# Assuming $c < rad^2(abc)$ implies The *abc* Conjecture true

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**Abstract**

In this paper, assuming that $c < rad^2(abc)$ is true, we give a proof of the *abc* conjecture by proposing the expression of the constant $K(\epsilon)$, then we approve that $\forall \epsilon > 0$, for $a, b, c$ positive integers relatively prime with $c = a + b$, we have $c < K(\epsilon).rad(abc)^{1+\epsilon}$. Some numerical examples are given.

**Résumé**

Assumant que la conjecture $c < rad^2(abc)$ est vraie, on donne une démonstration de la conjecture *abc* en proposant la constante $K(\epsilon)$. On approuve alors pour tout $\epsilon > 0$, et pour tout triplet $(a, b, c)$ avec $c = a + b$ et $a, b, c$ des entiers positifs relativement premiers, on a $c < K(\epsilon)rad^{1+\epsilon}(abc)$. Des exemples numériques sont présentés.

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*To the memory of my Father who taught me arithmetic*
*To my wife Wahida and my children Sinda and Mohamed Mazen*

## 1 Introduction and notations

Let $a$ a positive integer, $a = \prod_i a_i^{\alpha_i}$, $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of $a$ the integer $\prod_i a_i$ noted by $rad(a)$. Then $a$ is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

***Conjecture*** 1.1. ( ***abc*** **Conjecture**): Let $a, b, c$ positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists a constant $K(\epsilon)$ such that :

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{3}$$

$K(\epsilon)$ depending only of $\epsilon$.

The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. The difficulty to find a proof of the $abc$ conjecture is due to the incomprehensibility how the prime factors are organized in $c$ giving $a, b$ with $c = a + b$.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [1]. A conjecture was proposed that $c < rad^2(abc)$ [3]. It is the key to resolve the $abc$ conjecture. In my paper, assuming that $c < rad^2(abc)$, I give a proof of the $abc$ conjecture. The paper is organized as follows: in the second section, we give the proof of the $abc$ conjecture. In section three, we present some numerical examples.

## 2 The proof of the $abc$ conjecture

### 2.1 The Proof of the $abc$ Conjecture (1.1), Case :$\epsilon \geq 1$

Assuming that $c < rad^2(abc)$ is true, we have $\forall \epsilon \geq 1$:

$$c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \ \epsilon \geq 1 \tag{4}$$

We verify easily that $K(\epsilon) > 1$ for $\epsilon \geq 1$. Then the $abc$ conjecture is true.

### 2.2 The Proof of the $abc$ Conjecture (1.1), Case :$\epsilon < 1$

### 2.3 Case: $\epsilon < 1$

#### 2.3.1 Case: $c < R$

In this case, we can write :

$$c < R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \ \epsilon < 1 \tag{5}$$

here also $K(\epsilon) > 1$ for $\epsilon < 1$ and the $abc$ conjecture is true.

#### 2.3.2 Case: $c > R$

In this case, we confirm that :

$$c < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, 0 < \epsilon < 1 \tag{6}$$

If not, then $\exists \epsilon_0 \in ]0, 1[$, so that the triplets $(a, b, c)$ checking $c > R$ and:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0) \tag{7}$$

are in finite number. We have:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0}.c \geq R^{1-\epsilon_0}.R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow$$
$$R^{1-\epsilon_0}.c \geq R^2.K(\epsilon_0) > c.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0} > K(\epsilon_0) \tag{8}$$

As $c > R$, we obtain:

$$c^{1-\epsilon_0} > R^{1-\epsilon_0} > K(\epsilon_0) \Longrightarrow$$

$$c^{1-\epsilon_0} > K(\epsilon_0) \Longrightarrow c > K(\epsilon_0)^{\left(\dfrac{1}{1-\epsilon_0}\right)} \tag{9}$$

We deduce that it exists an infinity of triples $(a, b, c)$ verifying (7), hence the contradiction. Then the proof of the *abc* conjecture is finished. We obtain that $\forall \epsilon > 0$, $c = a + b$ with $a, b, c$ relatively coprime, $a > b \geq 2$:

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \quad with \quad K(\epsilon) = e^{\left(\dfrac{1}{\epsilon^2}\right)} \tag{10}$$

Q.E.D

# 3 Examples

In this section, we are going to verify some numerical examples.

## 3.1 Example 1

The example is given by:

$$1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6 \tag{11}$$

$a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47\,045\,880 \Rightarrow \mu_a = 2 \times 3 \times 7 = 42$ and $rad(a) = 2 \times 3 \times 5 \times 7 \times 127$,
$b = 1 \Rightarrow \mu_b = 1$ and $rad(b) = 1$,
$c = 19^6 = 47\,045\,880 \Rightarrow rad(c) = 19$. Then $rad(abc) = rad(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506\,730$..

We have $c > rad(ac)$ but $rad^2(ac) = 506\,730^2 = 256\,775\,292\,900 > c = 47\,045\,880$.

### 3.1.1 Case $\epsilon = 0.01$

$c < K(\epsilon).rad(ac)^{1+\epsilon} \Longrightarrow 47\,045\,880 \overset{?}{<} e^{10000}.506\,730^{1.01}$. The expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{\frac{1}{0.0001}} = e^{10000} = 8,7477777149120053120152473488653e + 4342 \tag{12}$$

We deduce that $c \ll K(0.01).506\,730^{1.01}$ and the equation (10) is verified.

### 3.1.2 Case $\epsilon = 0.1$

$K(0.1) = e^{\frac{1}{0.01}} = e^{100} = 2,6879363309671754205917012128876e + 43 \Longrightarrow c < K(0.1) \times 506\,730^{1.01}$. And the equation (10) is verified.

#### 3.1.3 Case $\epsilon = 1$

$K(1) = e \Longrightarrow c = 47\,045\,880 < e.rad^2(ac) = 697\,987\,143\,184, 212.$ and the equation (10) is verified.

#### 3.1.4 Case $\epsilon = 100$

$$K(100) = e^{0.0001} \Longrightarrow c = 47\,045\,880 \overset{?}{<} e^{0.0001}.506\,730^{101} = \\ 1, 5222350248607608781853142687284e + 576$$

and the equation (10) is verified.

### 3.2 Example 2

We give here the example of Eric Reyssat [1], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \tag{13}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$. For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{9999.99} = 8, 7477777149120053120152473488653e + 4342 \tag{14}$$

Let us verify (10):

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow c = 6436343 \overset{?}{<} K(0.01) \times (3 \times 109 \times 2 \times 23)^{1.01} \Longrightarrow \\ 6436343 \ll K(0.01) \times 15042^{1.01} \tag{15}$$

Hence (10) is verified.

### 3.3 Example 3

The example of Nitaj about the ABC conjecture [1] is:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79 \tag{16}$$
$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311 \tag{17}$$
$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953 \tag{18}$$
$$rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110 \tag{19}$$

#### 3.3.1 Case 1

we take $\epsilon = 100$ we have:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} e^{0.0001}.(2.3.5.7.11.13.41.79.311.953)^{101} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 2, 7657949971494838920022381186039e + 1359$$

then (10) is verified.

### 3.3.2 Case 2

We take $\epsilon = 0.5$, then:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow \tag{20}$$

$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} e^4.(2.3.5.7.11.13.41.79.311.953)^{1.5} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 8\,450\,961\,319\,227\,998\,887\,403,9993 \tag{21}$$

We obtain that (10) is verified.

### 3.3.3 Case 3

We take $\epsilon = 1$, then

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$

$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} (2.3.5.7.11.13.41.79.311.953)^2 \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 831\,072\,936\,124\,776\,471\,158\,132\,100 \tag{22}$$

We obtain that (10) is verified.

## 3.4 Example 4

It is of Ralf Bonse about the ABC conjecture [3] :

$$2543^4.182587.2802983.85813163 + 2^{15}.3^{77}.11.173 = 5^{56}.245983 \tag{23}$$
$$a = 2543^4.182587.2802983.85813163$$
$$b = 2^{15}.3^{77}.11.173$$
$$c = 5^{56}.245983$$
$$rad(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163$$
$$rad(abc) = 1.5683959920004546031461002610848e + 33 \tag{24}$$

### 3.4.1 Case 1

For example, we take $\epsilon = 10$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{0.01} = 1,007815740428295674320461741677 9$$

Let us verify (10):

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \overset{?}{<}$$
$$e^{0.01}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{11}$$
$$\Longrightarrow 3.4136998783296235160378273576498e + 44 <$$
$$1,4236200596494908176008120925721e + 365 \tag{25}$$

The equation (10) is verified.

### 3.4.2 Case 2

We take $\epsilon = 0.4 \Longrightarrow K(\epsilon) = 12,182473474251512159126256696 08$, then:

$$\begin{gathered}c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \overset{?}{<} \\ e^{6.25}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{1.4} \\ \Longrightarrow 3.4136998783296235160378273576498e+44 < \\ 3,6255465680011453642792720569685e+47\end{gathered} \tag{26}$$

And the equation (10) is verified.

Ouf, end of the mystery!

## 4 Conclusion

Assuming $c < R^2$ true, we have given an elementary proof of the *abc* conjecture in the two cases $c = a' + 1$ and $c = a + b$, confirmed by some numerical examples. We can announce the important theorem:

**Theorem 4.1.** *Let $a, b, c$ positive integers relatively prime with $c = a + b$ and assuming that $c < R^2$ holds, then for each $\epsilon > 0$, there exists $K(\epsilon)$ such that :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{27}$$

*where $K(\epsilon)$ is a constant depending of $\epsilon$ proposed equal to* $e^{\left(\frac{1}{\epsilon^2}\right)}$.

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