The Collatz Conjecture - a proof

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Abstract

Originated by Lothar Collatz in 1937 [1], the conjecture states: given the recursive function, y=3x+1 if x is odd, or y=x/2 if x is even, for any positive integer x, y will equal 1 after a finite number of steps. This analysis examines number form and uses a tree type graph to prove the process.

1. examples

An example for a random selection of 7, using the original method:

S=(7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1)

An example for a random selection of 12, using the original method:

S= (12, 6, 3, 10, 5, 16, 8, 4, 2, 1)

2. functions

The recursive function is replaced with function d for odd values (2n-1), with

$$d(x) = 3x+1 = u = 2^k y \qquad (2.0)$$

and function e for even values, which removes all factors of 2,

$$e(u) = y \qquad (2.1)$$

The function e can be defined as a short program with a loop that repeatedly divides u by 2 until the output is an odd integer. This eliminates the redundancy and clutter of repeated division by 2.
After k divisions by 2, u = y, an odd integer. The value of y becomes the input x, and the cycle is repeated until y=1. The application of e(d(7)) results in S=(7 11 17 13 5 1), the revised format used in this analysis, with the understanding of a 2^k factor between each pair of odd integers. Notation is upper case for sets, lower case for elements of a set.

3. reverse sequences

If all sequences converge to the value 1, then it should be possible to form all reverse sequences, diverging from 1. For this purpose the odd integers are classified into 3 subsets, 0 mod 3, 1 mod 3, and 2 mod 3, labeled as Y_0, Y_1, and Y_2.

$Y_0 = \{3\ 9\ 15\ 21\ 27\ ...\}$
$Y_1 = \{1\ 7\ 13\ 19\ 25\ ...\}$
$Y_2 = \{5\ 11\ 17\ 23\ 29\ ...\}$.

Rearranging (2.0), we can find x, given y, while requiring y to be a (1 mod 3) value.
If $y \equiv 1 \bmod 3$, then k is even and if $y \equiv 2 \bmod 3$, then k is odd.

$$x = (2^k y-1)/3 = (u-1)/3. \tag{2.2}$$

Varying k in (2.2) with y = 1, is shown in table 1.

table 1					
k	2	4	6	8	...
u	4	16	64	256	...
x	1	5	21	85	...

Varying k in (2.2) with y = 5, is shown in table 2.

table 2					
k	1	3	5	7	...
u	10	40	160	640	...
x	3	13	53	213	...

There are multiple combinations of x and k, that produce a given y. The x terms for each y, form an unlimited set and transform to y via the function e(d(x)). They are labeled as generation 1, 2, 3, ... etc. in order of increasing values. The gen-1 terms are defined as base terms and the remaining terms branching or b-terms which allow a horizontal growth for each y. They are functionally equivalent to the base term since $e(2^k y) = y$ for all k. The B notation for y=1 and y=5 is:
$B_1 = \{1\ 5\ \mathbf{21}\ 85\ 341\ ...\}$ and $B_5 = \{\mathbf{3}\ 13\ 53\ \mathbf{213}\ ...\}$, with (0 mod 3) terms in bold.
The u terms are related by a factor of 4, since that maintains the (1 mod 3) state of u.
If y = 3x+1, and 4y = 3w+1, then w = (4y-1)/3 = 4x+1.
Then the relation of successive b-terms is

$$x_{r+1} = 4x_r+1 \tag{2.3}$$

The use of n in the general forms denotes the position of an odd integer within its ordered subset.
Substituting the general form for y_1 in (2.2)

$$x_1 = (4(6n-5)-1)/3 = 8n-7 \tag{2.4}$$

as elements of X_1.

Substituting the general form for y_2 in (2.2)

$$x_2 = (2(6n-1)-1)/3 = 4n-1 \qquad (2.5)$$

as elements of X_2.

3.1 terminating values for reverse sequences

There is a corresponding x for each y, from (2.4) and (2.5), but there is still the remaining odd x of the form 8n-3, labeled as X_3. Rearranging gives

$$x_3 = 8n-3 = 4(2n-1)+1 \qquad (2.6)$$

Comparing (2.3) and (2.6), reveals there is a b-term for each odd integer.
Since the function d cannot produce (0 mod 3) output, an element from Y_0 can only begin a descending sequence S of odd integers, which implies, a reverse sequence R will terminate. The one exception is (1 1). This case is the only positive solution to $3x+1 = 4^k x$, a simple loop.
In ascending mode, using B_1 = {1 5 **21** 85 341 ...}, the next term is 1. R = (1 1) and terminates,

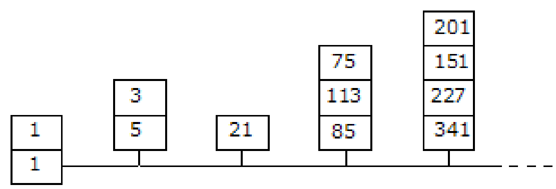

fig. 1

Fig. 1 shows available options via B_1. The b-terms allow bypassing the next cell by forming a new branch. Using the next available term from B_1, R = (1 5).
From B_5 = {**3** 13 53 **213** ...}, R = (1 5 **3**), the sequence R terminates.

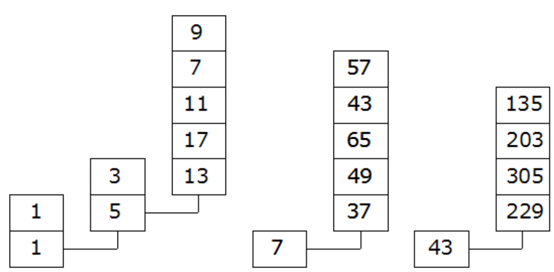

fig. 2

Remaining with the current R and B_5, the next (gen-2) term 13, allows a new branch and extension of R, as shown in fig.2. A new branch can be formed from any term in the current branch except (0 mod 3). In the example, the next to last term is arbitrarily selected. Using the b-terms for each successive x, extends the branch vertically to the next termination value 9. This process is repeated with 7, 43, 203, etc., and can be extended without limit.

B_1 ={1 5 **21** 85 341 **1365** ...},	(1 5)
B_5 ={**3** 13 53 **213** ...},	(1 5 13)
B_{13} = {17 **69** 277 **1109**...},	(1 5 13 17)
B_{17} = {11 **45** 181 **725**...},	(1 5 13 17 11)
B_{11} = {7 29 **117** 469...},	(1 5 13 17 11 7)
B_7 = {**9** 37 149 **597**...},	(1 5 13 17 11 7 37)
...	(1 5 13 17 11 7 37...)

A reverse sequence R of any length can be formed using the b-terms which allow tree expansion.

4. even integer selection

All reverse sequences for even integer selection, can be formed by appending a 2^k progression times an odd integer, presented here as a list, using 1, 3, 5, 7, 9, ...

(2 4 8 16 32 ...)
(6 12 24 48 ...)
(10 20 40 80 ...)
(14 28 56 122 ...)
(18 36 72 144 ...)
...

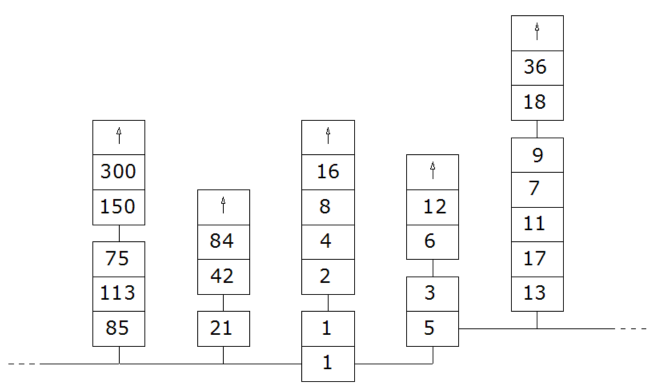

fig.3

This provides a means of extending the Y_0 termination values to sequences of unlimited length as shown in fig.3. It also provides a second indirect method of selecting an odd integer.

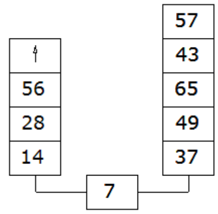

fig.4

Each term from Y_1 and Y_2 now have an additional branch of even integers, fig.4.

5. x-y correspondence

		↑	↑		↑	↑	↑		↑	↑	↑		↑	↑	...	gen
		85	213		469	597	725		981	1109	1237		1493	1621	...	4
A	X3	21	53		117	149	181		245	277	309		373	405	...	3
		5	13		29	37	45		61	69	77		93	101	...	2
	X2		3		7		11		15		19		23		...	1
	X1	1				9				17				25	...	1
	u	4	10		22	28	34		46	52	58		70	76	...	1
B	4u	16	40		88	112	136		184	208	232		280	304	...	2
	16u	64	160		352	448	544		746	832	928		1120	1216	...	3
	↓	↓	↓		↓	↓	↓		↓	↓	↓		↓	↓	...	
C	Y2		5		11		17		23		29		35		...	
	Y1	1				7				13				19	...	

fig.5

Fig.5 is a summary of x to y correspondence. The B_x extend vertically in the X3 section. Remaining in the same column, an odd integer x is selected from section A. An application of d(x) yields u in section B, with a matching generation index. An application e(u) yields y in section C.

6. Collatz graph

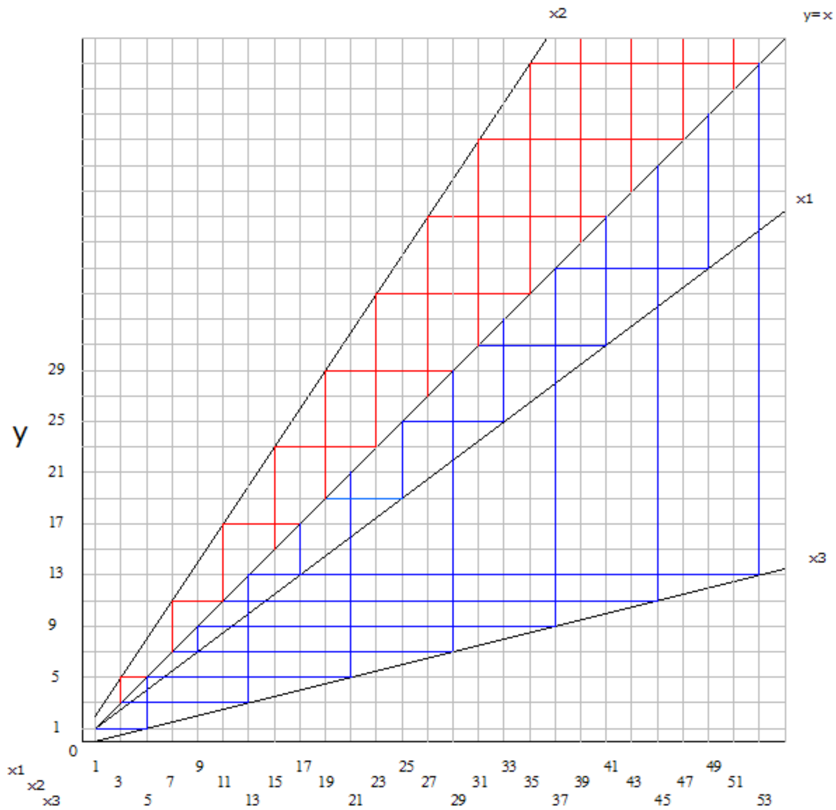

fig.6

Fig.6 graphs the difference Δ of x values, initially on the diagonal y=x, after applying d and e functions (par.2). Grid spacing us 2 units.
For X_1, Δ = (6n-5) - (8n-7) = -2n+2 (blue)
For X_2, Δ = (6n-1) - (4n-1) = 2n (red)
For X_3, Δ = (2n-1) - (8n-3) = -6n+2 (blue)
Blue horizontal lines from X_3 values beyond 53 are not shown for clarity.

7.1 additional factors

Is it possible for an element of X_2 to increase with a continuous number of steps?
Using (2.0) and (2.1), 3(4mn-1)+1 = 12mn -2, so y = 6mn-1, with m a multiple of 2.
With m=2, successive cycles yield y = (12n-1, 18n-1), and y decreases.
With m=4, successive cycles yield y = (24n-1, 36n-1, 54n-1), and y decreases.
With m=8, successive cycles yield y = (48n-1, 72n-1, 108n-1, 162n-1), and y decreases.
As x_2 increases there are longer finite sequences of increases, interrupted by a decrease, when the form becomes a member of 6n-1. If m = 2^k, sequence length = k.

8. the tree

Using the tree analogy, when ascending from 1, there is a one to many relationship from y to x, via the B_x branching, presenting an unlimited number of possible sequences, as shown in section 3 and 4. Descending from a randomly selected term x, there is a one to one relationship from x to y. The current x determines the next term, therefore the entire sequence is pre-determined by the initial selection.

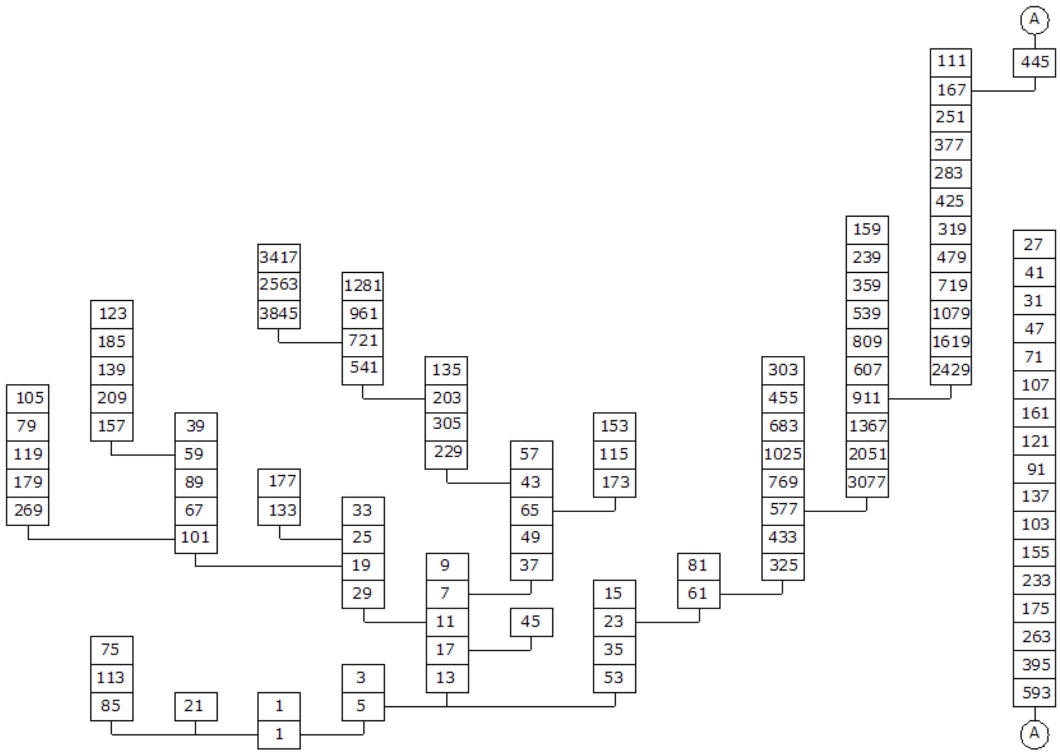

fig.7

Fig.7 shows the initial growth of the tree for odd integers only, from a 'trunk' of 1, vertically with each branch terminating in a (0 mod 3) value, and horizontally via the B_x as demonstrated in section 3. The sequence for x=27 is revealed as a composite of 7 branches to the right. Two dimensions is not adequate to visualize a partial tree with all odd integer and even integer branches.

conclusion

All odd integers, have a corresponding b-term as shown in section 3, therefore all sequences merge in descending mode.. The Collatz conjecture applies only to finite length sequences, in the descending mode.

reference

1. Wikipedia.org/Collatz Conjecture, Mar 2018