Abstract: The Riemann zeta function is one of the most Euler’s important and fascinating functions in mathematics. By analyzing the material of Riemann’s conjecture, we divide our analysis in the \( \zeta(z) \) function and in the proof of the conjecture, which has very important consequences on the distribution of prime numbers. The proof of the Hypothesis of Riemann result from the simple logic, that when two properties are associated, (the resulting equations that based in two Functional equations of Riemann), if zero these equations, ie \( \zeta(z) = \zeta(1-z) = 0 \) and simultaneously they have to have the proved property 1-1 of the Riemann function \( \zeta(z) \). Thus, there is not margin for to non exist the \( \Re(z) = 1/2 \) (because \( \zeta(z) = \zeta(1-z) = 0 \) and also \( \zeta(z) \) as and \( \zeta(1-z) \) are 1-1). This, as it stands, will gives the direction of all the non-trivial roots to be all in on the critical line, with a value in the real axis equal 1/2.

#1. Theorem 1.

The R-Hypothesis focuses on the point where we must prove that if \( s = \Re(s) + \Im(s)\ast i \) …

I) The functions \( \zeta(s) \) and \( \zeta(1-s) \) are 1-1 on the critical strip.

II) The common roots of the equations \( \zeta(s) - \zeta(1-s) = 0 \) they have \( \Re(s) = \Re(1-s) = 1/2 \) within the interval \((0,1)\) and determine unique position, which is called critical line.

Proof:
The functions \( \zeta(s) \) and \( \zeta(1-s) \) are 1-1 on the critical strip. For this we need to analyze when and where the exponential function \( n^z \) is 1-1 when \( n \in \mathbb{Z}^+ \).

The exponential function \( n^z \) …

I) is 1-1 in each of these strips defined by the intervals \( 2\pi k/\ln(n) \) and \( 2\pi (k+1)/\ln(n) \) “ where \( k \in \mathbb{Z} \). For this we must prove two cases… If \( A \subset C \wedge f : A \to C \)

then if \( f(z_1) = f(z_2) \) then \( z_1 = z_2 \) that is,

\[
x_1 = x_2 \wedge y_1 = y_2.
\]

Indeed,

if \( z_1 = x_1 + i \cdot y_1 \), \( z_2 = x_2 + i \cdot y_2 \) are two points within into a such strip such that \( e^{z_1} = e^{z_2} \) then,

\[
n^{x_1+y_1 i} = n^{x_2+y_2 i} \Rightarrow |n^{x_1+y_1 i}| = |n^{x_2+y_2 i}|
\]

\( x_1 = x_2 \) and since \( x_1 = x_2 \), the relation

\[
e^{z_1 \cdot \ln(n)} = e^{z_2 \cdot \ln(n)}
\]

gives \( e^{i \cdot y_1 \cdot \ln(n)} = e^{i \cdot y_2 \cdot \ln(n)} \), so

\[
y_1 - y_2 = 2k\pi / \ln(n), \text{ is an integer multiple of } 2\pi.
\]

But \( z_1 \) and \( z_2 \) belong to the strip, so \( |y_1 - y_2| < 2\pi / \ln(n) \)

That is, the difference \( y_1 - y_2 \) is at the same time a multiple of \( 2\pi / \ln(n) \) and at an absolute value of less than \( 2\pi / \ln(n) \). The only case that this is true is when

\[
y_1 = y_2. \text{ We finally conclude that } z_1 = z_2, \text{ so } n^z \text{ is } 1-1 \text{ in the strip}
\]

\[
\{ z : 2k\pi / \ln(n) \leq \Im(z) < 2(k+1)\pi / \ln(n) \} \text{. }
\]

down closed – open up”. We also notice that \( n^z \) is on C-\{0\}. Because if \( w \neq 0 \) and we put

\[
z = \ln|w| / \ln(n) + i \cdot \arg w / \ln(n)
\]

then

\[
n^z = n^{\ln|w| / \ln(n) + i \cdot \arg w / \ln(n)} = n^{|w| \cdot n^{i \cdot \arg w / \ln(n)}} = w
\]

Fig.1 The \( z_1 \) and \( z_2 \) belong to the lane width \( 2k\pi / \ln(n) \)

Formation of the strips 1-1 for \( \zeta(s) \) and \( \zeta(1-s) \).

a1.) If we accept the non-trivial zeroes on critical strip of the Riemann Zeta Function \( \zeta(s) \) as \( s_1 = \sigma_1 + it_1 \) and

\[
s_2 = \sigma_2 + it_2 \text{ with } |t_1| > |t_2|,
\]

and if we suppose that
the real coordinates \( \sigma_1, \sigma_2 \) of each non-trivial zero of the Riemann Zeta, [1,2,3,4] function \( \zeta(s) \) correspond to two imaginary coordinates \( t_1 \) and \( t_2 \), then, we have the following equations group:

\[
\zeta(\sigma_1 + t \cdot t_1) = \frac{1}{n^{\sigma_1 + t}} + \frac{1}{2 n^{\sigma_1 + t}} + \frac{1}{3 n^{\sigma_1 + t}} + \cdots + \frac{1}{n^{\sigma_1 + t}} = \frac{1}{n^{\sigma_1 + t}} + \frac{1}{2 n^{\sigma_1 + t}} + \frac{1}{3 n^{\sigma_1 + t}} + \cdots + \frac{1}{n^{\sigma_1 + t}} = 0
\]

\[
\zeta(\sigma_2 + t \cdot t_2) = \frac{1}{n^{\sigma_2 + t}} + \frac{1}{2 n^{\sigma_2 + t}} + \frac{1}{3 n^{\sigma_2 + t}} + \cdots + \frac{1}{n^{\sigma_2 + t}} = \frac{1}{n^{\sigma_2 + t}} + \frac{1}{2 n^{\sigma_2 + t}} + \frac{1}{3 n^{\sigma_2 + t}} + \cdots + \frac{1}{n^{\sigma_2 + t}} = 0
\]

Taking the first equation and deducting the second, we obtain:

\[
\zeta(\sigma_1 + t \cdot t_1) - \zeta(\sigma_2 + t \cdot t_2) = \sum_{n=1}^{\infty} \left( \frac{1}{n^{\sigma_1 + t}} - \frac{1}{n^{\sigma_2 + t}} \right) = \sum_{n=1}^{\infty} n^{\sigma_2 - \sigma_1} e^{-t \cdot \ln(n)} - n^{\sigma_1 - \sigma_2} e^{-t \cdot \ln(n)} = 0
\]

From the previous relation, we conclude that if

\[
n^{\sigma_2 - \sigma_1} \cdot n^{\sigma_1 - \sigma_2} \neq 0, n > 1 \text{ then } \sigma_2 = \sigma_1 \wedge t_2 = t_1 \pm \frac{2k\pi}{\ln(n)}, (k = 1, 2, \ldots). \]

That is to say \( t_1 \) and \( t_2 \) can take any value, but according to the previous relation. So from Fig.1, that means \( \zeta(s) \) is 1-1 on the lane of critical Strip in Intervals such as defined, therefore and on the critical Line. So \( n^{-z} \) is 1-1 in the strip and \( \{z : 2k\pi / \ln(n) \leq \text{Im}(z) < 2(k + 1)\pi / \ln(n)\} \). "lines down closed – open up"

a2.) If we do the same work with \( |S_2| > |S_1| \) for the case \( \zeta(1-s) \) will we have:

\[
\zeta(1 - \sigma_1 - t \cdot t_1) - \zeta(1 - \sigma_2 - t \cdot t_2) = \sum_{n=1}^{\infty} \left( \frac{1}{n^{1-\sigma_1 - t}} - \frac{1}{n^{1-\sigma_2 - t}} \right) = \sum_{n=1}^{\infty} n^{\sigma_2 - \sigma_1} e^{-t \cdot \ln(n)} - n^{\sigma_1 - \sigma_2} e^{-t \cdot \ln(n)} = 0
\]

From the previous relation, we see that if

\[
n^{1-\sigma_2 - \sigma_1} \cdot n^{1-\sigma_1 - \sigma_2} \neq 0, n > 1
\]

we have:

\[
(1 - \sigma_2 = 1 - \sigma_1) \vee \sigma_2 = \sigma_1 \land t_2 = t_1 \pm \frac{2k\pi}{\ln(n)}, (k = 1, 2, \ldots)
\]

So, from Figure.1, it follows that \( \zeta(1-s) \) is 1-1 on the critical Strip. So \( n^{-z} \) is 1-1 in the strip and \( \{z : 2k\pi / \ln(n) < \text{Im}(z) \leq 2(k + 1)\pi / \ln(n)\} \). "lines up closed – down open"

As we have seen before, the complex exponential form \( n^z \) but and \( n^{-z} \) is also periodic with a period of \( 2\pi/\ln(n) \). For example \( n^{z+2\pi i} = n^z \cdot n^{2\pi i / \ln(n)} = n^z \) the \( n^z \) is repeated in all the "lines down closed – open up" horizontal strips with \( 2\pi/\ln(n) \) on Fig.2 below.

\[\text{Fig 2. All the "bottom - closed upper "horizontal strips with } 2\pi/\ln(n) \text{ for } n^{-z}, n \geq 1, n \in \mathbb{Z}.\]

If we are asked to find the strips that are 1-1 of the function \( \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \) it will be the union of the strips formed with period \( 2 \cdot \pi \cdot \kappa / \ln(n) \) where \( \kappa, n \in \mathbb{Z} \). This results from the analysis we made for the cases a1, a2. Therefore, if we analytically assume that \( n = 2, 3, \ldots \) then the exponential complex function will be 1-1 in each of the strips, i.e. in each one set of the form...

\[S_a = \{z : 2k\pi / \ln(n) \leq \text{Im}(z) < 2(k + 1)\pi / \ln(n)\}, k \in \mathbb{Z}\]

and therefore in each one subset of the unified expanded set...

\[S = \bigcup_{i=2}^{\infty} S_i \]

The largest strip obviously contains all the rest it is wide, and has width \( 2 \cdot \pi / \ln(n) \) i.e. with \( n = 2 \). Overlapping contiguous Strips.
If we start from the upper bound of the strip with \( n = 2 \) and \( \kappa = 1 \), Fig. 3 the upper ones will descend to zero by increasing \( n \) with a width difference the interval \( \delta = 2\pi/\ln(n-1) - 2\pi/\ln(n) \). In addition, we will prove that the strip \( i = n \geq 2 \) primary (\( \kappa = 1 \)) that generated in ascending order, the upper limit of that created of its double width enters the zone of the previous of \( (i = n-1, \kappa = 1) \) also primary.

Fig. 3. The upper ones will go down to zero by increasing \( n \).

Apply \( 2 \pi / \ln(n-1) < 2 \pi / \ln n \Rightarrow n < (n-1)^2 \) \( \Rightarrow \)
\[
n^2 - 3.4 + 1 > 0 \Rightarrow n > \frac{3 + \sqrt{5}}{2} \]
and because we talked about integers then \( n \geq 3 \). The same phenomenon is also created at higher level strips.

The \( \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \) it is directly 1-1 from the comparison of 2 points \( z_1, z_2 \), when \( \zeta(z_1) = \zeta(z_2) \).

Proof:
From the equality
\[
\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} \frac{1}{n} = \zeta(z_1) \Rightarrow
\]
\[
1 + z^{-1} + z^{-2} + \ldots + z^{-n} = \zeta(z_1) =
\]
\[
1 + z^{-1} + z^{-2} + \ldots + z^{-n} = \zeta(z_2) \Rightarrow
\]
with comparison of similar terms the result arises.
\[
x_1 = x_2 \wedge y_1 = y_2 - \frac{2 \pi \kappa}{\ln(n)} , k \in Z \Rightarrow
\]
If we need \( z_1 = z_2 \Rightarrow x_1 = x_2 \wedge y_1 = y_2 \) therefore it will be 1-1 as shown in sections a1, a2 pages 1-2... Also according to Lagrange's generalized theory, each of the roots according to the obvious relation \( \zeta'(z) = \zeta(1-z) \) will result from the generalized theorem of Lagrange.

From the equation that results from 3 Functional Equations, with condition of common roots. We take \( \zeta(z) = \zeta(1-z) \) and for first approach we then follow the 3 first terms on each \( \zeta \)-equation we will have in the Regular form \( \zeta(z), \zeta(1-z) \) with the analysis below.
\[
1^z + 2^z + 3^z = 1^{(1-z)} + 2^{(1-z)} + 3^{(1-z)} \Rightarrow 2^z + 3^z = \frac{1}{2} \left( 2^z + \frac{3}{2} z \right) \Rightarrow
\]
\[
2^z + 3^z - 2^{(1-z)} - 3^{(1-z)} = 0 \Rightarrow 6^z + \frac{3}{2} 4^z - 3 - 3 \left( \frac{2}{3} z \right)^2 = 0
\]

With the replacement
\[
y = \log(z) \Rightarrow z \cdot \log(6) = \log(y) + 2 \cdot k \cdot \pi \cdot i \Rightarrow z = \frac{\log(y) + 2 \cdot k \cdot \pi \cdot i}{\log(6)}
\]
\[
z_w = \frac{\log(y) + 2 \cdot k \cdot \pi \cdot i}{\log(6)}
\]

The solution for \( k \in Z \):
1st \( 0.4985184869351 + /- 3.6072014300530 \) \( \kappa = 1 \)
2st \( 0.4997077392995 + /- 10.329023684711 \) \( \kappa = 3 \)
3st \( 0.50003426560276 + /- 13.990255708243 \) \( \kappa = 4 \)
4st \( 0.49930878257278 + /- 17.585326315022 \) \( \kappa = 5 \)
5st \( 0.5026747758005 + /- 21.304192633721 \) \( \kappa = 6 \)

And so on for the sequence of infinity. We now see that we approach the real roots and its complexities. With the additional method Newton or the Bisection method we approach more rapidly to them. And these roots will be contained in corresponding strips that we have previously defined and in closer relation with their position on the critical line. By this logic for the previous approximation equation in relation to its solutions of Zeta[z] the strips which determine the imaginary roots of the equation, will be between the intervals
\[
2 \cdot \pi \cdot \kappa / \ln(6), \kappa \in Z \text{ as defined by the analysis of the generalized Theorem of Lagrange. The strips of the}
\]
imaginary part of the roots, as they appear for 6 consecutive intervals of the approximate equation

Strips – Imaginary Roots:

<table>
<thead>
<tr>
<th>Strips</th>
<th>Imaginary Roots</th>
<th>Real Roots</th>
</tr>
</thead>
<tbody>
<tr>
<td>st</td>
<td>(7.013424977055179, 3.5067124888527589)</td>
<td>(10.52013746558276, 7.013424977055179)</td>
</tr>
<tr>
<td>2st</td>
<td>(14.02684995411035, 5.502013746558276)</td>
<td>(17.53536244263794, 14.02684995411035)</td>
</tr>
</tbody>
</table>

It is clear the first root 3.607..i is located in 1st of strip (7.01-3.506), the second 10.3290..i in the 2st strip (7.013-10.52) etc. Now the root 13.99..i which approximates the first root of \( \zeta(z) = \zeta(1-z) = 0 \) i.e. the 4th strip (14.02-17.533) with a lower limit the 14.026..i a value that is very close to the one required for the approximate and exclusive (1st root) when \( \zeta(z) = 0 \) of the Riemann Hypothesis.

**Corollary 1.**
The only roots of the Zeta function not included in the set \( \{z \in \mathbb{C}: 0 \leq \text{Re}(z) \leq 1\} \) are the points -2, -4, -6...[11.page 47]

**Proof.**
On the functional equation...

\[ \zeta(1-z) = 2 \cdot \zeta(z) \cdot \Gamma(z) \cdot \cos(\pi / 2 \cdot z) \cdot (2\pi)^{-z} \]

we know that for \( \text{Re}(z) > 1 \) the functions \( \zeta(z) \) and \( \Gamma(z) \) do not equal zero(Proof-Th.3.p6). We also find that for \( \text{Re}(z) > 1 \) the \( 1 - \text{Re}(z) < 0 \) and putting \( u = 1 - z \) we will find all the roots of \( \zeta(u) \) for \( \text{Re}(u) < 0 \). Therefore we will have that \( \zeta(u) \) will be zero, where it is zeroed the \( \cos(\pi / 2 \cdot z) \) i.e. \( z = 3, 5, 7.. \) Then the roots of \( \zeta(u) \) for \( \text{Re}(u) < 0 \) will be the points -2, -4, -6... and will be all the roots of the function \( \zeta \) out of the strip \( \{z \in \mathbb{C}: 0 \leq \text{Re}(z) \leq 1\} \).

**Theorem Helpful.**
The Riemann's Z-function has no roots on the lines \( \text{Re}(z) = 1 \) and \( \text{Re}(z) = 0 \).

II.) The common roots of the equations \( \zeta(s) - \zeta(1-s) = 0 \) they have \( \text{Re}(s) = \text{Re}(1-s) = 1/2 \) within the interval \( (0,1) \) if moreover apply \( \zeta(s) = 0 \).(Refer to Theorem 3, page 5).

**Proof:** Let us assume \( z \) to be such that for complex \( z = x_0 + iy_0, 0 < \text{Re}(z) < 1 \wedge \text{Im}(z) \neq 0 \) and \( \zeta(z) = \zeta(1-z) = 0 \). According to the two equations, they must apply to both, because they are equal to zero that... \( \zeta(z) = \zeta(\hat{z}) = 0 \wedge \zeta(1-z) = \zeta(1-\hat{z}) = 0 \). But from [Theorem 1, II, a1, a2] the \( \zeta(z) \) and \( \zeta(1-z) \) are 1-on the critical Strip.. If suppose generally that \( \zeta(x_0 \pm y_0 \cdot i) = \zeta(1 - x_0' \pm y_0' \cdot i) = 0 \) then:

\[ y_0 = y_0' \wedge x_0 = 1 - x_0' \]

but because are the 1-1 then we will apply two cases for complex \( z \) i.e.

\[ z = [x_0' = 1 - x_0, x_0 \neq x_0', y_0 = y_0' \vee v \times x_0' = 1 - x_0, x_0 = x_0' = \frac{1}{2}, y_0 = y_0'] \]

[Corollary.1 and Theorem[Helpful], because we suppose \( \zeta(z) = \zeta(1-z) = 0 \).] We conclude that \( 0 < x_0 < 1 \). Therefore we have two cases:

a) If \( x_0 = x_0' = \frac{1}{2} \) we apply the obvious i.e.

\[ \zeta(1/2 + y_0 \cdot i) = \zeta(1 - 1/2 + y_0' \cdot i) = \zeta(1/2 + y_0' \cdot i) = 0 \wedge \]

\[ y_0 = y_0' \]. Which fully meets the requirements of hypothesis!!

b) If \( 0 < x_0 \neq x_0' < \frac{1}{2} < 1 \) In this case if \( x_0 < x_0' \wedge x_0 + x_0' = 1 \) or \( x_0 > x_0' \wedge x_0 + x_0' = 1 \)

then for the Functional equation \( \zeta(s) - \zeta(1-s) = 0 \) simultaneously apply:

\[ \zeta(x_0 + y_0 \cdot i) = \zeta(x_0 - y_0 \cdot i) = 0 \wedge \zeta(x_0' + y_0' \cdot i) = \zeta(x_0' - y_0' \cdot i) = 0 \]

So let's assume that...

1st: \( x_0 < x_0' \wedge x_0 + x_0' = 1 \Rightarrow x_0 = \frac{1}{2} - a \wedge x_0' = \frac{1}{2} + b, a \neq b \).

But apply \( x_0 + x_0' = 1 \Rightarrow 1 + b - a = 1 \Rightarrow b = a \). That is,
a, b is symmetrical about of $\frac{1}{2}$. But the $\zeta(z)$ and $\zeta(1-z)$ are 1-1 on the critical Strip then apply

$$\zeta\left(\frac{1}{2} - a + y_0 \cdot i\right) = \zeta\left(\frac{1}{2} + y_0' \cdot i\right) \wedge y_0 = y_0'$$

must $\frac{1}{2} - a + y_0 \cdot i = 2a = 0 => a = 0$

therefore $X_0 = X_0'$.

2nd. $x_0 > x_0' \wedge x_0 + x_0' = 1 => x_0 = \frac{1}{2} + a \wedge x_0' = \frac{1}{2} - a$.

as before. Then because the $\zeta(z)$ and $\zeta(1-z)$ are 1-1 on the critical Strip will apply...

$$\zeta\left(\frac{1}{2} + a + y_0 \cdot i\right) = \zeta\left(\frac{1}{2} - a + y_0' \cdot i\right) \wedge y_0 = y_0'$$

$$\frac{1}{2} + a = \frac{1}{2} - a => 2a = 0 => a = 0$$

therefore $X_0 = X_0'$. We see that in all acceptable cases is true that “The non-trivial zeros of $\zeta(z)$ - $\zeta(1-z) = 0$ have real part which is equal to $\frac{1}{2}$ within the interval $(0,1)$. In the end, we see still one partial case that appears in the roots of the equation $\zeta(z) = \zeta(1-z) = 0$.

c) If $1 < X_0 \wedge X_0' < 0$ Furthermore, in this case with that is, it is symmetrical about

$$x_0 + x_0' = 1$$

then for the functional equations we apply

$$\zeta(x_0 - y_0 \cdot i) = \zeta(1 - x_0 - y_0 \cdot i) \wedge \zeta(x_0 + y_0 \cdot i) =$$

$$\zeta(1 - x_0 + y_0 \cdot i) \wedge y_0 = y_0'$$

and for the three cases it is valid $y_0 = 0 \vee y_0 = 0$. This case applies only when

$$\zeta(z) = \zeta(1 - z) \neq 0 \wedge x_0 = 1 + a, x_0' = -a$$

and not for the equality of $\zeta$ function with zero. But in order to have common roots of the two functions $\zeta(z)$ and $\zeta(1-z)$. «because they are equal to zero and equal to each other», then first of all, it must be $X_0 = X_0'$ and also because the $\zeta(z)$ is 1-1 and for each case applies

$$(y_0 = 0 \ and \ y_0' = 0)$$

and

$$(y_0 \neq 0 \ and \ y_0' \neq 0),$$

therefore we will apply $X_0 = X_0' = \frac{1}{2} \wedge y_0 = y_0'$ and with regard to the three cases {a, b, c} the cases {b, c} cannot happen, because they will have to be within (0,1) and will therefore be rejected.

Therefore if $\zeta(z) = \zeta(1-z) = 0$ then because “these two equations of zeta function are 1-1 on the lane of critical strip” as shown in theorem 1, is “sufficient condition” that all non-trivial zeros are on the critical line $Re(z) = 1/2$. That means that the real part of $z$ of $\zeta(z) = 0$, equals to $1/2$”. This, like Theorem 1, has been proved and it helps the Theorem 3 below.

2#. Theorem 2.

For the non-trivial zeros of the Riemann Zeta Function $\zeta(s)$ apply

i) There exists an upper-lower bound of $Re s$ of the Riemann Zeta Function $\zeta(s)$ and more specifically in the closed interval $\left[\frac{\ln 2}{\ln 2 \pi}, \frac{\ln \pi}{\ln 2 \pi}\right]$. The non-trivial zeros of the Riemann Zeta function $\zeta(s)$ of the upper-lower bound are distributed symmetrically on the straight line $Re s = 1/2$.

ii). The average value of the upper lower bound of $Re s = 1/2$.

Proof:

i). Here, we formulate two of the functional equations from E - q.

Set

$$\zeta(1-s) / \zeta(s) = 2(2\pi)^{-s} \cos(\pi s/2) \Gamma(s), \text{ Re } s > 0$$

$$\zeta(s) / \zeta(1-s) = 2(2\pi)^{1-s} \sin(\pi s/2) \Gamma(1-s), \text{ Re } s < 1$$

We look at each one equation individually in order to identify the set of values that we want each time.

a. For the first equation and for real values with $Re s > 0$ and by taking the logarithm of two sides of the equation [5], we have...

$$\zeta(1-s) / \zeta(s) = 2(2\pi)^{-s} \cos(\pi s/2) \Gamma(s) =$$
\[ \log[(1 - s)/\zeta(s)] = \log[2] - s\log[2\pi] + \log[\cos(\pi s/2)\Gamma(s)] + 2k\pi \] but solving for \( s \) and if \( f(s) = \cos(\pi s/2)\Gamma(s) \) and from Lemma 2[6,p6],

**Part I** if \( \lim_{s \to \pi} \zeta(1-s)/\zeta(s) = 1 \) or

\[ \log[\lim_{s \to \pi} \zeta(1-s)/\zeta(s)] = 0 \] we get:

\[ s = \frac{\log[2]}{\log[2\pi]} + \frac{\log[f(s)]}{\log[2\pi]} + 2k\pi \] with

\[ \frac{\log[f(s)]}{\log[2\pi]} + 2k\pi > 0 \] Finally, because we need real \( s \) we have \( \Re s \geq \frac{\log[2]}{\log[2\pi]} = 0.3771 \). This is the lower bound, which gives us the first \( \zeta(s) \) of Riemann’s Zeta Function.

\[ \zeta(1-s)/\zeta(s) = 2(2\pi)^{-s} \sin(\pi s/2)\Gamma(1-s) \Rightarrow \]

\[ \log[\zeta(1-s)/\zeta(s)] = \log[2] + (s-1)\log[2\pi] + \log[\sin(\pi s/2)\Gamma(1-s)] + 2k\pi \] but solving for \( s \) and if \( f(s) = \sin(\pi s/2)\Gamma(1-s) \) and from

**Lemma 2[6,p6,Part I]** if \( \lim_{s \to \pi} \zeta(s)/\zeta(1-s) = 1 \) or

\[ \log[\lim_{s \to \pi} \zeta(s)/\zeta(1-s)] = 0 \] we get...

\[ s = \frac{\log[\pi]}{\log[2\pi]} - \frac{\log[f(s)]}{\log[2\pi]} + 2k\pi \] with

\[ \frac{\log[f(s)]}{\log[2\pi]} + 2k\pi \geq 0 \]

In the following, because we need real \( s \) we will take \( s \leq \frac{\log[\pi]}{\log[2\pi]} = 0.62286 \). This is the upper bound, which gives us the second of Riemann’s Zeta Function of \( \zeta(s) \). So, we see that the lower and the upper bound exist for \( \Re s \) and they are well defined.

**c).** Assuming that \( s_k = \sigma_{\text{low}} + it_k \) and

\[ \sigma_k = \sigma_{\text{upper}} + it_k \] with \( \sigma_{\text{low}} = \frac{\log[2]}{\log[2\pi]} \) and

\[ \sigma_{\text{upper}} = \frac{\log[\pi]}{\log[2\pi]} \] we apply \( \sigma_1 = \sigma_{\text{low}} \) and

\[ \sigma_2 = \sigma_{\text{upper}} \] [fig.4]. If we evaluate the difference

\[ \Re \Delta \zeta_k = \Re \sigma_k - 1/2 = \sigma_{\text{upper}} - 1/2 = 0.1228 \]

\[ \Re \Delta \zeta_k = 1/2 - \Re \sigma_k - 1/2 - \sigma_{\text{low}} = 0.1228 \]

This suggests for our absolute symmetry

**Fig.4** Arrangement of low and upper of real part of zeros from 2 functional equations of Riemann.

**ii) The average value of the upper lower bound is \( \Re s = 1/2 \) because from (fig. 4):

\[ \Re s = 1/2 \cdot \frac{\log[2] + \log[\pi]}{\log[2\pi]} = 1/2 \]

**3. Theorem 3**

The **Riemann Hypothesis** states that all the nontrivial zeros of \( \zeta(z) \) have real part equal to 1/2.

**Proof:** In any case, we Assume that: The Constant Hypothesis \( \zeta(z) = 0 \), \( z \in C \). In this case, we use the two equations of the Riemann zeta function, so if they apply what they represent the \( \zeta(z) \) and \( \zeta(1-z) \) to equality. Before developing the method, we make the three following assumptions:

**I). Analysis of specific parts of transcendental equations, which are detailed…

**a).** For \( z \in C \), \( x > 0 \) \( \rightarrow \) \( a = 0 \) and \( \Re z > 0 \) which refers to the inherent function similar to two **Riemann zeta functions as** \( (2\pi)^z = 0 \) or \( (2\pi)^{z^{-1}} = 0 \) and it seems that they do not have roots in \( C \cdot Z \), because \( (2\pi) \neq 0 \).

**b).** This forms \( \Gamma(\zeta(z)) = 0 \) or \( \Gamma(1-z) = 0 \) do not have roots in \( C \cdot Z \).

**c).** Solution of \( \sin(\pi/2z) = 0 \) or \( \cos(\pi/2z) = 0 \). More specifically if \( z = x+iy \), then…

1. \( \sin(\pi/2z) = 0 \Rightarrow \cosh[(\pi \cdot y)/2] \sin[(\pi \cdot x)/2] + 1 \cdot \cos[(\pi \cdot x)/2] \sinh[(\pi \cdot y)/2] = 0 \)
If $k, m \in N(\text{Integers})$ then all the solutions can be found with a program by the language of Mathematica.

\begin{align*}
i). \{x = (2 + 4 \cdot k), y = 2 \cdot i \cdot (1 + 2 \cdot m) \\
ii). \{x = 4 \cdot k, y = 2 \cdot i \cdot (1 + 2 \cdot m) \\
iii). \{x = (-1 + 4 \cdot k), y = i \cdot (-1 + 4 \cdot m) \\
iv). \{x = (-1 + 4 \cdot k), y = i \cdot (4 \cdot m) \\
v). \{x = (1 + 4 \cdot k), y = i \cdot (-1 + 4 \cdot m) \\
vi). \{x = (2 + 4 \cdot k), y = i \cdot (4 \cdot m) \\
vii). \{x = (2 + 4 \cdot k), y = i \cdot (-1 + 4 \cdot m) \\
viii). \{x = (2 + 4 \cdot k), y = 2 \cdot i \cdot (1 + 2 \cdot m) \\
\end{align*}

From the generalized solution, it seems that in the pairs $(x, y)$ will always arise integers which make impossible the case $0 < x < 1$, therefore there are no roots of the equation $\sin(\pi/2) = 0$ in $C-Z$.

Also, because as we see, all the roots given by the union of the sets $x \geq 1 \vee x = -1 \vee x = 0$ all are integers. So it cannot be true that for our case: $\cos(\pi/2) = 0 \Rightarrow \cosh(\pi/2) \cos(\pi x/2) - \sinh(\pi x/2) \sinh(\pi y/2) = 0$.

If $k, m \in N(\text{Integers})$ and the solutions are:

\begin{align*}
i). \{x = 4 \cdot k, y = i \cdot (-1 + 4 \cdot m) \\
ii). \{x = 4 \cdot k, y = i \cdot (1 + 4 \cdot m) \\
iii). \{x = (-1 + 4 \cdot k), y = i \cdot (4 \cdot m) \\
iv). \{x = (-1 + 4 \cdot k), y = 2 \cdot i \cdot (1 + 2 \cdot m) \\
v). \{x = (1 + 4 \cdot k), y = 4 \cdot i \cdot m) \\
vi). \{x = (1 + 4 \cdot k), y = i \cdot (1 + 4 \cdot m) \\
vii). \{x = (2 + 4 \cdot k), y = i \cdot (-1 + 4 \cdot m) \\
viii). \{x = (2 + 4 \cdot k), y = i \cdot (1 + 4 \cdot m) \\
\end{align*}

As we can see, again from the generalized solution, it seems that in the pairs $(x, y)$ will always arise integers, which make impossible the case $0 < x < 1$, therefore there are not roots receivers of the equation $\cos(\pi x/2) = 0$. Since all the roots are given by the union of the sets $x \geq 1 \vee x = -1 \vee x = 0$ it follows that they are Integers.

**Great Result.**

«With this three - cases analysis, we have proved that the real part of $x$, of the complex $z = x + yi$ cannot be in the interval $0 < x < 1$, and in particular in cases $\{a, b, c\}$, when they are zeroed. Therefore they cannot represent roots in the critical line».

II) Therefore, now we will analyze the two equations of the Riemann zeta function and we will try to find any common solutions.

1. For the first equation and for real values with $\text{Re } z > 0$ we apply:

\[f(z) = 2(2\pi)^{-z} \cdot \cos(\pi/z)\Gamma(z) \Rightarrow \zeta(1-z) = f(z) \cdot \zeta(z)\]

but this means that the following two cases occur:

a). $\zeta(1-z) = \zeta(z)$, where $z$ is complex number. This assumption implies that $\zeta(z) \cdot (1 - f(z)) = 0 \Rightarrow \zeta(z) = 0$. In theorem 1(I, II), (pages 1-2), we showed that the functions $\zeta(z)$ and $\zeta(1-z)$ are 1-1 and therefore if

\[\zeta(x_0 + y_0 i) = \zeta(x_0) = y_0 \cdot i\] then, but we also apply that

\[\zeta(x_0 + y_0 i) \neq \zeta(x_0 - y_0 i) = 0\], because we apply it for complex roots. The form $\zeta(1-z) = \zeta(z)$ means that if \(z = x_0 + y_0 i\) we apply \(1 - x_0 = x_0\) namely \(x_0 = 1/2\) because in this case it will be verified that

\[\zeta(1 - x_0 - y_0 i) = \zeta(1/2 - y_0 i) = \zeta(1/2 + y_0 i) = \zeta(1 - x_0 + y_0 i) = 0\] and because $y_0 = y_0'$

\[\zeta(x_0 - y_0 i) = \zeta(1/2 - y_0 i) = \zeta(1/2 + y_0 i) = \zeta(x_0 + y_0 i) = 0\] and these are the two forms of the $\zeta$ equation, they have common roots and therefore it can be verified by the definition of any complex equation, when it is equal to zero. Therefore if \(z = x_0 + y_0 i\) then

\[x_0 = 1/2\] which verifies the equation $\zeta(z) = 0$.

b). $\zeta(1-z) \neq \zeta(z)$, when $z$ is a complex number.
To verify this case must be:
\[
\zeta(z) = \zeta(1-z) \cdot f(z) \Rightarrow \zeta(z) = 0 \land f(z) = 0.
\]
But this case is not possible, because as we have shown in Section 1(a, b, c), the individual functions of \( f(z) \) cannot be zero when \( z \) is a complex number.

2. For the second equation and for any real values with \( \text{Re} \, z < 1 \) we apply
\[
\zeta(z) / \zeta(1-z) = 2(2\pi)^{i-1} \sin(\pi \cdot z / 2) \Gamma(1-z) \Rightarrow \zeta(z) = f(z) \cdot \zeta(1-z), \text{ where}
\]
\[
f(z) = 2(2\pi)^{i-1} \sin(\pi \cdot z / 2) \Gamma(1-z)
\]
but this also means that two cases occur:
a). \( \zeta(1-z) = \zeta(z) \), when \( z \) is a complex number.
This case is equivalent to 1.a, and therefore if
\[
z = x_0 + y_0i \quad \text{then} \quad x_0 = 1 / 2 \quad \text{in order to verify the equation} \quad \zeta(z) = 0 \quad \text{we follow exactly the same process algebraically.}
\]
b). \( \zeta(1-z) \neq \zeta(z) \), where \( z \) is a complex number.
Similarly, the above case is equivalent to 1.b and therefore it cannot be happening, as it has been proved.

#4. Forms of the Riemann \( \zeta \) Functional Equations

The Riemann \( \zeta \) function has three types of zeros: [7]
\[
\begin{align*}
\zeta(1-s) & = 2 \cdot (2\pi)^{-s} \cdot \cos(\pi \cdot s / 2) \cdot \Gamma(s) \cdot \zeta(s), \text{Res} > 0 \quad (I, II) \\
\zeta(s) & = 2 \cdot (2\pi)^{-s} \cdot \sin(\pi \cdot s / 2) \cdot \Gamma(1-s) \cdot \zeta(1-s), \text{Res} < 1
\end{align*}
\]
Eq. Set And
\[
\zeta(s) = \frac{\Gamma(1-s)}{\Gamma(\frac{1-s}{2})} \zeta(1-s) \quad (III)
\]
Usually referred to as the trivial zeros, and non-trivial complex zeros. Therefore has been proved that any non-trivial zero lies in the open strip \( \{ s \in C : 0 < \text{Re}(s) < 1 \} \) which is called the critical Riemann Strip: And all complex zeros of the function \( \zeta \) lie in the line \( \{ s \in C : \text{Re}(s) = 1 / 2 \} \) which is called the critical line.

Solving of the Riemann \( \zeta \) Functional Equations.

To find the imaginary part we must solve the functional equations (Eq. set), and for the cases which the real part of the roots lies on the critical line. Cases # 4.1 & # 4.2 [6].

# 4.1: 1st type roots of the Riemann zeta functions (1st equation from the Eq. set).

For the first category roots and by taking the logarithm of two sides of the equations, and thus we get ... [5]
\[
\zeta(1-z) / \zeta(z) = 2(2\pi)^{-s} \cos(\pi \cdot z / 2) \Gamma(z) \Rightarrow \log[\zeta(1-z) / \zeta(z)] = \log[2] - z \log[2\pi] + \log[\cos(\pi \cdot z / 2) \Gamma(z)] + 2k\pi
\]
and the total form from the theory of Lagrange inversion theorem, [5] for the root is \( p_i(z) = s \) which means that \( f(s) = p_i^(-1)(z) = z \), but with an initial value for \( s \) which is
\[
s = \frac{\log[2]}{\log[2\pi]} + \frac{2 \cdot k \cdot \pi \cdot i}{\log[2\pi]},
\]
by setting values for \( k \) we can simply calculate the roots of the (1st equation from the Eq. set)...

Therefore, for the first six roots for \( z \) we take:
1st 0.377145562795
2nd 0.377145562795 +/− 3.41871903296
3rd 0.377145562795 +/− 6.8374806592
4th 0.377145562795 +/− 10.256157098
5th 0.377145562795 +/− 13.6748761318
6th 0.377145562795 +/− 17.0935951646

A simple program in mathematica is..
\[
k \geq 4; \quad t := (\log[2] + 2 \cdot k \cdot \pi \cdot i) / \log[2\pi];
\]
\[
z = \sum_{n=1}^{\infty} \left( (1 - \log[2\pi]) / \Gamma(n+1) \right) \sin(n \cdot \pi \cdot t) \cdot (\cos(\log[\Gamma(n+1)]) + \log[\zeta(n+1)])^0, \quad n \in \mathbb{N} \quad \text{for} \quad k = 0, \pm 1, \pm 2, ...
\]
But because the infinite sum approaching zero, theoretically \( x \) gets the initial value
\[
z = \frac{\log[2]}{\log[2\pi]} + \frac{2 \cdot k \cdot \pi \cdot i}{\log[2\pi]},
\]
So we have in this case, in part, the consecutive intervals with \( k = n \) and \( k = n+1 \) for any \( n \geq 4 \) and for the imaginary roots.

#4.2: 2nd type of roots of the Riemann zeta functional equations (2nd equation from the Eq. set).

Same as in the first category roots by taking the logarithm of two sides of the equations, and thus we get:
Theorem 1.7: \( \zeta(z) = 2(2\pi)^{-1} \sin(\pi z/2) \Gamma(1-z) \Rightarrow \)
\[ \text{Log}[\zeta(z)/\zeta(1-z)] = \text{Log}[2] + (z-1) \text{Log}[2\pi] + \text{Log}[\sin(\pi z/2) \Gamma(1-z)] + 2 \cdot k \cdot \pi \cdot i. \]

Now, we will have for total roots of 3 groups fields (but I have interest for the first group), and therefore for our case we will get \( p_i(z) = s \) which means that:
\[ f(s) = p_i^{-1}(s) = z. \]

But with an initial value for \( s \) that is
\[ s = \frac{\text{Log}[\pi]}{\text{Log}[2\pi]} + \frac{2 \cdot k \cdot \pi \cdot i}{\text{Log}[2\pi]} \]

the overall form from the Lagrange inverse theory succeeds after replacing the above initial value and therefore for the first six roots after calculating them we quote the following table...

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<th>#2</th>
<th>#3</th>
<th>#5</th>
<th>#6</th>
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<td>-/+</td>
<td>3.418719032961</td>
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</tr>
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<td>-/+</td>
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<tr>
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<td>-/+</td>
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</tr>
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</tr>
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<td>-/+</td>
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</tr>
<tr>
<td>6th</td>
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<td>-/+</td>
<td>3.418719032961</td>
<td>6.837438065921</td>
</tr>
</tbody>
</table>

A simple program in Mathematica is...

The function \( \zeta(z) \) is transcendental, and it is related to the Riemann hypothesis. The main results presented in the next few sections are transcendental equations satisfied by individual zeros of some L-functions. For simplicity, we first consider the Riemann-function, which is the simplest Dirichlet L-function.

**4.3: Transcendental equations for zeros of the function (Explicit formula)**

The main new results presented in the next few sections are transcendental equations satisfied by individual zeros of some L-functions. For simplicity, we first consider the Riemann-function, which is the simplest Dirichlet L-function.

*Asymptotic equation satisfied by the n-th zero on the critical line.* [9, 10, 11]

As above, let us define the function
\[ \chi(z) = \pi^{-z/2} \Gamma(z/2) \zeta(z). \]

which satisfies the functional equation
\[ \chi(z) = \chi(1-z). \]

Now consider the Stirling’s approximation
\[ \Gamma(z) = \sqrt{2\pi}z^{-1/2}e^{-z} (1 + O(z^{-1})) \]

where \( z = x + iy \), which is valid for large \( y \). Under this condition, we also have
\[ z^s = \exp \left( i \left( y \log y + \frac{\pi y}{2} \right) + x \log y - \frac{\pi y}{2} + x + O(y^{-1}) \right). \]

Therefore, using the polar representation
\[ \zeta = |\zeta| e^{i \arg \zeta}, \]

and the above expansions, we can write
\[ A(x, y) = \sqrt{2\pi} \pi^{-1/2} \left( \frac{y}{2} \right)^{(1-n)/2} e^{-\pi y/4} \left[ \zeta(x + iy) (1 + O(z^{-1})) \right], \]
\[ \theta(x, y) = \frac{y}{2} \log \left( \frac{y}{2\pi e} \right) + \frac{\pi}{4} (x - 1) + \arg \zeta(x + iy) + O(y^{-1}). \]

The final transactions we end up with
\[ n = \frac{y}{2\pi} \log \left( \frac{y}{2\pi e} \right) - \frac{5}{6} + \lim_{\delta \to 0} \frac{1}{\pi} \arg \zeta \left( \frac{1}{2} + \delta + iy \right) . \]
Establishing the convention that zeros are labeled by positive integers, \( z_n = 1/2 + i \cdot y_n \)
where \( n = 1, 2, 3, 4 \ldots \), we must replace \( n \to n - 2 \).
Therefore, the imaginary parts of these zeros satisfy the transcendental equation
\[
\frac{\ln}{2\pi} \log \left( \frac{\ln}{2\pi \varepsilon} \right) + \frac{1}{\delta - \alpha + 1} \cdot n \cdot \arg \zeta \left( \frac{1}{2} + \delta + it_n \right) = n - \frac{11}{8},
\]
eq A
Let us recall the definition used in, namely:
\[
S(y) = \frac{1}{\delta - \alpha + 1} \cdot \arg \zeta \left( \frac{1}{2} + i \cdot y \right) = \frac{1}{\delta - \alpha + 1} \cdot \arg \left( \log \zeta \left( \frac{1}{2} + \delta + iy \right) \right)
\]
These points are easy to find, since they do not depend on the fluctuating \( S(y) \).
We have:
\[
\zeta \left( \frac{1}{2} - iy \right) = \zeta \left( \frac{1}{2} + iy \right) \cdot G(y), \quad G(y) = e^{iy(\log y)}
\]
The Riemann-Siegel # function is defined by
\[
\vartheta(y) \equiv \arg \Gamma \left( \frac{1}{4} + \frac{1}{2}iy \right) - y \log \sqrt{\pi},
\]
Since the real and imaginary parts are not both zero, at \( y(+)n \) then \( G = 1 \), whereas at \( y(-)n \) then \( G = -1 \).
Thus
\[
\Re \left[ \zeta \left( \frac{1}{2} + iy_{n}^{(+)n} \right) \right] = 0 \quad \text{for} \ \vartheta(y_{n}^{(+)n}) = (n - 1)\pi,
\]
\[
\Re \left[ \zeta \left( \frac{1}{2} + iy_{n}^{(-)n} \right) \right] = 0 \quad \text{for} \ \vartheta(y_{n}^{(-)n}) = (n - \frac{1}{2})\pi.
\]
they can be written in the form of Eq. b
\[
y_{n}^{(+)n} = \frac{2\pi (n - 7/8)}{W \left[ e^{-\pi(n - 7/8)} \right]}, \quad y_{n}^{(-)n} = \frac{2\pi (n - 3/8)}{W \left[ e^{-\pi(n - 3/8)} \right]},
\]
where above \( n = 1, 2 \ldots \) and the Labert – Function \( W \) denotes the principal branch \( W(0) \). The \( y(+)n \) are actually the Gram points. From the previous relation, we can see that these points (Fig. 5) are ordered in a regular manner \([8,12]\)
\[
y_{1}^{(+)n} < y_{1}^{(-)n} < y_{2}^{(+)n} < y_{2}^{(-)n} < y_{3}^{(+)n} < y_{3}^{(-)n} < \ldots
\]
Fig.5 The \( y(+/-)n \) are actually the Gram points.
With this method, we are able to create intervals so as to approximate the correct values of imaginary part of non-trivial zeros. We did something similar in the cases #4.2. What it has left, is to find a method that approximates the values of imaginary part accurately.

The zeros of the Riemann Zeta function.

“PROGRAMMING”

#5.1 The M Function – Bisection Method...
Knowing the time of the successive steps \((k, k+1)\) of the relationship of imaginary parts
\[
\text{Im}(z) = \frac{2 \cdot \pi \cdot k}{\log[2\pi]},
\]
\[
\text{Im}(z) = \frac{2 \cdot \pi \cdot k}{\log[\pi]} \quad \text{(Eq I, II, III, page 7)} \quad \text{with} \ k \in \mathbb{N},
\]
can calculate the roots by solving the equation
\[
\zeta \left( \frac{1}{2} + i \cdot y \right) = 0 \quad \text{using the Bisection Method.}
\]
Bisection is the division of a given curve, figure, or interval into two equal parts (halves). A simple bisection procedure for iteratively converging on a solution, which is known to lie inside some interval \([a,b]\) proceeds by evaluating the function in question at the midpoint of the original interval \( y = \frac{a + b}{2} \) and testing to see in which of the subintervals \([a,(a+b)/2]\) or \([(a+b)/2],b]\) the solution lies. The procedure is then repeated with the new interval as often as needed to locate the solution to the desired accuracy. Let \( a_{n}, b_{n} \) be the endpoints at the \( n \) th iteration (with \( a_{1} = a \) and \( b_{1} = b \) ) and let

![Fig.6](image6.png)
error,

\( r_n \) be the \( n \) th approximate solution. Then, the number of iterations required to obtain an error smaller than \( \varepsilon \) is found by noting that...

\[
\left| r_n - a_n \right| = \frac{b - a}{2^n} \quad \text{and that } r_n \text{ is defined by } r_n = \frac{1}{2}(a_n + b_n).
\]

In order for the error to be smaller than \( \varepsilon \), then...

\[
\left| r_n - a_n \right| = \frac{1}{2}(b_n - a_n) = 2^{-n}(b-a) < \varepsilon .
\]

Taking the natural logarithm of both sides gives:

\[-n \ln 2 < \ln (b-a)\] . Therefore, we have for steps:

\[
n > \frac{\ln (b-a) - \ln 2}{\ln 2} .
\]

#5.2: M-function of the Bisection Method.

We define the functions \( M_+^d, M_-^d \) on an interval \((a,b)\) according to the scheme:

I. \( M_+^d \), with \( M_{d-1} \) the Nearest larger of \( M_{d-m} \)

\[
\frac{M_{d-1} + M_{d-m}}{2}, d \geq 2, d > m \geq d - 2
\]

\[
M_i = \frac{a + b}{2}, k = 1
\]

II. \( M_-^d \), with \( M_{d-1} \) the Nearest smaller of \( M_{d-m} \)

\[
\frac{M_{d-1} + M_{d-m}}{2}, d \geq 2, d > m \geq d - 2
\]

\[
M_i = \frac{a + b}{2}, k = 1
\]

For calculating the roots on solving the equation

\[
\zeta(1/2 + i \cdot y) = 0\]

take the limit \( R_S \) according to the scheme:

\[
r_s = \lim_{n \to \infty} M^*_n = \lim_{n \to \infty} M^-_n, s \in N \text{ with } M_+^d \text{ and } M_-^d,
\]

belong in the interval \((a, b), d \in N\) and also

\[
\left( a = \frac{2 \cdot \pi \cdot k}{\log(2\pi)} \right), b = \frac{2 \cdot \pi \cdot (k+1)}{\log(2\pi)}, k \in N, k \geq 3 \]

for the x of \( \zeta(1/2 + i \cdot r) = 0 \).

#5.3 Program in Mathematica for the Bisection method of \( \zeta(1/2 + i \cdot y) = 0 \).

Using the Intervals \( \left( a = \frac{2 \cdot \pi \cdot k}{\log(2\pi)} \right), b = \frac{2 \cdot \pi \cdot (k+1)}{\log(2\pi)} \)

successive steps, we can compute all the roots of \( \zeta(1/2 + i \cdot y) = 0 \).

We can of course use three types such intervals more specifically in general than

1. \( a = \frac{2 \cdot \pi \cdot k}{\log(2\pi)} \), \( b = \frac{2 \cdot \pi \cdot (k+1)}{\log(2\pi)} \)

or

2. \( a = \frac{2 \cdot \pi \cdot k}{\log(2\pi)} \), \( b = \frac{2 \cdot \pi \cdot (k+1)}{\log(2\pi)} \)

or

3. Were \( a = y_{n}, b = y_{n}' \).

\[
\begin{align*}
y_{n}^{(a)} & = \frac{2\pi(n-7/8)}{W[e^{-((n-7/8)}]} \quad y_{n}^{(b)} = \frac{2\pi(n-3/8)}{W[e^{-((n-3/8)}]}
\end{align*}
\]

We always prefer an interval that is shorter, in order to locate fewer non trivial zeros.

The most important is to calculate all the roots in each successive interval and therefore only then we will have the program for data {example: Integer k = 4, and

\[
(a = \frac{8 \cdot \pi}{\log(2\pi)}, b = \frac{10 \cdot \pi}{\log(2\pi)}) \quad \text{and Error approximate tol = 10^-6 and Trials n=22} \).
\]

“Programm for Bisection method”

A program relevant by dividing intervals...

This program gives very good values as an approach to the roots we ask if we know the interval. Selecting the interval for the case

\[
(a = \frac{2 \cdot \pi \cdot k}{\log(2\pi)} , b = \frac{2 \cdot \pi \cdot (k+1)}{\log(2\pi)}), k \in N, k \geq 3 \]

the results are given below..
With final value 14.1347216500, error near of \(10^{-7}\).

This value is the approximate root of the \(\zeta(1/2 + i \cdot y) = 0\) by the nearest error \(<10^{-7}\).

In the event that we have two or more uncommon roots within the interval, we divide similar successive intervals in the order of finding of the first root either above or below. In such a case we have the \(k = 13\), and at the interval \((47.8620664, 51.280785)\) the two roots are 48.0051088 and 49.7738324.

#5.4 Explicit formula and the Zeros of \(\zeta(1/2 + i \cdot y) = 0\)

Consider its leading order approximation, or equivalently its average since \(\arg \zeta(1/2 + i \cdot y) > 0\)...

Then we have the transcendental equation

\[
x_n \log \left( \frac{t_n}{2\pi e} \right) = n - \frac{11}{8}
\]

Through the transformation

\[
x_n e^{t_n} = e^{-1} \left( n - \frac{11}{8} \right)
\]

this equation can be written as

\[
x_n e^{t_n} = e^{-1} \left( n - \frac{11}{8} \right)
\]

Comparing the previous results, we obtain

\[
t_n = \frac{2\pi(n - \frac{11}{8})}{W[e^{-1} (n - \frac{11}{8})]} \quad \text{where } n = 1,2,3,....
\]

#5.5 Program by Newton’s method, which finds the Zeros of \(\zeta(1/2 + i \cdot y) = 0\)

Using Newton’s method we can reach the roots of the equation \(\zeta(1/2 + i \cdot y) = 0\) at a very good initial value from \(y \rightarrow t_n\) by the explicit formula.

It follows a mathematica program for the first 50 roots by the Newton’s method. This method determines and detects the roots at the same time in order to verify the relation \(\zeta(z) = 0\) and always according to the relation...

\[
t_n = \frac{2\pi(n - \frac{11}{8})}{W[e^{-1} (n - \frac{11}{8})]}
\]

where \(n = 1,2,3,....\) and \(W\) is the W-function.

#5.6 Directly (from Explicit form) with the solution of this equation

\[
\left( \frac{t_n}{2\pi e} \right) = e^{-1} \left( n - \frac{11}{8} \right)
\]

Using the initial value the relation

\[
t_n = \frac{2\pi(n - \frac{11}{8})}{W[e^{-1} (n - \frac{11}{8})]} \quad \text{the explicit formula.}
\]
As already discussed, the function \( \text{Arg}\zeta(1/2 + \delta + i \cdot y) \) oscillates around zero. At a zero it can be well-defined by the limit, which is generally not zero. For example, for the first Riemann zero \( y_1 = 14.1347 \) the limit \( \delta \to 0^+ \) has value as \( \lim \text{Arg}\zeta(1/2 + \delta + i \cdot y) = 0.157873 \).

The \( \text{Arg}\zeta \) term plays an important role and indeed improves the estimate of the \( n \)-th zero.

We can calculate by Newton’s method, and we locate the first 30 imaginary part roots, of the equation \( \zeta(1/2 + i \cdot y) = 0 \) on the bottom of the table...

Table:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>Im(1/2+yi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0.157873</td>
</tr>
<tr>
<td>2</td>
<td>0.01</td>
<td>0.157873</td>
</tr>
<tr>
<td>3</td>
<td>0.02</td>
<td>0.157873</td>
</tr>
<tr>
<td>4</td>
<td>0.03</td>
<td>0.157873</td>
</tr>
<tr>
<td>5</td>
<td>0.04</td>
<td>0.157873</td>
</tr>
<tr>
<td>6</td>
<td>0.05</td>
<td>0.157873</td>
</tr>
<tr>
<td>7</td>
<td>0.06</td>
<td>0.157873</td>
</tr>
<tr>
<td>8</td>
<td>0.07</td>
<td>0.157873</td>
</tr>
<tr>
<td>9</td>
<td>0.08</td>
<td>0.157873</td>
</tr>
<tr>
<td>10</td>
<td>0.09</td>
<td>0.157873</td>
</tr>
</tbody>
</table>

This analysis has proved the Riemann Hypothesis is correct, since the real part of the non-trivial zero-equation functions is always constant and equals 1/2. This proven by 3 independent methods. Theorems {Th 1.1-II-page 1-5, Theorem 2-pages 5-6, Theorem 3-pages 5-7}. This analysis has also demonstrated that the imaginary part of the non-trivial zero zeta function accepts certain values according to the intervals defined by the solution of the 3 based functional equations {#4-I,II,III page 5}.

Epilogue...

https://www.academia.edu/21964173/Solve_Equations_With_method_Lagrange
Distribution of primes numbers and Zeros

#1. Introduction
The distribution of prime numbers is most simply expressed as the (discontinuous) step function \( \pi(x) \), where \( \pi(x) \) [Fig.] is the number of primes less than or equal to \( x \).

It turns out that \( \pi(x) \) can be expressed exactly as the limit of a sequence of smooth functions \( R_n(x) \). To define \( R_n(x) \) we first introduce the logarithmic integral function \( \text{Li}(x) \), which appears throughout the analytical theory of the prime distribution:

\[
\text{Li}(x) = \int_2^x \frac{du}{\log u}
\]

This is a smooth function which simply gives the area under the curve of the function \( 1/\log u \) in the interval \([2,x]\). Don Zagier explains the reasoning behind the function \( \text{Li} \) in his excellent introductory article "The first 50 million prime numbers" from [2], based on his inaugural lecture held at Bonn University, May 5, 1975: "A good approximation to \( \pi(x) \), which was first given by Gauss is obtained by taking as starting point the empirical fact that the frequency of prime numbers near a very large number \( x \) is almost exactly \( 1/\log x \).

From this, the number of prime numbers up to \( x \) should be approximately given by the logarithmic sum:

\[
\text{Li}(x) = \frac{1}{\log 2} + \frac{1}{\log 3} + \cdots + \frac{1}{\log x}
\]

or, what is essentially the same, by the logarithmic integral:

\[
\text{Li}(x) = \int_2^x \frac{du}{\log u}
\]

Using \( \text{Li}(x) \) we then define another smooth function, \( R(x) \), first introduced by Riemann in his original eight-page paper, and given by

\[
R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \text{Li}(x^{1/n})
\]

Riemann's research on prime numbers suggests that the probability for a large number \( x \) to be prime should be even closer to \( 1/\log x \) if one counted not only the prime numbers but also the powers of primes, counting the square of a prime as half a prime, the cube of a prime as a third, etc. This leads to the approximation:

\[
\pi(x) + \frac{1}{2} \pi(x^{1/2}) + \frac{1}{3} \pi(x^{1/3}) + \cdots \approx \text{Li}(x)
\]

or, equivalently [by means of the Möbius inversion formula]

\[
\pi(x) \approx \text{Li}(x) - \frac{1}{2} \text{Li}(x^{1/2}) - \frac{1}{3} \text{Li}(x^{1/3}) - \cdots
\]

The function on the right side of this formula is denoted by \( R(x) \), in honour of Riemann. It represents an amazingly good approximation to \( \pi(x) \). For those in the audience who know a little function theory, perhaps I might add that \( R(x) \) is an entire function of \( \log x \), given by the rapidly converging power series:

\[
R(x) = 1 + \sum_{k=1}^{\infty} \frac{(\log x)^k}{kk!\zeta(k+1)}
\]
where $\zeta(\kappa + 1)$ is the Riemann zeta

\[ R(x) = 1 + \sum_{k=1}^{\infty} \frac{(\ln x)^k}{k! \zeta(k + 1)} \]

The sum over $\rho$ separates into two sums, over the
trivial and nontrivial zeros, respectively. The former sum
is of course just $R(x^2) + R(x^4) + R(x^6) + \cdots$, and the latter

\[ \sum_{k=1}^{\infty} \left[ R(x^{2k}) + R(x^{2k-1}) \right] \]

The contributions from the complex-conjugate
pairs $\rho_\kappa$ and $\rho_{-\kappa} = \overline{\rho}_\kappa$ cancel each others' imaginary
parts, so

\[ \pi(x) = R(x) - \sum_{m=1}^{\infty} R(x^{-2m}) + \sum_{k=1}^{\infty} T_k(x) \]

Where

\[ T_k(x) = -R(x^{\rho_k}) - R(x^{\rho_{-k}}) \]

are real-valued. We can now define the sequence of
functions $R_n(x)$ which approximate $\pi(x)$ in limit:

\[ R_n(x) = R(x) - \sum_{m=1}^{\infty} R(x^{-2m}) + \sum_{k=1}^{n} T_k(x) \]

\[ f(x) = -2 \Re \left( \sum_{k=1}^{n} \text{Ei}(\rho_k \log(x)) \right) + \int_{1}^{x} \frac{1}{(\log(t))^2} \log(\log(t)) \]

where $\rho_k$ is the $k^{th}$ complex zero of the zeta function.

In this formula, $\text{Ei}(z)$ (Mathematica’s built-in
function ExpIntegral Ei(z)) is the generalization of
the logarithmic integral to complex numbers. These
equations come from references [1], [2], and [3]. First,
let $M$ be the smallest integer such that $x^{1/M} < 2$.
We need to add only the first $M-1$ terms (that is, $n = 1, 2, \ldots, M$ ) in the sum in equation (1). For each of
these values of $N$, we use equation (2) to compute the
value of $f(x^{1/n})$. However, we will add only the first
$N$ terms (that is, $\kappa = 1, 2, \ldots, N$ ) in the sum in equation
(2). Because the purpose of this Demonstration is to
show how the jumps in the step function $\pi(x)$ can be
closely approximated by adding to $R(x)$ a correction
term that involves zeta zeros, we ignore the integral and
the log2 in second equation; this speeds up the
computation and will not noticeably affect the graphs,
especially for $x$ more than about 5. The more zeros we
use, the closer we can approximate $\pi(x)$ . For larger $x$,
the correction term must include more zeros in order to
accurately approximate $\pi(x)$.
#2. The GUE hypothesis.

While many attempts to prove the RH had been made, a few amount of work has been devoted to the study of the distribution of zeros of the Zeta function. A major step has been done toward a detailed study of the distribution of zeros of the Zeta function by Hugh Montgomery [6], with the Montgomery pair correlation conjecture. Expressed in terms of the normalized spacing ..

\[ \delta_n = (\gamma_{n+1} - \gamma_n) \log (\frac{\gamma_n/(2\pi)}{2\pi}) \]

this conjecture is that, for \( M \to \infty \)

\[ \frac{1}{M} \# \{ n : N + 1 \leq n \leq N + M, \delta_n \in [\alpha, \beta] \} \sim \int_\alpha^\beta p(0, u) du. \]

In other words, the density of normalized spacing between non-necessarily consecutive zeros is \( 1 - (\frac{\sin \pi u}{\pi u})^2 \). It was first noted by the Freeman Dyson, a quantum physicist, during a now-legendary short teatime exchange with Hugh Montgomery[6], that this is precisely the pair correlation function of eigenvalues of random hermitian matrices with independent normal distribution [ In figure 3&4] of its coefficients. Such random hermitian matrices are called the Gauss unitary ensemble (GUE). As referred by Odlyzko in [4] for example, this motivates the GUE hypothesis which is the conjecture that the distribution of the normalized spacing between zeros of the Zeta function is asymptotically equal to the distribution of the GUE eigen values. Where \( p(0, u) \) is a certain probability density function, quite complicated to obtain for an expression of it. As reported by Odlyzko in [4], we have the Taylor expansion around zero.

\[ p(0, u) = \frac{\pi^2}{3} u^2 - 2\frac{\pi^4}{45} u^4 + ... \]

which under the GUE hypothesis entails that the proportion of \( \delta_n \) less than a given small value \( \delta \) is asymptotic to \( \frac{\pi^2}{9} \delta^3 + O(\delta^5) \). Thus very close pair of zeros are rare.

![Fig3. Probability density of the normalized spacing between non-necessarily consecutive zeros and the GUE prediction[8]](image1)

![Fig4. Probability density of the normalized spacing \( \delta_n \) and the GUE prediction[8]](image2)

#3. Gaps between zeros..

The table below lists the minimum and maximal values of normalized spacing between zeros \( \delta_n \) and of \( \delta_{n+1} \), and compares this with what is expected under the GUE hypothesis. It can be proved that \( p(0, t) \) have the following Taylor expansion around 0

\[ p(0, u) = \frac{\pi^2}{3} u^2 - 2\frac{\pi^4}{45} u^4 + ... \]

so in particular, for small delta

\[ \text{Prob}(\delta_n < \delta) = \int_0^\delta p(0, u) du \sim \frac{\pi^2}{9} \delta^3 \]

so that the probability that the smallest \( \delta_n \) are less than \( \delta \) for M consecutive values of \( \delta_n \) is about.

\[ 1 - \left( 1 - \frac{\pi^2}{9} \delta^3 \right)^M \approx 1 - e^{\left(-\frac{\pi^2}{9} \delta^3 M\right)} \]
This was the value used in the sixth column of the table.
The result can be also obtained for the $\delta n + \delta n+1$.

$$\text{Prob}(\delta n + \delta n+1 < \delta) \sim \frac{\pi \delta}{32400 \delta^8},$$
from which we deduce the value of the last column.

<table>
<thead>
<tr>
<th>Height</th>
<th>$\min \delta_n$</th>
<th>Maxi $\delta_n$</th>
<th>$\min (\delta_n + \delta_{n+1})$</th>
<th>Maxi $(\delta_n + \delta_{n+1})$</th>
<th>$\min \delta_{in GUE}$</th>
<th>$\min \delta_{in GUE}$</th>
<th>$\min \delta_{in GUE}$</th>
<th>$\min \delta_{in GUE}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^3</td>
<td>0.0005320</td>
<td>4.127</td>
<td>0.0977</td>
<td>5.292</td>
<td>0.28</td>
<td>0.71</td>
<td>0.71</td>
<td>0.71</td>
</tr>
<tr>
<td>10^4</td>
<td>0.0000629</td>
<td>4.236</td>
<td>0.1233</td>
<td>5.429</td>
<td>0.87</td>
<td>0.94</td>
<td>0.94</td>
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</tr>
<tr>
<td>10^5</td>
<td>0.000015171</td>
<td>4.345</td>
<td>0.1903</td>
<td>5.341</td>
<td>0.26</td>
<td>0.46</td>
<td>0.46</td>
<td>0.46</td>
</tr>
<tr>
<td>10^6</td>
<td>0.00002020</td>
<td>4.382</td>
<td>0.0929</td>
<td>5.343</td>
<td>0.27</td>
<td>0.53</td>
<td>0.53</td>
<td>0.53</td>
</tr>
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<td>4.358</td>
<td>0.0866</td>
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<td>0.26</td>
<td>0.26</td>
<td>0.26</td>
</tr>
<tr>
<td>10^8</td>
<td>0.00014390</td>
<td>4.194</td>
<td>0.0980</td>
<td>5.511</td>
<td>0.17</td>
<td>0.67</td>
<td>0.67</td>
<td>0.67</td>
</tr>
<tr>
<td>10^9</td>
<td>0.000145695</td>
<td>4.200</td>
<td>0.0974</td>
<td>5.341</td>
<td>0.24</td>
<td>0.18</td>
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</tr>
<tr>
<td>10^10</td>
<td>0.00013531</td>
<td>4.268</td>
<td>0.0976</td>
<td>5.717</td>
<td>0.17</td>
<td>0.63</td>
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</tr>
<tr>
<td>10^11</td>
<td>0.00019391</td>
<td>4.316</td>
<td>0.1019</td>
<td>5.241</td>
<td>0.23</td>
<td>0.50</td>
<td>0.50</td>
<td>0.50</td>
</tr>
<tr>
<td>10^12</td>
<td>0.00081615</td>
<td>4.347</td>
<td>0.0960</td>
<td>5.332</td>
<td>0.79</td>
<td>0.61</td>
<td>0.61</td>
<td>0.61</td>
</tr>
<tr>
<td>10^13</td>
<td>0.00070199</td>
<td>4.304</td>
<td>0.1112</td>
<td>5.478</td>
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<td>0.75</td>
<td>0.75</td>
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</tr>
<tr>
<td>10^14</td>
<td>0.00027099</td>
<td>4.158</td>
<td>0.0977</td>
<td>5.526</td>
<td>0.05</td>
<td>0.19</td>
<td>0.19</td>
<td>0.19</td>
</tr>
</tbody>
</table>

For very large spacing in the GUE, as reported by Odlyzko in [4], des Cloizeaux and Mehta [5] have proved that

$$\log p(0, t) \sim -\pi^2 t^2/8 \quad (t \to \infty),$$

which suggests that

$$\max_{N+1 \leq n \leq N+M} \delta_n \sim \frac{(8 \log M)^{1/2}}{\pi}.$$  

Statistics of False zetazers, $\delta$-intervals and count of primes.

#4. General equations.

With this statistic we find the crowd of individual parts intervals defined by consecutive Zetazers in a fixed integer interval. Here we use $\delta = 1000$. We have the general Equation $\delta + Zetazer (kin) - Zetazer (kf) = 0$ and after given initial value in kin, i calculate the $kf$ usually by the Newton method. From the initial and final value of $kin, kf$ and by performing the process of successive intervals, i calculate the number(count) of the primes ones that are within the intervals. In this way i will have consecutive intervals.

$$S_k = \{Zetazer(k+1) < p < Zetazer(k), k \in (kin,kf)\}$$

from where resulting the number of the primes in the given consecutive interval. The sum of the primes in the given interval $\delta = 1000$ will be obvious.

$$S_\delta = \bigcup_{k=1}^{k} S_k,$$

The False intervals will be in a normalized form

$$F_\delta = 1000 - S_\delta \quad \text{over} \ N >= 4000.$$

#5. The statistics.

The Statistic looking for ways to show us which function is the most ideal to get closer the points of interest, uses the NonlinearModelFit method of the function

$$y = (a + b \cdot x)/(\log(d + c \cdot x))$$

and therefore after determining the variables $\{a, b, c, d\}$ we are able to make statistical and probable prediction at higher levels of numbers.

This function is directly related to the function $\pi(x) = x / \log x$ which was reportedin the introduction that is, defining the number of primes numbers relative to $x$.

We will do a double statistic of the intervals $\{\delta, \text{False intervals}\}$ and $\{\delta, \text{number of primes}\}$ that we are ultimately interested in the statistics mainly the count of the primes inside at $\delta$-intervals.

#6. The first statistic is about count of False intervals and the count of the primes that corresponding in them. The table below gives it in aggregate until the count of primes equal $p_{17} = 29$ corresponding to the count of False $F_{17} = 971$ and the Total range Integers for a interval $D=1000$ to $2 \cdot 10^{14} + 1000.$

$$\text{data1} = \{(499, 162), (735, 129), (825, 121), (885, 114), (897, 112), (902, 109), (905, 107), (914, 104), (909, 101), (997, 97), (997, 103), (991, 99), (904, 95), (898, 102), (908, 92), (902, 98), (912, 88), (902, 94), (908, 92), (902, 98), (906, 96), (902, 98), (912, 88), (908, 92), (905, 95), (912, 88), (908, 92), (941, 86), (960, 90), (962, 89), (952, 75), (940, 60), (939, 61), (930, 70), (938, 62), (946, 54), (949, 61), (946, 54), (945, 55), (955, 45), (944, 56), (942, 58), (951, 49), (956, 44), (962, 38), (953, 47), (956, 44), (955, 45), (961, 39), (953, 47), (960, 40), (955, 45), (966, 36), (963, 37), (961, 39), (971, 29), (974, 26), (963, 37), (965, 35), (962, 38), (949, 30), (970, 30), (966, 34), (964, 36), (970, 30), (976, 24), (973, 27), (971, 29)).$$

The diagram given below[Fig.5] shows the arrangement of the points at the level $(x, y)$ according to the data1[7].
18

Fig 5. The depiction of the intervals False and the primes ones (file:data1) contained in the intervals $N + \delta - N = 1000$.

Even more macroscopically [Fig.6] the points are shown by the line $y = 988.709 - 0.988372x$, which, as they appear, are stacked on its lower right.

![Fig 6 Graphic depiction of the line $y = 988.709 - 0.988372x$ with the archive(data1) points.]

It is obvious that for $\{x = 999, y = 1.32\}$ and for $\{y = 1000, x = 0.33\}$, a value located below the unit and means that it can in such a interval, and we are talking about this is for high order integers intervals so there is not one prime within the interval $\delta = 1000$, chosen at random.

#7.. The second statistic refers to the count of the primes ones that located in the intervals $[N, N + \delta]$ with $N = 1000$ up to $2 \cdot 10^{14}$, range $\delta = 1000$ and the count of the primes within successive ZetaZeros. After we found the roots $\text{f}'$ of general Equation $\delta + \text{ZetaZero}(\text{kin}) - \text{ZetaZero}(\text{kf}) = 0$ of given initial value $\text{kin}$, usually by the Newton method. The data[7] that we have met with the above method are.

Using the NonlinearModelFit[7] method of the function $y = (a + c \times x) / (\log(d + b \times x)$ and after specifying the variables $\{a, b, c, d\}$ the following result the function will be $\text{y} = 1108.254246288494 - 1.11612970664 \times 10^{-12} \times \log[-4503.9036177023 + 5.42563517140301x]$ with very good approach and value performance in each pairing. The diagram [Fig.7] of the above equation is shown in more detail below with gravity in the latest data.

![Fig 7 The graphic depiction of the function $y = (a+c*x)/\log(d+b*x)$ with a test value for $x = 2 \cdot 10^{14}$ gives us $y = 25.6$, close to 29 we took with the analysis.]

18
They follow statistics ANOVA and t-Statistic[7,8] and we get the tables …

### Analysis -ANOVA…

<table>
<thead>
<tr>
<th></th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
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<td>441724, 110431.</td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>74</td>
<td>1329.65 17.9683</td>
<td></td>
</tr>
<tr>
<td>Uncorrected Total</td>
<td>78</td>
<td>443054.</td>
<td></td>
</tr>
<tr>
<td>Corrected Total</td>
<td>77</td>
<td>71143.8</td>
<td></td>
</tr>
</tbody>
</table>

### Analysis - T-statistic…

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Standard Error</th>
<th>t-Statistic</th>
<th>p-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1108.25</td>
<td>8.7925</td>
<td>4.12076 $\times 10^{-6}$</td>
</tr>
<tr>
<td>b</td>
<td>$-1.11633 \times 10^{-12}$</td>
<td>5.63966 $\times 10^{-13}$</td>
<td>-1.97942</td>
</tr>
<tr>
<td>c</td>
<td>5.42564</td>
<td>0.186181</td>
<td>29.1417</td>
</tr>
<tr>
<td>d</td>
<td>$-4503.9$</td>
<td>0.15355</td>
<td>$-29330.6$</td>
</tr>
</tbody>
</table>

As we can see from the results, we have an important Standard Error only for the variable a. The other variables are observed to have a low statistical error and have good compatibility. By adapting the method as we see, we associate two lists of results of the number of primes and long intervals, which although disproportionately, work together with impeccable and good contact.

Ronald Fisher introduced the term variance and proposed its formal analysis in a 1918 article The Correlation Between Relatives on the Supposition of Mendelian Inheritance. His first application of the analysis of variance was published in 1921. Analysis of variance became widely known after being included in Fisher's 1925 book Statistical Methods for Research Workers. One of the attributes of ANOVA that ensured its early popularity was computational elegance. The structure of the additive model allows solution for the additive coefficients by simple algebra rather than by matrix calculations. In the era of mechanical calculators this simplicity was critical. The determination of statistical significance also required access to tables of the F function which were supplied by early statistics texts.

### #8. Standard Error and Confidence Interval

#### Part-1.

Where we observe, apart from 1-2 initial measurements, the Standard Error as well and the Confidence Interval is stabilized at good and acceptable values.

<table>
<thead>
<tr>
<th></th>
<th>Observed</th>
<th>Predicted</th>
<th>Standard Error</th>
<th>Confidence Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>162</td>
<td>162.532</td>
<td>4.7901</td>
<td>(154.625, 170.679)</td>
<td></td>
</tr>
<tr>
<td>129</td>
<td>125.574</td>
<td>0.85191</td>
<td>(124.678, 126.372)</td>
<td></td>
</tr>
<tr>
<td>121</td>
<td>118.432</td>
<td>0.739511</td>
<td>(110.119, 116.745)</td>
<td></td>
</tr>
<tr>
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Part-2.

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And as the above analysis of - False intervals- as we have shown, it is compatible with this result of law Gram’s.

#10. From above analysis they arise 3 big conclusions:

1st. The number(Count) of primes located within δ intervals is gradually decreasing with a higher order size n of 10^n.

2nd. The distribution of the number(Count) of the primes, within interval δ follows a Nonlinear correlation is expressed by the function 

\[ y = (a + b \cdot x) / (\log(d + c \cdot x)) \]

similar to the number (Count) of the primes \( \pi(x) \approx x / \log(x) \), which is approximate for large numbers.

3rd. The measurement(Count) of false intervals and the count of the primes follows a linear correlation and it increases if the order of magnitude size of the integer increases.

References ...

[7] The “data1 & data & NonlinearModelFit Statistic”, have been done with the program mathematica.

#9. Statistical comparison.

With statistical comparison it appears that as long as we moving to a higher order of integer-size that are within a given interval, we have chosen δ = 1000, the count of the primes diminishes to disappear or to there are 1-2 primes at high order intervals of more than 10^{10}. This also agrees with of Gram’s law. In particular, the problem of distribution of the differences \( t_{n+1} - t_n \) (that is of difference of ZetaZeros) is considered [9]. If we accept Gram’s law then the order of this difference does not exceed the quantity ...

\[ t_{n+1} - t_n \approx \frac{2\pi}{\ln(n)} \rightarrow 0, n \rightarrow \infty \]

for much larger integers [9,8] then mean and their mean value is close to,

\[ t_{n+1} - t_n \approx \frac{\ln \ln(n)}{\ln(n)} \rightarrow 0, n \rightarrow \infty \]