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Assuming $c < R.exp(\frac{3\sqrt[3]{2}}{2}Log^{2/3}R)$ - A New Conjecture - Implies The *abc* Conjecture True

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To the memory of my Father who taught me arithmetic
To my wife Wahida, my daughter Sinda and my son Mohamed Mazen

Abstract

In this paper about the *abc* conjecture, we propose a new conjecture about an upper bound for c as $c < R.exp(\frac{3\sqrt[3]{2}}{2}Log^{2/3}R)$. Assuming the last condition holds, we give the proof of the *abc* conjecture by proposing the expression of the constant $K(\epsilon)$, then we approve that $\forall \epsilon > 0$, for a, b, c positive integers relatively prime with $c = a + b$, we have $c < K(\epsilon).rad^{1+\epsilon}(abc)$. Some numerical examples are given.

1. *Introduction and notations*

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1.1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{1.2}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 1. (***abc* Conjecture**): Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists a constant $K(\epsilon)$ such that :

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{1.3}$$

$K(\epsilon)$ depending only of ϵ.

The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [**2**]. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$.

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We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [**1**]. A conjecture was proposed that $c < rad^2(abc)$ [**3**]. It is the key to resolve the abc conjecture. In my paper, I propose the constant $K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}$ and assuming that $c < R.exp(\frac{3\sqrt[3]{2}}{2}Log^{2/3}R)$ the new conjecture more stronger than $c < R^2$. In my proof of the abc conjecture, we will find that c must verify $c < R.exp(\frac{3\sqrt[3]{2}}{2}Log^{2/3}R)$ so we will obtain that the abc conjecture is true. The paper is organized as follows: in the second section, we give the proof of the abc conjecture. In sections three and four, we present some numerical examples respectively for the cases $c = a+1$ and $c = a+b$.

2. *The Proof of the abc Conjecture*

Let a, b, c (respectively a, c) positive integers relatively prime with $c = a+b, a > b, b \geq 2$ (respectively $c = a+1$, $a \geq 2$). We note $R = rad(abc)$ in the case $c = a+b$ or $R = rad(ac)$ in the case $c = a+1$. I propose the constant $K(\epsilon)$ as:

$$K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} > 1, \forall\, \epsilon > 0 \tag{2.1}$$

2.1. *Case $c < R$:*

As $c < R \Longrightarrow c < K(\epsilon).R^{1+\epsilon}, \forall \epsilon > 0$ since $K(\epsilon) > 1$ and the conjecture (1) is verified.

2.2. *Case $c = R$*

Case to reject as a, b, c (respectively a, c) are relatively prime.

2.3. *Case $R < c$*

In this case, we have $c/R > 1 \Longrightarrow Log(c/R) > 0$. Let for $\forall \epsilon > 0$:

$$y(\epsilon) = \frac{1}{\epsilon^2} + (1+\epsilon)LogR - Logc \tag{2.2}$$

Our main task is give the proof that $y(\epsilon) > 0 \Longrightarrow \dfrac{1}{\epsilon^2} + (1+\epsilon)LogR > Logc$, then $\Longrightarrow Logc < \dfrac{1}{\epsilon^2} + (1+\epsilon)LogR$ and we obtain $c < e^{\left(\frac{1}{\epsilon^2}\right)}.R^{1+\epsilon} = K(\epsilon).R^{1+\epsilon}, \forall \epsilon > 0$.

We have also:

$$lim_{\epsilon \longrightarrow 0} y(\epsilon) = +\infty \tag{2.3}$$

$$lim_{\epsilon \longrightarrow +\infty} y(\epsilon) = +\infty \tag{2.4}$$

For $\epsilon > 0$, the function derivative $y'(\epsilon)$ is given by:

$$y'(\epsilon) = -\frac{2}{\epsilon^3} + LogR = \frac{\epsilon^3 LogR - 2}{\epsilon^3} \tag{2.5}$$

$y'(\epsilon) = 0$ gives:

$$\epsilon_0 = \sqrt[3]{\frac{2}{LogR}} \leq \sqrt[3]{\frac{2}{Log6}} \approx 1.03733 \tag{2.6}$$

If $R \nearrow$, then $\epsilon_0 \to 0$. For $\epsilon = \epsilon_0$, we obtain:

$$y(\epsilon_0) = \frac{1}{\epsilon_0^2} + (1+\epsilon_0)LogR - Logc = LogR + \frac{3}{2}\sqrt[3]{2}Log^{2/3}R - Logc \tag{2.7}$$

$y(\epsilon_0)$ is positive if $LogR + \frac{3}{2}\sqrt[3]{2}Log^{2/3}R - Logc > 0$. So we assume that :

$$c < R.exp(\frac{3}{2}\sqrt[3]{2}Log^{2/3}R) \Longrightarrow y(\epsilon_0) > 0 \tag{2.8}$$

Then the new conjecture proposed by us is :

$$\boxed{c < R.exp(\frac{3}{2}\sqrt[3]{2}Log^{2/3}R)} \tag{2.9}$$

From (2.3-2.4), the point $(\epsilon_0, y(\epsilon_0))$ is the minimum of the curve $y(\epsilon)$ for all $\epsilon > 0$. Then $y(\epsilon) > 0$ and the proof of the *abc* conjecture is finished. We obtain that $\forall \epsilon > 0$, $c = a + b$ with a, b, c relatively coprime:

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \quad with \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \quad \epsilon > 0 \tag{2.10}$$

REMARK 1. We verify that for $R.exp(\frac{3}{2}\sqrt[3]{2}Log^{2/3}) < R^{1+2/3}$ for R large, $R > 7\,830\,169\,545$.

REMARK 2. Nowadays, we know numerically [**1**] that $\frac{Logc}{LogR} \leq 1.629912 < 1 + 2/3 \approx 1.666667$. All the numerical examples below verify $c < R^{1+2/3}$, so, I would suggest that $c < R^{1+2/3}$ as a new open conjecture that it is more difficult than $c < R^2$.

In the two following sections, we are going to verify some numerical examples.

3. *Examples : Case $c = a + 1$*

EXAMPLE 1. The example is given by:

$$1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6 \tag{3.1}$$

$a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47\,045\,880 \Rightarrow \mu_a = 2 \times 3 \times 7 = 42$ and $rad(a) = 2 \times 3 \times 5 \times 7 \times 127$, in this example, $\mu_a < rad(a)$.
$c = 19^6 = 47\,045\,881 \Rightarrow rad(c) = 19$. Then $R = rad(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506\,730$.
We have $c > R$ and $R.exp(\frac{3}{2}\sqrt[3]{2}Log^{2/3}R) = 18\,800\,185\,299.081 > c = 47\,045\,881$.

3.0.1. *Case $\epsilon = 0.01$* $c < K(\epsilon).rad(ac)^{1+\epsilon} \Longrightarrow 47\,045\,881 \overset{?}{<} e^{10000}.506\,730^{1.01}$. The expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{\frac{1}{0.0001}} = e^{10000} = 8.7477777149120053120152473488653e + 4342 \tag{3.2}$$

We deduce that $c \ll K(0.01).506\,730^{1.01}$ and the equation (2.10) is verified.

3.0.2. *Case $\epsilon = 0.1$* $K(0.1) = e^{\frac{1}{0.01}} = e^{100} = 2.6879363309671754205917012128876e + 43 \Longrightarrow c < K(0.1) \times 506\,730^{1.01}$, and the equation (2.10) is verified.

3.0.3. *Case $\epsilon = 1$* $K(1) = e \Longrightarrow c = 47\,045\,881 < e.rad^2(ac) = 697\,987\,143\,184.212$ and the equation (2.10) is verified.

3.0.4. *Case* $\epsilon = 100$

$$K(100) = e^{0.0001} \Longrightarrow c = 47\,045\,881 \overset{?}{<} e^{0.0001}.506\,730^{101} = 1.52223502486076087818531426872 84e+576$$

and the equation (2.10) is verified.

EXAMPLE 2. We give here the example 2 from $https://nitaj.users.lmno.cnrs.fr$:

$$1 + 3^7 \times 7^5 \times 13^5 \times 17 \times 1831 = 2^{30} \times 5^2 \times 127 \times 353^2 \tag{3.3}$$

$a = 3^7 \times 7^5 \times 13^5 \times 17 \times 1831 = 424\,808\,316\,456\,140\,799 \Rightarrow rad(a) = 3 \times 7 \times 13 \times 17 \times 1831 = 8497671 \Longrightarrow$ $\mu_a > rad(a)$,
$b = 1, c = a + 1 = 424\,808\,316\,456\,140\,800 \Longrightarrow rad(c) = 2 \times 5 \times 127 \times 353$. Then $R = rad(ac) = 8497671 \times 448310 = 3\,809\,590\,886\,010 < c$. We obtain $R.exp(\frac{3}{2}\sqrt[3]{2}Log^{2/3}R) = 210\,209\,917\,628\,130\,447\,085.912 > c$, then $c < R.exp(\frac{3}{2}\sqrt[3]{2}Log^{2/3}R)$.

For example, we take $\epsilon = 0.5$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{1/0.25} = e^4 = 54.59800313096579789056 \tag{3.4}$$

Let us verify (2.10):

$$c \overset{?}{<} K(\epsilon).rad(ac)^{1+\epsilon} \Longrightarrow c = 424\,808\,316\,456\,140\,800 \overset{?}{<} K(0.5) \times (3\,809\,590\,886\,010)^{1.5}$$
$$\Longrightarrow 424\,808\,316\,456\,140\,800 < 405\,970\,304\,762\,905\,691\,174.98260818045 \tag{3.5}$$

Hence (2.10) is verified.

4. *Examples : Case* $c = a + b$

EXAMPLE 3. We give here the example of Eric Reyssat [**1**], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \tag{4.1}$$

$a = 3^{10}.109 = 6\,436\,341 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109 \Longrightarrow \mu_a > rad(a)$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $R = rad(abc) = 2 \times 3 \times 109 \times 23 = 15042 < c$. Let us verify $c < R.exp(\frac{3}{2}\sqrt[3]{2}Log^{2/3}R)$. We obtain : $c = 6\,436\,343 < 77\,532\,979.756$.

For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{9999.99} = 8.7477777149120053120152473488653e+4342 \tag{4.2}$$

Let us verify (2.10):

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow c = 6436343 \overset{?}{<} K(0.01) \times (3 \times 109 \times 2 \times 23)^{1.01} \Longrightarrow$$
$$6436343 \ll K(0.01) \times 15042^{1.01} \tag{4.3}$$

Hence (2.10) is verified.

EXAMPLE 4. The example of Nitaj about the ABC conjecture [**1**] is:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79 = 11\,297$$
$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311 = 89\,257$$
$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953$$
$$R = rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110 < c$$

We have also $\mu_a > rad(a), \mu_b > rad(b) > rad(a)$ and $\mu_b < \mu_a$. We find $c < R.exp(\frac{3}{2}\sqrt[3]{2}Log^{2/3}R) = 3\,614\,932\,048\,440\,771\,457\,890.631$.

4.0.1. *Case 1* we take $\epsilon = 100$ we have:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} e^{0.0001}.(2.3.5.7.11.13.41.79.311.953)^{101} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 2.7657949971494838920022381186039e + 1359$$

then (2.10) is verified.

4.0.2. *Case 2* We take $\epsilon = 0.5$, then:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow \tag{4.4}$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} e^{4}.(2.3.5.7.11.13.41.79.311.953)^{1.5} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 8\,450\,961\,319\,227\,998\,887\,403, 9993 \tag{4.5}$$

We obtain that (2.10) is verified.

4.0.3. *Case 3* We take $\epsilon = 1$, then

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} e.(2.3.5.7.11.13.41.79.311.953)^{2} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 831\,072\,936\,124\,776\,471\,158\,132\,100 \times e \tag{4.6}$$

We obtain that (2.10) is verified.

Example 5. It is of Ralf Bonse about the ABC conjecture [**3**] :

$$2543^4.182587.2802983.85813163 + 2^{15}.3^{77}.11.173 = 5^{56}.245983 \tag{4.7}$$
$$a = 2543^4.182587.2802983.85813163$$
$$b = 2^{15}.3^{77}.11.173$$
$$c = 5^{56}.245983 = 3.4136998783296235160378273576498e + 44$$
$$R = rad(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163$$
$$R = 1,5683959920004546031461002610848e + 33 < c \tag{4.8}$$

We have also: $\mu_a < rad(a), \mu_b > rad(b) > \mu_a, \mu_c > rad(c)$ and $\mu_b < \mu_c$. The calculate of $A = R.exp(\frac{3}{2}\sqrt[3]{2}Log^{2/3}R$ gives:

$$A = 9.5054989139840681669171835013874e + 47 > c$$

4.0.4. *Case 1* For example, we take $\epsilon = 10$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{0.01} = 1.007815740428295674320461741677$$

Let us verify (2.10):

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \overset{?}{<}$$
$$e^{0.01}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{11}$$
$$\Longrightarrow 3.4136998783296235160378273576498e + 44 <$$
$$1.4236200596494908176008120925 72e + 365 \tag{4.9}$$

The equation (2.10) is verified.

4.0.5. *Case 2* We take $\epsilon = 0.4 \Longrightarrow K(\epsilon) = 12.182473474251512159126256696O8$, then:

$$\begin{aligned} & c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \overset{?}{<} \\ & e^{6.25}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{1.4} \\ & \Longrightarrow 3.41369987832962351603782735764 98e+44 < \\ & 3.62554656800114536427927205696 85e+47 \end{aligned} \tag{4.10}$$

And the equation (2.10) is verified.

5. *Conclusion*

Assuming $c < R.exp(\frac{3}{2}\sqrt[3]{2}Log^{2/3}R)$, we have given an elementary proof of the *abc* conjecture, confirmed by some numerical examples. We can announce the theorem:

THEOREM 5.1. *Let a, b, c positive integers relatively prime with $c = a + b$, and assuming $c < R.exp(\frac{3}{2}\sqrt[3]{2}Log^{2/3}R)$ is true, then for each $\epsilon > 0$, there exists $K(\epsilon)$ such that :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{5.1}$$

where $K(\epsilon)$ is a constant depending of ϵ proposed as :

$$K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} \quad , \epsilon > 0$$

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