 0$

$$\xi(1-s) = \xi(s) = 0$$

$\therefore \xi(1-s) = 0$ —— ①

H.M. Edwards [1, p. 39]

$$\xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \quad \text{—} ②$$

where ξ is entire function, ρ ranges over all roots ρ of $\xi(\rho) = 0$ (*)

$\xi(1-s) = 0$ (from ①)

$$\Rightarrow \xi(0) \prod_{\rho} \left[1 - \frac{(1-s)}{\rho}\right] = 0$$

$$\Rightarrow \xi(0) \prod_{\rho} \left(\frac{\rho + s - 1}{\rho}\right) = 0$$

$\xi(0) = \frac{1}{2}$ ([2, p. 37, Theorem 2.11])

$$\Rightarrow \prod_{\rho} \left(\frac{\rho + s - 1}{\rho}\right) = 0$$

$$\prod_{\rho} \frac{\rho + s - 1}{\rho} = 0 \quad \text{—} \ (3)$$

let, $s = \sigma + it$

& $\rho = a + ib$

$$(3) \Rightarrow \prod_{\rho = a+ib} \frac{a+ib+\sigma+it-1}{a+ib} = 0$$

$$\Rightarrow \prod_{\rho = a+ib} \frac{(a+\sigma-1)+i(b+t)}{a+ib} = 0$$

$$\Rightarrow \left| \prod_{\rho = a+ib} \frac{(a+\sigma-1)+i(b+t)}{a+ib} \right|^2 = 0$$

$$\Rightarrow \prod_{\rho = a+ib} \left| \frac{(a+\sigma-1)+i(b+t)}{a+ib} \right|^2 = 0$$

$$\Rightarrow \prod_{\rho = a+ib} \frac{(a+\sigma-1)^2+(b+t)^2}{a^2+b^2} = 0$$

$$\Rightarrow \prod_{\rho = a+ib} \frac{(a-\sigma+2\sigma-1)^2+(b+t)^2}{a^2+b^2} = 0$$

$$\Rightarrow \prod_{\rho = a+ib} \frac{(a-\sigma)^2+(2\sigma-1)^2+2(a-\sigma)(2\sigma-1)+(b+t)^2}{a^2+b^2} = 0$$

$$\Rightarrow \prod_{\rho = a+ib} \frac{(a-\sigma)^2+(2\sigma-1)(2\sigma-1+2a-2\sigma)+(b+t)^2}{a^2+b^2} = 0$$

$$\prod \frac{(a-\sigma)^2+(b+t)^2+(2\sigma-1)(2a-1)}{a^2+b^2} = 0 \quad \text{—} \; (4)$$

$\rho = a+ib$

$\because$ The Critical strip (where R.Z.F. has non-trivial zeros) is ~~defined~~ $0 < Re(s) < 1$

$0 < Re(s) < 1$; $s = \sigma + it$

$0 < \sigma < 1$

2 Cases $\rightarrow$ $0 < \sigma \leq \frac{1}{2}$ & $\frac{1}{2} \leq \sigma < 1$

Case 1 :- $0 < \sigma \leq 1/2$

If $\rho = a+ib$

Claim :- $0 < a \leq \frac{1}{2}$

We Prove this by contradiction.

Let $a \notin (0, \frac{1}{2}]$

$\because$ $0 < a < 1$ ($\because \rho = a+ib$ is a non trivial zero of $\xi(s)$)

$\hookrightarrow$ [3] $\rightarrow$ Jonathan Sondow

$\therefore$ $\frac{1}{2} < a < 1$

$$\xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right)$$

$$\xi(s) = \frac{1}{2} \prod_{\rho=a+ib} 1 - \left(\frac{\sigma+it}{a+ib}\right)$$

$$\xi(s) = \frac{1}{2} \prod_{\rho=a+ib} \frac{(a-\sigma)+i(b-t)}{a+ib}$$

$$\xi(s) = \frac{1}{2} \prod_{\rho=a+ib} \frac{(a-\sigma)+i(b-t)}{a+ib}$$

$\because \; 0 < \sigma \leq \frac{1}{2}$

$-\frac{1}{2} \leq -\sigma < 0$ —— (5)

$\because \; \frac{1}{2} < a < 1$ —— (6)

Adding (5) & (6)

$0 < a - \sigma < 1 \Rightarrow a - \sigma \neq 0$

$$\xi(s) = \frac{1}{2} \prod \frac{(a-\sigma)+i(b-t)}{a+ib}$$

$(a-\sigma) \neq 0$

$$\Rightarrow \frac{(a-\sigma)+i(b-t)}{a+ib} \neq 0$$

$$\Rightarrow \prod_{\rho=a+ib} \frac{(a-\sigma)+i(b-t)}{a+ib} \neq 0$$

$$\Rightarrow \frac{1}{2} \prod_{a+ib} \frac{(a-\sigma)+i(b-t)}{a+ib} \neq 0$$

$\Rightarrow \; \xi(s) \neq 0$

Which is a contradiction since we assumed that $\xi(s) = 0$

$\therefore$ Our assumption that $a \notin [0, \frac{1}{2}]$ is wrong

$\therefore \; a \in [0, \frac{1}{2}]$

$a \in \left[0, \frac{1}{2}\right]$

$0 < a \le \frac{1}{2}$

(4) $\Rightarrow \prod_{\rho = a+ib} \frac{(a-\sigma)^2 + (b+t)^2 + (2\sigma - 1)(2a-1)}{a^2+b^2} = 0$

$\Rightarrow \prod_{\rho = a+ib} \frac{(a-\sigma)^2 + (b+t)^2 + (1-2\sigma)(1-2a)}{a^2+b^2} = 0$ — (7)

$\because \; 0 < \sigma \le 1/2$ (by case 1)

$\Rightarrow 1 - 2\sigma \geq 0$ — (8)

& $\because \; 0 < a \le 1/2$

$\Rightarrow 1 - 2a \geq 0$ — (9)

$\therefore$ ~~By, (8) & (8)~~ From (8) & (9)

$\Rightarrow (1-2\sigma)(1-2a) \geq 0$ — (10)

(7) $\Rightarrow \prod_{a+ib} \frac{(a-\sigma)^2 + (b+t)^2 + (1-2\sigma)(1-2a)}{a^2+b^2} = 0$

$\Rightarrow \frac{(a_0-\sigma)^2 + (b_0+t)^2 + (1-2\sigma)(1-2a_0)}{a_0^2+b_0^2} = 0$ — (11)

for some $a_0 + ib_0 \in \mathbb{C}$

$\because$ By (10) $(1-2\sigma)(1-2a) \geq 0$; $0 < a \le \frac{1}{2}$

$\Rightarrow (1-2\sigma)(1-2a_0) \geq 0$

$\therefore$ (11) $\Rightarrow (a_0-\sigma)^2 = 0$, $(b_0+t)^2 = 0$ and

$(1-2\sigma)(1-2a_0) = 0$

$(a_0 - \sigma)^2 = 0 \quad \Rightarrow a_0 = \sigma$

Putting $a_0 = \sigma$ in $(1-2\sigma)(1-2a_0) = 0$

$\Rightarrow (1-2\sigma)(1-2\sigma) = 0$

$\Rightarrow (1-2\sigma)^2 = 0$

$\Rightarrow 1-2\sigma = 0$

$\Rightarrow \boxed{\sigma = \frac{1}{2}}$

$Re(s) = \frac{1}{2}$ Pd.

Case 2 $\quad \frac{1}{2} \leq \sigma < 1$

Claim :- $\frac{1}{2} \leq a < 1$, $\rho = a+ib$ [3] Jonathan Sondow

$0 < a < 1 \quad \left[\because \rho = a+ib \text{ is a non trivial zero of } \xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \right]$ [3]

We prove the claim by contradiction

let us assume that $a \notin [\frac{1}{2}, 1)$.

$\because 0 < a < 1$

$\therefore 0 < a < \frac{1}{2}$

$$\xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right)$$

$s = \sigma + it$

$\rho = a + ib$

$$\xi(\sigma + it) = \xi(0) \prod_{\rho = a+ib} \frac{(a-\sigma) + i(b-t)}{a+ib}$$

$\frac{1}{2} \leq \sigma < 1$

$-1 < -\sigma \leq -\frac{1}{2}$

$0 < a < \frac{1}{2}$ (assumption

$-1 < a - \sigma < 0$

$a - \sigma \neq 0$

$\xi(\sigma + it) = \xi(0) \prod \frac{(a-\sigma) + i(b-t)}{a+ib}$

$(a-\sigma) + i(b-t) \neq 0$

$\frac{(a-\sigma) + i(b-t)}{a+ib} \neq 0$

$\prod_{\rho = a+ib} \frac{(a-\sigma) + i(b-t)}{a+ib} \neq 0$

$\xi(0) \prod_{\rho = a+ib} \frac{(a-\sigma) + i(b-t)}{a+ib} \neq 0$

$\Rightarrow \xi(\sigma + it) \neq 0$

which is a Contradiction $\because$ we had $\xi(\sigma + it) = 0$

$\therefore$ Our Assumption that $a \notin [\frac{1}{2}, 1)$ is wrong

$\therefore \quad a \in [\frac{1}{2}, 1)$

$\frac{1}{2} \leq a < 1$

& $\frac{1}{2} \leq \sigma < 1$ (By case 2)

$\frac{1}{2} \le \sigma < 1 \Rightarrow 2\sigma - 1 \geq 0$

$\frac{1}{2} \le a < 1 \Rightarrow 2a - 1 \geq 0$

By (4),

$$\prod_{\rho = a+ib} \frac{(a-\sigma)^2 + (b+t)^2 + (2\sigma-1)(2a-1)}{a^2+b^2} = 0 \quad \text{--- (12)}$$

$\because \quad 2\sigma - 1 \geq 0 \ \& \ 2a - 1 \geq 0$

$\therefore \quad (2\sigma-1)(2a-1) \geq 0 \quad \text{--- (13)}$

$$(4) \Rightarrow \prod \frac{(a-\sigma)^2 + (b+t)^2 + (2\sigma-1)(2a-1)}{a^2+b^2} = 0$$

$$\Rightarrow \frac{(a_1-\sigma)^2 + (b_1+t)^2 + (2\sigma-1)(2a_1-1)}{a_1^2+b_1^2} = 0$$

for some $a_1 + ib_1 \in \mathbb{C}$

$$(a_1-\sigma)^2 + (b_1+t)^2 + (2\sigma-1)(2a_1-1) = 0$$

$\because$ By (13) $(2\sigma-1)(2a-1) \geq 0 \ \forall a \in [\frac{1}{2}, 1)$

$\Rightarrow (2\sigma-1)(2a_1-1) \geq 0 \ \forall a_1 \in [\frac{1}{2}, 1)$

$$(a_1-\sigma)^2 + (b_1+t)^2 + (2\sigma-1)(2a_1-1) = 0$$

$(a_1-\sigma)^2 = 0 \ \& \ (b_1+t)^2 = 0 \ \& \ (2\sigma-1)(2a_1-1) = 0$

$\Rightarrow (a_1-\sigma)^2 = 0 \quad \& \ b_1 = -t \quad \& \ (2\sigma-1)(2a_1-1) = 0$

$\Rightarrow a_1 = \sigma \qquad (2\sigma-1)(2\sigma-1) = 0$

$a_1 = \sigma \Rightarrow$

$\Rightarrow (2\sigma-1)^2 = 0$

$\Rightarrow \boxed{\sigma = \frac{1}{2}}$

References

1.) E.C. Titchmarsh - The Theory of the R.Z. Function
2.) Equivalents of R.H. - Arithmetic Equivalents
Kevin Broughan
3.) A Monotonicity Property of Riemann's Xi Function
& a Reformulation of the R.H.: Jonathan
Sondow and ~~Chris~~ Cristian Dumitrescu.
Periodica Hungarica - May 2010