$\xi(s)$ i.e the Riemann's Xi function satisfies ②
the functional equation

$$\xi(1-s) = \xi(s)$$

Let $\xi(s) = 0$

Claim :- $Re(s) = \frac{1}{2}$.

$\because \xi(s) = 0$

$\xi(1-s) = \xi(s) = 0$

$\therefore \xi(1-s) = 0$ —— ①

H.M. Edwards [4, p. 39]

$$\xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \quad —— ②$$

where ξ is entire function, ρ ranges over all roots ρ of $\xi(\rho) = 0$ (*)

$\xi(1-s) = 0$ (from ①)

$$\Rightarrow \xi(0) \prod_{\rho} \left[1 - \frac{(1-s)}{\rho}\right] = 0$$

$$\Rightarrow \xi(0) \prod_{\rho} \left(\frac{\rho + s - 1}{\rho}\right) = 0$$

$\xi(0) = \frac{1}{2}$ ([2, p. 37, Theorem 2.11)

$$\Rightarrow \prod_{\rho} \left(\frac{\rho + s - 1}{\rho}\right) = 0$$

$$\prod_{\rho} \frac{\rho+s-1}{\rho} = 0 \quad \text{—} \ (3)$$

let, $s = \sigma + it$

& $\rho = a + ib$

$$(3) \Rightarrow \prod_{\rho=a+ib} \frac{a+ib+\sigma+it-1}{a+ib} = 0$$

$$\Rightarrow \prod_{\rho=a+ib} \frac{(a+\sigma-1)+i(b+t)}{a+ib} = 0$$

$$\Rightarrow \left| \prod_{\rho=a+ib} \frac{(a+\sigma-1)+i(b+t)}{a+ib} \right|^2 = 0$$

$$\Rightarrow \prod_{\rho=a+ib} \left| \frac{(a+\sigma-1)+i(b+t)}{a+ib} \right|^2 = 0$$

$$\Rightarrow \prod_{\rho=a+ib} \frac{(a+\sigma-1)^2+(b+t)^2}{a^2+b^2} = 0$$

$$\Rightarrow \prod_{\rho=a+ib} \frac{(a-\sigma+2\sigma-1)^2+(b+t)^2}{a^2+b^2} = 0$$

$$\Rightarrow \prod_{\rho=a+ib} \frac{(a-\sigma)^2+(2\sigma-1)^2+2(a-\sigma)(2\sigma-1)+(b+t)^2}{a^2+b^2} = 0$$

$$\Rightarrow \prod_{\rho=a+ib} \frac{(a-\sigma)^2+(2\sigma-1)(2\sigma-1+2a-2\sigma)+(b+t)^2}{a^2+b^2} = 0$$

$$\prod_{\rho=a+ib} \frac{(a-\sigma)^2+(b+t)^2+(2\sigma-1)(2a-1)}{a^2+b^2} = 0 \quad - (4)$$

(where R.Z.F. has non-trivial zeros)

$\because$ The Critical strip is ~~defined~~ $0 \le Re(s) \le 1$

$$0 \le Re(s) \le 1 \; ; \; s = \sigma + it$$

$$0 \le \sigma \le 1$$

2 Cases $\rightarrow$ $0 \le \sigma \le \frac{1}{2}$ & $\frac{1}{2} \le \sigma < 1$

Case 1 :- $0 \le \sigma \le 1/2$

If $\rho = a+ib$

Claim :- $0 \le a \le \frac{1}{2}$

We Prove this by contradiction.

Let $a \notin [0, \frac{1}{2}]$

$\because$ $0 \le a \le 1$ ($\because \rho = a+ib$ is a non trivial zero of $\xi(s)$)

$\hookrightarrow$ [3] $\rightarrow$ Jonathan Sondow

$\therefore$ $\frac{1}{2} < a \le 1$

$$\xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right)$$

$$\xi(s) = \frac{1}{2} \prod_{\rho=a+ib} 1 - \left(\frac{\sigma+it}{a+ib}\right)$$

$$\xi(s) = \frac{1}{2} \prod_{\rho=a+ib} \frac{(a-\sigma)+i(b-t)}{a+ib}$$

$$\xi(s) = \frac{1}{2} \prod_{\rho = a+ib} \frac{(a-\sigma) + i(b-t)}{a+ib}$$

$\because \; 0 \leq \sigma \leq \frac{1}{2}$

$-\frac{1}{2} \leq -\sigma \leq 0$ —— (5)

$\because \; \frac{1}{2} < a \leq 1$ —— (6)

Adding (5) & (6)

$0 < a - \sigma \leq 1 \Rightarrow a - \sigma \neq 0$

$$\xi(s) = \frac{1}{2} \prod \frac{(a-\sigma) + i(b-t)}{a+ib}$$

$(a-\sigma) \neq 0$

$\Rightarrow \frac{(a-\sigma) + i(b-t)}{a+ib} \neq 0$

$\Rightarrow \prod_{\rho = a+ib} \frac{(a-\sigma) + i(b-t)}{a+ib} \neq 0$

$\Rightarrow \frac{1}{2} \prod_{a+ib} \frac{(a-\sigma) + i(b-t)}{a+ib} \neq 0$

$\Rightarrow \xi(s) \neq 0$

Which is a contradiction since we assumed that $\xi(s) = 0$

$\therefore$ Our assumption that $a \notin [0, \frac{1}{2}]$ is wrong

$\therefore \; a \in [0, \frac{1}{2}]$

$a \in \left[0, \frac{1}{2}\right]$

$0 \leq a \leq \frac{1}{2}$

$$④ \Rightarrow \prod_{\rho = a+ib} \frac{(a-\sigma)^2 + (b+t)^2 + (2\sigma-1)(2a-1)}{a^2+b^2} = 0$$

$$\Rightarrow \prod_{\rho = a+ib} \frac{(a-\sigma)^2 + (b+t)^2 + (1-2\sigma)(1-2a)}{a^2+b^2} = 0 \quad \text{—— ⑦}$$

$\because \; 0 \leq \sigma \leq 1/2$ (by case 1)

$\Rightarrow \; 1 - 2\sigma \geq 0$ —— ⑧

& $\because \; 0 \leq a \leq 1/2$

$\Rightarrow \; 1 - 2a \geq 0$ —— ⑨

$\therefore$ ~~By, ⑧ & ⑧~~ From ⑧ & ⑨

$\Rightarrow \; (1-2\sigma)(1-2a) \geq 0$ —— ⑩

$$⑦ \Rightarrow \prod_{a+ib} \frac{(a-\sigma)^2 + (b+t)^2 + (1-2\sigma)(1-2a)}{a^2+b^2} = 0$$

$$\Rightarrow \frac{(a_0-\sigma)^2 + (b_0+t)^2 + (1-2\sigma)(1-2a_0)}{a_0^2+b_0^2} = 0 \quad \text{—— ⑪}$$

for some $a_0 + ib_0 \in \mathbb{C}$

$\because$ By ⑩ $(1-2\sigma)(1-2a) \geq 0$; $0 < a \leq \frac{1}{2}$

$\Rightarrow (1-2\sigma)(1-2a_0) \geq 0$

$\therefore$ ⑪ $\Rightarrow (a_0-\sigma)^2 = 0$, $(b_0+t)^2 = 0$ and

$(1-2\sigma)(1-2a_0) = 0$

Generated by CamScanner

$(a_0 - \sigma)^2 = 0 \Rightarrow a_0 = \sigma$

Putting $a_0 = \sigma$ in $(1-2\sigma)(1-2a_0) = 0$

$$\Rightarrow (1-2\sigma)(1-2\sigma) = 0$$

$$\Rightarrow (1-2\sigma)^2 = 0$$

$$\Rightarrow 1-2\sigma = 0$$

$$\Rightarrow \boxed{\sigma = \frac{1}{2}}$$

$Re(s) = \frac{1}{2}$ Pd.

<u>Case 2</u> $\frac{1}{2} \le \sigma \le 1$

<u>Claim:-</u> $\frac{1}{2} \le a \le 1$, $\rho = a+ib$

$\because 0 \le a \le 1$ $\left[\because \rho = a+ib \text{ is a non trivial zero of } \xi(s) = \xi(0)\prod_{\rho}\left(1-\frac{s}{\rho}\right)\right]$ [3] Jonathan Sondow [3]

We prove the claim by contradiction
let us assume that $a \notin [\frac{1}{2}, 1]$.

$\because 0 \le a \le 1$

$\therefore 0 \le a < \frac{1}{2}$

$$\xi(s) = \xi(0)\prod_{\rho}\left(1-\frac{s}{\rho}\right)$$

$s = \sigma + it$
$\rho = a+ib$

$$\xi(\sigma+it) = \xi(0)\prod_{\rho = a+ib}\frac{(a-\sigma)+i(b-t)}{a+ib}$$

Generated by CamScanner

$$\frac{1}{2} \leq \sigma \leq 1$$

$$-1 \leq -\sigma \leq -\tfrac{1}{2}$$

$$0 \leq a < \frac{1}{2} \quad \text{(assumption}$$

$$-1 \leq a-\sigma < 0$$

$$a-\sigma \neq 0$$

$$\xi(\sigma+it) = \xi(0) \prod \frac{(a-\sigma)+i(b-t)}{a+ib}$$

$$(a-\sigma)+i(b-t) \neq 0$$

$$\frac{(a-\sigma)+i(b-t)}{a+ib} \neq 0$$

$$\prod_{f=a+ib} \frac{(a-\sigma)+i(b-t)}{a+ib} \neq 0$$

$$\xi(0) \prod_{f=a+ib} \frac{(a-\sigma)+i(b-t)}{a+ib} \neq 0$$

$\Rightarrow \xi(\sigma+it) \neq 0$

which is a Contradiction $\because$ we had $\xi(\sigma+it) = 0$

$\therefore$ Our Assumption that $a \notin [\frac{1}{2}, 1]$ is wrong

$\therefore a \in [\frac{1}{2}, 1]$

$$\frac{1}{2} \leq a \leq 1$$

$\& \ \frac{1}{2} \leq \sigma \leq 1$ (By Case 2)

$\frac{1}{2} \le \sigma \le 1 \Rightarrow 2\sigma - 1 \ge 0$

$\frac{1}{2} \le a \le 1 \Rightarrow 2a - 1 \ge 0$

By (4),

$$\prod_{\rho = a+ib} \frac{(a-\sigma)^2 + (b+t)^2 + (2\sigma-1)(2a-1)}{a^2+b^2} = 0 \quad \text{—— (12)}$$

$\because 2\sigma - 1 \ge 0$ & $2a - 1 \ge 0$

$\therefore (2\sigma-1)(2a-1) \ge 0$ —— (13)

(4) $\Rightarrow \prod \frac{(a-\sigma)^2 + (b+t)^2 + (2\sigma-1)(2a-1)}{a^2+b^2} = 0$

$\Rightarrow \frac{(a_1-\sigma)^2 + (b_1+t)^2 + (2\sigma-1)(2a_1-1)}{a_1^2+b_1^2} = 0$

for some $a_1 + ib_1 \in \mathbb{C}$

$(a_1-\sigma)^2 + (b_1+t)^2 + (2\sigma-1)(2a_1-1) = 0$

$\because$ By (13) $(2\sigma-1)(2a-1) \ge 0 \quad \forall a \in [\frac{1}{2}, 1]$

$\Rightarrow (2\sigma-1)(2a_1-1) \ge 0 \quad \forall a_1 \in [\frac{1}{2}, 1]$

$(a_1-\sigma)^2 + (b_1+t)^2 + (2\sigma-1)(2a_1-1) = 0$

$(a_1-\sigma)^2 = 0$ & $(b_1+t)^2 = 0$ & $(2\sigma-1)(2a_1-1) = 0$

$\Rightarrow (a_1-\sigma)^2 = 0$ & $b_1 = -t$ & $(2\sigma-1)(2a_1-1) = 0$

$\Rightarrow a_1 = \sigma$

$a_1 = \sigma \Rightarrow (2\sigma-1)(2\sigma-1) = 0$

$\Rightarrow (2\sigma-1)^2 = 0$

Generated by CamScanner

2) $\boxed{\sigma = \frac{1}{2}}$

References

1.) E.C. Titchmarsh - The Theory of the R.Z. Function

2.) Equivalents of R.H. - Arithmetic Equivalents Kevin Broughan

3.) A Monotonicity Property of Riemann's Xi Function & a Reformulation of the R.H. Jonathan Sondow and ~~Chris~~ Cristian Dumitrescu. Periodica Hungarica - May 2010

4.) H.M Edwards - Riemann's Zeta function - Academic Press