A four circle problem and division by zero

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Abstract. We generalize a problem involving four circles and a triangle, and consider some limiting cases of the problem by division by zero.

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1. Introduction

We generalize the following problem involving four circles and a triangle in [20]. The same sangaku problem was proposed in 1826 and cited in [19] and [1] with no solution. Some limiting cases of the problem will be considered by division by zero [6].

![Figure 1](image)

Problem 1. For a triangle $EFG$ with incircle $\alpha$, $\delta$ is the circle passing through $E$ and $F$ and touching $\alpha$, $\gamma$ is the incircle of the curvilinear triangle made by $\delta$ and the sides $EF$ and $GE$, and $\beta$ is the circle touching $\delta$ and $FG$ at the midpoint from the side opposite to $\alpha$. Let $a$, $b$ and $c$ be the radii of $\alpha$, $\beta$ and $\gamma$, respectively. Show $a^2 = 4bc$.

A similar sangaku problem considering the case $|EF| = |GE|$ can be found in [2, p. 302].

2. Generalization

The problem assumes that $\alpha$ is the incircle of $EFG$, but we show that the condition is inessential. We consider the following figure (see Figure 2): For a chord $FG$ of a circle $\delta$, $M$ is the midpoint of $FG$, $\beta$ is a circle touching $\delta$ and $FG$ at $M$, $\alpha$ is a circle touching $\delta$ and the chord $FG$ from the side opposite to $\beta$, $f$ and $g$ are the tangents of $\alpha$ from the points $F$ and $G$, respectively, $\gamma$ is the circle lying on the same side of $FG$ as $\alpha$ and touching $\delta$ externally and $f$ and $g$ from the same side
as \( \alpha \). Let \( a, b, c \) and \( d \) be the radii of \( \alpha, \beta, \gamma \) and \( \delta \), respectively. We denote this configuration by \( S \).

We use a rectangular coordinate system with origin \( M \) such that the center of \( \alpha \) has coordinate \((x_a, a)\) for a real number \( x_a \). Firstly we consider a special case in which \( f \) and \( g \) are parallel (see Figure 3).

**Theorem 1.** The following statements are equivalent for \( S \).

(i) The lines \( f \) and \( g \) are parallel.

(ii) The center of \( \alpha \) lies on the circle of diameter \( FG \).

(iii) \( a = 4b \).

**Proof.** We may assume that the point \( G \) has coordinates \((k, 0)\), and \( f \) and \( g \) have equations \( x + m_1y + k = 0 \) and \( x + m_2y - k = 0 \), respectively for real numbers \( m_1 \) and \( m_2 \). Since \( f \) and \( g \) touch \( \alpha \), we have

\[
(1) \quad m_1 = \frac{a^2 - (k + x_a)^2}{2a(k + x_a)}, \quad m_2 = \frac{a^2 - (k - x_a)^2}{2a(k - x_a)}.
\]

Notice that \( k^2 - x_a^2 \neq 0 \), since \( k^2 - x_a^2 = 0 \) implies that \( \alpha \) touches \( FG \) at \( F \) or \( G \). The lines \( f \) and \( g \) are parallel if and only if \( m_1 = m_2 \), which is equivalent to

\[
(2) \quad a^2 + x_a^2 = k^2.
\]

This proves the equivalence of (i) and (ii), since the left side equals the square of the distance between the center of \( \alpha \) and \( M \) (see Figure 3). While the square of the distance between the centers of \( \delta \) and \( \alpha \) equals

\[
(3) \quad x_a^2 + (d - 2b - a)^2 = (d - a)^2.
\]

And the power of the origin with respect to \( \delta \) equals

\[
(4) \quad -2b(2d - 2b) = -k^2.
\]

Eliminating \( d \) from (3) and (4), we get \( xa^2 + 4ab = k^2 \), which implies

\[
a^2 + xa^2 - k^2 = a(a - 4b).
\]

Hence (2) and \( a = 4b \) are equivalent, i.e., (i) and (iii) are equivalent. \( \square \)

**Corollary 1.** One of the three relations \( 4b < a < c \), \( 4b = a = c \), \( 4b > a > c \) holds for \( S \).
Figures 2, 3 and 4 show the cases \(4b > a > c\), \(4b = a = c\) and \(4b < a < c\), respectively. The next theorem is a generalization of Problem 1.

**Theorem 2.** The following statements hold.
(i) The relation \(a^2 = 4bc\) holds.
(ii) One of the internal common tangents of \(\alpha\) and \(\gamma\) is parallel to \(FG\).

**Proof.** We use the same notation as in the proof of Theorem 1. If \(f\) and \(g\) are parallel, we get \(a = c\). Therefore we get \(a^2 = 4bc\) by Theorem 1. We assume that \(f\) and \(g\) intersect. We denote the point of intersection by \(E\), which has coordinates

\[
(x_e, y_e) = \left( \frac{k(m_1 + m_2)}{m_1 - m_2}, \frac{-2k}{m_1 - m_2} \right).
\]

Substituting (1) in (5), we get

\[
(x_e, y_e) = \left( x_a - \frac{2a^2 x_a}{a^2 - k^2 + x_a^2}, 2a - \frac{2a^3}{a^2 - k^2 + x_a^2} \right).
\]

The square of the distance between the centers of \(\delta\) and \(\gamma\) equals

\[
x_c^2 + (d - 2b - y_e)^2 = (c + d)^2,
\]

where \((x_e, y_e)\) are the coordinates of the center of \(\gamma\). Since \(E\) is the external center of similitude of \(\alpha\) and \(\gamma\), we get

\[
\frac{-cx_a + ax_e}{a - c} = x_e, \quad \frac{-ca + ay_e}{a - c} = y_e.
\]

Eliminating \(x_a, x_e, y_e, x_e, y_e\) and \(d\) from (3), (4), (6), (7), (8), we get

\[
(a^2 - 4bc)j(k) = 0,
\]

where \(j(k) = 4(a - 4b)b^2 - (4b - c)k^2\). If \(j(k) = 0\), we have \(k^2 = 4(a - 4b)b^2/(4b - c) > 0\). This implies \(a < 4b < c\) or \(c < 4b < a\). However this contradicts Corollary 1. Therefore we get \(j(k) \neq 0\), which implies \(a^2 = 4bc\).

We prove (ii). If \(f\) and \(g\) are parallel, the centers of \(\alpha, \gamma\) and \(M\) are collinear, i.e., \(x_a/a = x_c/y_c\). Eliminating \(b, c, k, x_a, x_c\) from the equations \(x_a/a = x_c/y_c\), \(a = c, a = 4b\), (3), (4) and (7), we get

\[
(3a - y_e)((a + 4d)a + (4d - a)y_e) = 0.
\]

Therefore we get \(y_e = 3a = 2a + c\), since \((4d - a)y_e > 0\). If \(f\) and \(g\) intersect, we eliminate \(b, k, x_a, x_c, x_e, y_e\) from (3), (4), (6), (7), (8). Then we get

\[
(2a + c - y_e)((a + 4d)c + (4d - a)y_e) = 0.
\]

Therefore we get \(y_e = 2a + c\). This proves (ii). \(\square\)
There are several sangaku problems stating the next corollary [2, p. 312, p. 317, p. 419] (see Figure 5).

**Corollary 2.** For a semicircle $\delta$ with diameter $FG$, let $\alpha$ be the circle of radius $a$ touching $\delta$ and $FG$ at the midpoint. If $c$ is the inradius of the curvilinear triangle made by $\delta$ and the tangents of $\alpha$ from the points $E$ and $F$, then $a = 4c$.

The next corollary can be found in the sangaku hung in 1830 [3, p. 40], which is incorrectly cited in [1, p. 34] (see Figure 6).

**Corollary 3.** For the configuration $\mathcal{S}$, let $\alpha'$ be a circle of radius $a'$ touching the circle $\delta$ and its chord $FG$ from the side opposite to $\alpha$. If the inradius of the curvilinear triangle made by $\delta$ and the tangents of $\alpha'$ from $F$ and $G$ equals $c'$, then $a^2a'^2 = cc'|FG|^2$.

**Proof.** Let $b'$ be the radius of the circle touching $\delta$ and $FG$ at the midpoint from the side opposite to $\alpha'$. Then we have $a'^2 = 4b'c'$, while $|FG|^2 = 16bb'$ and $a^2 = 4bc$. Eliminating $b$ and $b'$ from the three equations, we get $a^2a'^2 = |FG|^2cc'$. \qed
3. Limiting cases with division by zero

In this section we fix the circle $\delta$ for $S$ and consider the case where one of the circles $\alpha$ and $\beta$ has radius 0 with the definition of division by zero [6]:

\begin{equation}
\frac{z}{0} = 0 \text{ for a complex number } z.
\end{equation}

Notice that the definition implies that lines have radius 0 as circles [17].

We now consider a simple case in which the centers of $\alpha$, $\beta$ and $\gamma$ are collinear for $S$ and use a rectangular coordinate system with origin at the point of tangency of $\beta$ and $\delta$ such that the center of $\delta$ has coordinates $(0, d)$. The point of tangency of $\gamma$ and $\delta$ and the tangent of $\delta$ at the point are denoted by $D$ and $t$, respectively (see Figure 7). Notice that $d = a + b$.

![Figure 7](image)

### 3.1. The case $b = 0$

Firstly we consider the case $b = 0$. Then $\beta$ is a point or a line. The circle $\alpha$ has an equation $x^2 + (y - (b + d))^2 = (b - d)^2$, which is arranged as

\begin{equation}
f_a(x, y) = (x^2 + (y - d)^2 - d^2) + 2b(2d - y) = 0.
\end{equation}

From $f_a = 0$, we get $x^2 + (y - d)^2 = d^2$ in the case $b = 0$. Also from $f_a/b = 0$ we get $y = 2d$ in the case $b = 0$ by (9). Hence $\alpha$ coincides with the circle $\delta$ or the line $t$ in the case $b = 0$.

The circle $\beta$ has an equation

\[ f_b(x, y) = (x^2 + y^2) - 2by = 0. \]

From $f_b = 0$ we get $x^2 + y^2 = 0$ in the case $b = 0$. Also from $f_b/b = 0$ we get $y = 0$ in the case $b = 0$ by (9). Hence $\beta$ coincides with the origin or the $x$-axis in this case.

The circle $\gamma$ has an equation $x^2 + (y - 2d - c)^2 = c^2$, where $c = (d - b)^2/(4b)$, which is arranged as

\[ f_c(x, y) = \frac{d^2}{2b}(2d - y) + \left( x^2 + \left( y - \frac{3d}{2} \right)^2 - \frac{d^2}{4} \right) + \frac{b}{2}(2d - y) = 0. \]
From $f_c = 0$ we get $x^2 + (y - 3d/2)^2 = (d/2)^2$ in the case $b = 0$. Also from each of $f_c b = 0$ and $f_c/b = 0$ we get $y = 2d$ in the case $b = 0$. Hence $\gamma$ coincides with the line $t$ or the circle of radius $d/2$ touching $\delta$ at $D$ in this case.

When $\beta$ approaches to the origin, the circles $\alpha$ and $\gamma$ approach to $\delta$ and $t$, respectively. Therefore we can consider that $\alpha$ and $\gamma$ coincide with $\delta$ and $t$, respectively when $\beta$ degenerates to the origin, (see Figure 8). The relation $a^2 = 4bc$ does not holds in this case, but $a^2/b = 4c$ and $a^2/c = 4b$ hold by (9), since the radius of $t$ equals 0.

When $\beta$ coincides with the $x$-axis, we can thereby consider that $\alpha$ and $\gamma$ coincides with $t$ and the circle of radius $d/2$ touching $\delta$ at $D$, respectively as the remaining case (see Figure 9). The relation $a^2 = 4bc$ holds in this case.

![Figure 8](image1)

![Figure 9](image2)

<table>
<thead>
<tr>
<th>case</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>relation of the radii</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\delta$</td>
<td>origin</td>
<td>$t$</td>
<td>$a^2/b = 4c$, $a^2/c = 4b$</td>
</tr>
<tr>
<td>2</td>
<td>$t$</td>
<td>$x$-axis</td>
<td>circle of radius $d/2$ touching $\delta$ at $D$</td>
<td>$a^2 = 4bc$</td>
</tr>
</tbody>
</table>

Table 1: $b = 0$.

We summarize the results in Table 1. The case 1 described in Figure 8 is supposable without (9). But (9) enable us to get the case by algebraic manipulation. On the other hand, the case 2 described in Figure 9 can not be obtained without (9). In this case $d = a + b$ does not hold, but still can be considered that $\alpha$ and $\beta$ touch. However we cannot attain a reasoned interpretation for this case at the current moment. A similar phenomenon, in which a circle of half the radius appears, can be found in [8].

### 3.2. The case $a = 0$.

We now consider the case $a = 0$. Substituting $b = d - a$ in (10), we get

$$f_a = (x^2 + (y - 2d)^2) + 2a(y - 2d) = 0.$$ 

Hence we get $x^2 + (y - 2d)^2 = 0$ or $y = 2d$ in the case $a = 0$. Therefore $\alpha$ coincides with $D$ or $t$ in this case. Similarly we have

$$f_b = (x^2 + (y - d)^2 - d^2) + 2ay = 0.$$ 

Therefore we get $x^2 + (y - d)^2 = d^2$ or $y = 0$ in the case $a = 0$. Hence $\beta$ coincides with $\delta$ or the $x$-axis in the case $a = 0$. Also we have

$$f_c = 2d(x^2 + (y - 2d)^2) + 2a(x^2 + (y - 2d)^2) + a^2(2d - y) = 0.$$ 

Therefore we get $x^2 + (y - 2d)^2 = 0$ or $y = 2d$ in the case $a = 0$. Hence $\gamma$ coincides with $D$ or $t$ in this case.
When $\alpha$ approaches to $D$, $\beta$ and $\gamma$ approach to $\delta$ and $D$, respectively. Hence we consider that $\beta$ and $\gamma$ coincide with $\delta$ and $D$, respectively when $\alpha$ coincides with $D$ (see Figure 10). As the remaining case $\beta$ and $\gamma$ coincide with the $x$-axis and $t$, respectively when $\alpha$ coincides with $t$ (see Figure 11).

We summarize the results in Table 2. The case 3 described in Figure 10 is supposable without (9). On the other hand, the case 4 described in Figure 11 can not be obtained without (9). However we cannot attain a reasoned interpretation for this case at the current moment.

![Figure 10](image1)

![Figure 11](image2)

<table>
<thead>
<tr>
<th>case</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>relation of the radii</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$D$</td>
<td>$\delta$</td>
<td>$D$</td>
<td>$a^2 = 4bc$</td>
</tr>
<tr>
<td>4</td>
<td>$t$</td>
<td>$x$-axis</td>
<td>$t$</td>
<td>$a^2 = 4bc$</td>
</tr>
</tbody>
</table>

Table 2: $a = 0$.  

For a brief introduction of division by zero with Wasan geometry see [14], and its application to Wasan geometry see [4], [8], [9], [10], [11], [12], [13], [15]. For an extensive reference of division by zero and division by zero calculus, see [17].

## 4. Incorrect Sangaku Problems

In [16] we have considered two incorrect sangaku problems in [5, p. 69, p. 123], each of which can also be found in [7] and [21], respectively.

![Figure 12](image3)

![Figure 13](image4)

Figure 12: The figures in [5], [21].

Figure 13: The figure in [5].

The problems and the answers are almost the same as Problem 1, i.e., they demand to show the relation $a^2 = 4bc$ for three circles $\alpha$, $\beta$ and $\gamma$ of radii $a$, $b$ and $c$, respectively. However the figures are slightly different as shown in Figures 12 and 13. The figure in [21] is also the same as Figure 12. It seems that those problems were correct and essentially the same as Problem 1 in the original context but the
figures were incorrectly transcribed in [5] and [21]. While the figure in [7] is the same as Figure 1, therefore the problem is correct.

REFERENCES