Proof of Ramanujan's identities for q-series and Theta function

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Abstract

This paper involves in providing proof of Ramanujan's identities for q-hypergeometric series and Theta function, more specifically we are going to derive the q-analogue of gauss summation of ordinary hypergeometric series.

Keywords- ***q-hypergeometric series, Ramanujan's Theta function, q-bionomial theorem .***

1.Introduction

During year 1903-1914 Ramanujan recorded most of his mathematical discoveries without proof in his notebooks. Although many of his discoveries were in the previous mathematical literature. Ramanujan begins with stating some mostly familiar theorems in the theory of q-series that we are going to derive today. Ramanujan also discovered some of Heine's famous theorems including the q-analogue of Gauss theorem.

Ramanujan's findings in the theory of theta function contains many of the classic properties in particular he rediscovered several theorems found in jacobi's fundamental nova[1]. Ramanujan also gives the famous triple product identity which has numerous applications in the field of theta function.

We can conclude our introduction with several remarks on the notation.
$f(a, b)$ is the Ramanujan theta function, $f(a, b) = \vartheta(z, \tau)$ where $ab = e^{2\pi i\tau}$and ${}^{a}/_{b} = e^{4iz}$and $\vartheta(z, \tau)$ denotes the classic theta function. It is assumed throughout the paper that $|q| < 1$, as usual for any complex number a follows the following notation,

$$(a)_k = (a; q)_k = (1 - a)(1 - aq)(1 - aq^2) \dots \dots \dots \dots \dots (1 - aq^{k-1})$$

$$(a)_\infty = (a; q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k)$$

We can define the basic hypergeometric series $({}_{s+1}\varphi_s)$ by the following equation

$$({}_{s+1}\varphi_s)\begin{bmatrix} a_1, a_2, a_3, \dots \dots, a_{s+1}, \\ b_1, b_2, b_3, \dots \dots, b_s \end{bmatrix}; x\Bigg] = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k(a_3)_k \dots \dots (a_{s+1})_k}{(b_1)_k(b_2)_k(b_3)_k \dots \dots (b_s)_k} \frac{x^k}{(q)_k}$$

Where $|x| < 1$ and $a_1, a_2, a_3, \dots \dots, a_{s+1}, b_1 b_2 b_3 \dots \dots b_s$ are arbitrary $(b_j)_k \neq 0, 1 \leq j \leq s, 0 \geq k \geq \infty$.

First we are going to derive the following two identities more generally known as q-bionomial theorem

$$\frac{(-b;q)_\infty}{(a;q)_\infty}=\sum_{k=0}^{\infty}\frac{\left(-b/a\,;q\right)_k (a)^k}{(q;q)_k}$$

Only when $|a|,|q|<1$, this identity was also discovered by Cauchy[2] and has been attributed to him.

Second identity may be written in the form

$$\frac{(at;q)_\infty}{(a;q)_\infty}=\sum_{k=0}^{\infty}\frac{(t;q)_k (a)^k}{(q;q)_k}$$

This identity is the modification of q-bionomial theorem. We are going to derive those above two identities from the q-analouge of Gauss summation of ordinary hypergeometric series i.e

$$\frac{(ab;q)_\infty (ac;q)_\infty}{(a;q)_\infty (abc;q)_\infty}=\sum_{k=0}^{\infty}\frac{\left(1/b\,;q\right)_k \left(1/c\,;q\right)_k (abc)^k}{(q;q)_k (a;q)_k}$$

Where $|abc|<1$.

2.Proofs

Proof of q-bionomial theorem from the q-analouge of Gauss summation of ordinary hypergeometric series can be completed by replacing b by $-a/b$, c by $1/c$, a by $-bc$ and then finally let c tends to zero in the q-analouge of Gauss summation of ordinary hypergeometric series, therefore consider

$$\frac{(ab;q)_\infty (ac;q)_\infty}{(a;q)_\infty (abc;q)_\infty}=\sum_{k=0}^{\infty}\frac{\left(1/b\,;q\right)_k \left(1/c\,;q\right)_k (abc)^k}{(q;q)_k (a;q)_k}$$

After substitution we get,

$$\frac{\left(abc/a\,;q\right)_\infty \left(-bc/c\,;q\right)_\infty}{(-bc;q)_\infty \left(abc/bc\,;q\right)_\infty}=\sum_{k=0}^{\infty}\frac{\left(-b/a\,;q\right)_k (c;q)_k \left(abc/bc\right)^k}{(q;q)_k (-bc;q)_k}$$

$$\frac{(ac;q)_\infty (-b;q)_\infty}{(-bc;q)_\infty (a;q)_\infty}=\sum_{k=0}^{\infty}\frac{\left(-b/a\,;q\right)_k (c;q)_k (a)^k}{(q;q)_k (-bc;q)_k}$$

Let c tends to 0, we get

$$\frac{(-b;q)_\infty}{(a;q)_\infty}=\sum_{k=0}^{\infty}\frac{\left(-b/a\,;q\right)_k (a)^k}{(q;q)_k}$$

Proof of modified q-bionomial theorem from the q-analouge of gauss summation of ordinary hypergeometric series can be completed by replacing b by $1/t$, c by $1/c$, a by atc and then finally let c tends to zero in the q-analouge of Gauss summation of ordinary hypergeometric series, therefore consider

$$\frac{(ab;q)_\infty(ac;q)_\infty}{(a;q)_\infty(abc;q)_\infty}=\sum_{k=0}^{\infty}\frac{\left(1/_b\,;q\right)_k\left(1/_c\,;q\right)_k(abc)^k}{(q;q)_k(a;q)_k}$$

After substitution we get,

$$\frac{\left(atc/_t\,;q\right)_\infty\left(atc/_c\,;q\right)_\infty}{(atc;q)_\infty\left(atc/_{tc}\,;q\right)_\infty}=\sum_{k=0}^{\infty}\frac{(t;q)_k(c;q)_k\left(atc/_{tc}\right)^k}{(q;q)_k(atc;q)_k}$$

$$\frac{(ac;q)_\infty(at;q)_\infty}{(atc;q)_\infty(a;q)_\infty}=\sum_{k=0}^{\infty}\frac{(t;q)_k(c;q)_k(a)^k}{(q;q)_k(atc;q)_k}$$

Finally let c tends to 0, we get

$$\frac{(at;q)_\infty}{(a;q)_\infty}=\sum_{k=0}^{\infty}\frac{(t;q)_k(a)^k}{(q;q)_k}$$

If we replace $-b$ by at in the q-bionomial theorem then we can get the modified q-binomial theorem, therefore consider

$$\frac{(-b;q)_\infty}{(a;q)_\infty}=\sum_{k=0}^{\infty}\frac{\left(-b/_a\,;q\right)_k(a)^k}{(q;q)_k}$$

After substitution we get

$$\frac{(at;q)_\infty}{(a;q)_\infty}=\sum_{k=0}^{\infty}\frac{\left(at/_a\,;q\right)_k(a)^k}{(q;q)_k}$$

$$\frac{(at;q)_\infty}{(a;q)_\infty}=\sum_{k=0}^{\infty}\frac{(t;q)_k(a)^k}{(q;q)_k}.$$

The famous Jacobi's triple product can be summarized as follow

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty$$

Jacobi's triple product is of great importance in the field of theta functions, it can be derived from Macdonald's identity of quintuple product which can be summarized as follows

$$1+\sum_{k=0}^{\infty}\frac{\left(1/_\alpha\,;q^2\right)_k(-\alpha q)^k}{(\beta q^2;q^2)_k}z^k+\sum_{k=0}^{\infty}\frac{\left(1/_\beta\,;q^2\right)_k(-\beta q)^k}{(\alpha;q^2)_k}z^{-k}$$

$$=\left\{\frac{(-qz;q^2)_\infty\left(-q/_z\,;q^2\right)_\infty}{(-\alpha qz;q^2)_\infty\left(-\beta q/_z\,;q^2\right)_\infty}\right\}\left\{\frac{(q^2;q^2)_\infty(\alpha\beta q^2;q^2)_\infty}{(\alpha q^2;q^2)_\infty(\beta q^2;q^2)_\infty}\right\}$$

Jacobi's triple product from Macdonald's identity of quintuple product can be derived by replacing qz by a, $^{q}/_{z}$ by b and $\alpha = \beta = 0$, calculation of this proof are not going to be derived here as it is out of our scope.

Conclusion

Proof of q-bionomial theorem from the q-analouge of Gauss summation of ordinary hypergeometric series is the main aim of this paper which is a new proof and cannot be found in in Ramanujan's notebook, though it is not the only method of deriving the q-bionomial theorem this was the more general and straight forward way of deriving the q-bionomial theorem.

Some of the Ramanujan's identities are more generally followed by substitutions in q-analouge of Dougall's theorem i.e

$$({}_8\varphi_7)\begin{bmatrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-N} \\ \sqrt{a}, -\sqrt{a}, {}^{aq}/_{b}, {}^{aq}/_{c}, {}^{aq}/_{d}, {}^{aq}/_{e}, aq^{N+1} ; q \end{bmatrix} = \frac{(aq;q)_N\left({}^{aq}/_{cd};q\right)_N\left({}^{aq}/_{bd};q\right)_N\left({}^{aq}/_{bc};q\right)_N}{\left({}^{aq}/_{b};q\right)_N\left({}^{aq}/_{c};q\right)_N\left({}^{aq}/_{d};q\right)_N\left({}^{aq}/_{bcd};q\right)_N}$$

Where N is a positive integer,and theorem of Watson[4] i.e

$$({}_8\varphi_7)\begin{bmatrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-N} & \frac{a^2q^{N+2}}{cdef} \\ \sqrt{a}, -\sqrt{a}, {}^{aq}/_{c}, {}^{aq}/_{d}, {}^{aq}/_{e}, {}^{aq}/_{f}, aq^{N+1} ; & \end{bmatrix}$$

$$= \frac{(aq;q)_\infty\left({}^{aq^{N+1}}/_{f};q\right)_\infty\left({}^{aq^{N+1}}/_{e};q\right)_\infty\left({}^{aq}/_{ef};q\right)_\infty}{\left({}^{aq}/_{e};q\right)_\infty\left({}^{aq}/_{f};q\right)_\infty(aq^{N+1};q)_\infty\left({}^{aq^{N+1}}/_{ef};q\right)_\infty}({}_4\varphi_3)\begin{bmatrix} {}^{aq}/_{cd}, e, f, q^{-N} \\ efq^{-N}/_{a}, {}^{aq}/_{c}, {}^{aq}/_{d} ; q \end{bmatrix}.$$

Refrences

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[5] *Ranamujan's Notebooks Part 3*, B1431 ruce C. Brendt. 1990.