A chain of circles touching a circle and its tangent and division by zero

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Abstract. We consider a chain of circles by division by zero.

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1. Introduction

We consider a chain of circles whose members touch a given circle and its fixed tangent by division by zero [2]:

$$\frac{z}{0} = 0 \quad \textit{for any real number } z. \tag{1}$$

Let α be a circle of radius a with a tangent t touching at a point O, and let A be the farthest point on α from t. We consider with a rectangular coordinate system with origin O such that the point A has coordinates $(2a, 0)$ (see Figure 1).

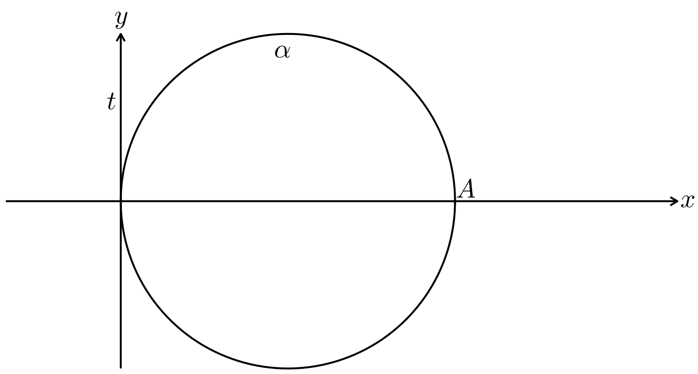

Figure 1.

2. Circles touching a given circle and its tangent

Theorem 1. *A proper circle touches the circle α externally and the line t from the same side as α if and only if its has center of coordinates $(a/z^2, \pm 2a/z)$ and radius a/z^2 for a real number $z \neq 0$.*

Proof. If a proper circle touches t from the same side as α, it has coordinates $(a/z^2, y)$ and radius a/z^2 for a real numbers $z \neq 0$ and y. It touches α externally if and only if $(a/z^2 - a)^2 + y^2 = (a/z^2 + a)^2$. The last equation is equivalent to $y = \pm 2a/z$. $\square$

We denote the circle of radius a/z^2 and center of coordinates $(a/z^2, 2a/z)$ by C_z for $z \neq 0$. Circles and lines are represented by the equation $e(x^2+y^2)-2fx-2gy+h=0$. If $e \neq 0$, the coordinates of the center and the radius are

$$\left(\frac{f}{e}, \frac{g}{e}\right), \quad \sqrt{\frac{f^2+g^2-eh}{e^2}}.$$

Therefore we can consider that *a line has center of coordinates* $(0,0)$ *and radius* 0 *as a circle* by (1) [17]. On the other hand, the circle C_z has an equation $(x-a/z^2)^2+(y-2a/z)^2 = a^2/z^4$, which is arranged as

$$C_z(x,y) = x^2+y^2-\frac{4a}{z}y+\frac{2a}{z^2}(2a-x)=0.$$

Therefore we get $x^2+y^2=0$, $y=0$ and $x=2a$, by $C_z(x,y)=0$, $zC_z(x,y)=0$, and $z^2C_z(x,y)=0$, respectively with (1). They express the origin, the line AO and the tangent of α at A, respectively. We denote them by C_∞, $C_{\overline{\infty}}$ and C_0, respectively (see Figure 2). Notice that C_0 has center of coordinates $(0,0)$ and radius 0. Someone may consider that $C_{\overline{\infty}}$ are orthogonal to α and t and does note touch them. But (1) implies $\tan(\pi/2)=0$. Therefore we can still consider that $C_{\overline{\infty}}$ touches α and t. We can also consider that C_∞ and $C_{\overline{\infty}}$ touch.

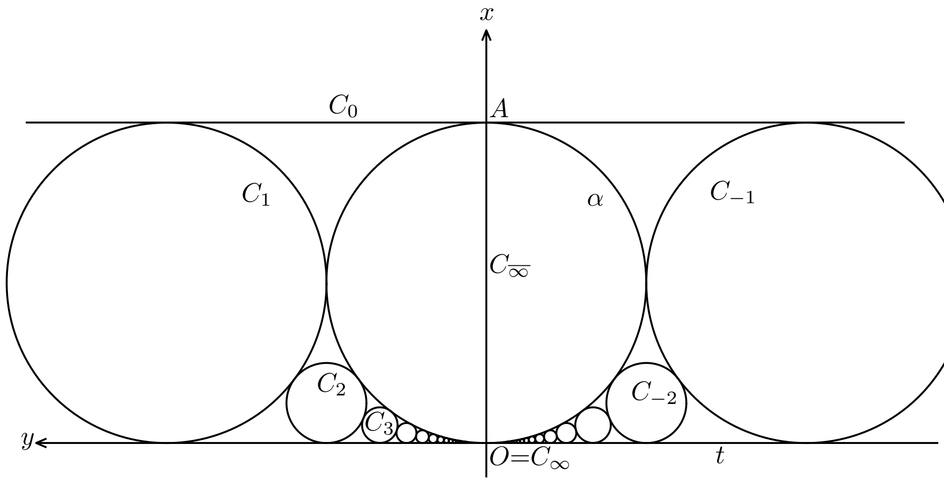

Figure 2.

Theorem 2. *The circles α_z and α_w touch if and only if $|z-w|=1$ for $w, z \in \mathbb{R}$.*

Proof. We assume $(w,z) \neq (0,0)$. Since

$$(a/z^2-a/w^2)^2+(2a/z-2a/w)^2-(a/z^2+a/w^2)^2 = \frac{4a^2}{w^2z^2}((w-z)^2-1),$$

C_z and C_w touch if and only if $|w-z|=1$. If $z=0$, C_z and C_w touch if and only if $w=\pm 1$. □

For more applications of division by zero to circle geometry, see [1], [3], [4, 5, 6, 7, 8, 9, 10, 11, 12] [13, 14, 15, 16]

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