$q$-analogs of sinc sums and integrals

Martin Nicholson

$q$-analogs of sum equals integral relations $\sum_{n \in \mathbb{Z}} f(n) = \int_{-\infty}^{\infty} f(x) dx$ for sinc functions and binomial coefficients are studied. Such analogs are already known in the context of $q$-hypergeometric series. This paper deals with multibasic ‘fractional’ generalizations that are not $q$-hypergeometric functions.

Surprising properties of sinc sums and integrals were first discovered by C. Stormer in 1895 [1,2]. The more general properties of band limited functions were known to engineers from signal processing and to physicists. For example, K.S. Krishnan viewed them as a rich source for finding identities [3]. R.P. Boas has studied the error term when approximating a sum of a band limited function with corresponding integral [5]. More recently these properties were studied and popularized in a series of papers [6–8].

sinc function is a special case of binomial coefficients

$$\left( \begin{array}{c} 2 \\ 1 + x \end{array} \right) = \frac{\Gamma(3)}{\Gamma(1 + x)\Gamma(1 - x)} = \frac{2 \sin \pi x}{\pi x} = 2 \text{sinc}(\pi x).$$

Therefore only sums with binomial coefficients will be studied in the following. It is known that binomial coefficients are band limited (e.g., see [10])

$$\left( \begin{array}{c} a \\ n \end{array} \right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 + e^{it})^{a} e^{-iut} dt,$$

i.e. their Fourier spectrum is limited to the band $|t| < \pi$. According to general theorems [5, 6] whenever Fourier spectrum of a function $f(x)$ is limited to the band $|t| < 2\pi$ one expects that

$$\sum_{n=-\infty}^{\infty} f(n) = \int_{-\infty}^{\infty} f(x) dx. \quad (1)$$

Bandwidth of a product of bandlimited functions is the sum of their bandwidths [8]. In case of binomial coefficients this together with the theorem mentioned above implies that

$$\sum_{n=-\infty}^{\infty} \left( \begin{array}{c} a \\ \alpha n \end{array} \right)^l = \int_{-\infty}^{\infty} \left( \begin{array}{c} a \\ \alpha x \end{array} \right)^l dx, \quad 0 < \alpha \leq \frac{2}{l}. \quad (2)$$

For a general band limited function the above formula would have been valid only when $\alpha < \frac{2}{l}$. The validity of (2) when $\alpha = \frac{2}{l}$ is explained by the fact that spectral density of binomial coefficient vanishes at boundary values $t = \pm \pi$.

$q$-analog of the Gamma function is defined as

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(xq^2; q)_\infty} (1 - q)^{1 - x}$$

and the $q$-binomial coefficients

$$\left[ \begin{array}{c} a \\ b \end{array} \right]_q = \frac{\Gamma_q(a + 1)}{\Gamma_q(b + 1) \Gamma_q(a - b + 1)}.$$
with the standard notations for the \( q \)-shifted factorials

\[
(a; q)_n = \prod_{k=0}^{n-1} (1 - a q^k), \quad (a_1, \ldots, a_r; q)_n = \prod_{k=1}^{r} (a_k; q)_n, \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - a q^k).
\]

In the limit \( q \to 1^- \) one has \( \Gamma_q(a) \to \Gamma(a) \), i.e. standard values of the Gamma function and binomial coefficients are recovered.\(^{[11]}\)

\( q \)-analog of the property of bandlimitedness has been studied in the literature \(^{[12]}\). This paper has a much more narrow scope and only deals with sums of binomial coefficients. We will find that \(^{[2]}\) with \( 0 < \alpha \leq 1/l \) has a very natural \( q \)-analog. However no such simple direct \( q \)-analog of \(^{[2]}\) with \( 1/l < \alpha \leq 2/l \) is known. Nevertheless there is a formula that in the limit \( q \to 1^- \) can be brought to the form \(^{[2]}\) after a series of simple steps.

In Theorem \(^{[2]}\) we will use a method of functional equations \(^{[13]}\) (see also \(^{[11]}\), sec. 5.2) combined with an idea to to G. Gasper \(^{[14]}\) to find a Laurent series for a certain integral of an infinite product. First we need the following theorem taken from the book \(^{[15]}\).

**Theorem 1.** Let

\[
F(z) = \int_{\gamma} f(\zeta, z) d\zeta,
\]

where the following conditions are satisfied

1. \( \gamma \) is an infinite piecewise continuous curve
2. the function \( f(\zeta, z) \) is continuous in \( (\zeta, z) \) at \( \zeta \in \gamma, \ z \in D \), where \( D \) is a domain in the complex \( z \) plane,
3. for each fixed \( \zeta \in \gamma \) the function \( f(\zeta, z) \) viewed as a function of \( z \) is regular in \( D \),
4. integral \(^{[3]}\) converges uniformly in \( z \in D', \) where \( D' \) is an arbitrary closed subdomain of \( D \).

Then \( F(z) \) is regular in \( D \).

**Lemma 1.** Let \( p \) and \( q \) two real numbers that satisfy \( 0 < p < q < 1 \), then

\[
F(z) = \int_{-\infty}^{\infty} \frac{(b q^z a q^{-\zeta}; p)_\infty}{(-z q^z, -q^{1-\zeta}/z; q)_\infty} d\zeta
\]

is regular in the half plane \( \Re z > 0 \).

**Proof.** Put in the theorem above \( f(\zeta, z) = \frac{(b q^\zeta, a q^{-\zeta}; p)_\infty}{(-z q^z, -q^{1-\zeta}/z; q)_\infty}, \ \gamma = (-\infty, +\infty), \) and \( D \) an arbitrary domain in the half plane \( \Re z > 0 \). Then (1),(2) and (3) are obviously satisfied. To prove (4) let \( p = e^{-\omega}, \ q = p^\alpha, \ \omega > 0, \ 0 < \alpha < 1 \) and consider the asymptotics of \( f(\zeta, z) \) when \( \zeta \to +\infty \). In this limit one has \( (b q^\zeta; p)_\infty \to 1, \ (z q^\zeta; q)_\infty \to 1 \). According to an asymptotic formula \(^{[11]}\), p. 118

\[
\Re [\ln(p^\alpha; p)_\infty] = \frac{\omega}{2}(\Re s)^2 + \frac{\omega}{2}(\Re s) + O(1), \quad \Re s \to -\infty,
\]

we have

\[
|(a q^{-\zeta}; p)_\infty| = |(p^{-\alpha \zeta - \omega^{-1} \ln a; p)_\infty| = O\left( |a|^\alpha q^{-(\alpha \zeta^2 - \zeta)/2}\right),
\]

\[
|-(q^{1-\zeta}/z; q)_\infty| = |(q^{1-\zeta + \alpha^{-1} \omega^{-1} \ln z; q)_\infty| = O\left(|q/z|^\zeta q^{-(\zeta^2 - \zeta)/2}\right).
\]

So

\[
f(\zeta, z) = O\left(|z a^\alpha/q|^\zeta q^{(1-\alpha) \zeta^2}/2\right), \quad \zeta \to +\infty.
\]
Similarly
\[ f(\zeta, z) = O\left(\left|b^\alpha / z\right|^{\zeta} q^{(1-\alpha)\zeta^2 / 2}\right), \quad \zeta \to -\infty. \]

It is now easy to see that the integral (*) converges. Hence according to Weierstrass M-Test integral \( F(z) \) converges uniformly in \( z \) when \( \text{Re} \ z \geq \delta > 0 \). As a result the function
\[ f(a, b, z) = \left(-z, -q/z; q\right)_\infty \int_0^\infty \frac{(bt/z, p/t; p)_\infty}{(-t, -q/t; q)_\infty} \frac{dt}{t} \]
is regular when \( \text{Re} \ z > 0 \)

\[ \text{Lemma 2.} \quad \text{The function} \]
\[ f(a, b, z) = \left(-z, -q/z; q\right)_\infty \int_0^\infty \frac{(bt, a/t; p)_\infty}{(-zt, -q/(zt); q)_\infty} \frac{dt}{t} \]

satisfies the functional equations
\[ f(a, b, z) = f(a, bp, z) - bf(a, bp, qz), \quad (4) \]
\[ f(a, b, z) = f(ap, b, z) - af(ap, b, z/q). \quad (5) \]

\[ \text{Proof.} \quad \text{After a series of simple manipulations of the infinite products we find} \]
\[ f(a, b, qz) = \left(-z, -1/z; q\right)_\infty \int_0^\infty \frac{(bt, a/t; p)_\infty}{(-qzt, -1/(zt); q)_\infty} \frac{dt}{t} \]
\[ = \frac{(-z, -q/z; q)_\infty}{z \ln \frac{1}{q}} \int_0^\infty \frac{z (bt, a/t; p)_\infty}{(-zt, -q/(zt); q)_\infty} \frac{dt}{t} \]
\[ = \frac{p}{b} \ln \frac{1}{q} \int_0^\infty \frac{bt}{p} \frac{(bt, a/t; p)_\infty}{(-zt, -q/(zt); q)_\infty} \frac{dt}{t} \]
\[ = \frac{p}{b} (f(a, b, z) - f(a, b/p, z)). \]

This is equivalent to \( [4] \). Similarly or using the first functional equation and the formula \( f(a, b, z) = f(b, a, q/z) \) we find
\[ f(a, b, z) = f(b, a, q/z) = f(b, ap, q/z) - af(b, ap, q^2/z) \]
\[ = f(ap, b, z) - af(ap, b, z/q), \]
as required.

\[ \text{Theorem 2.} \quad \text{Let} \quad p \quad \text{and} \quad q \quad \text{two complex numbers such that} \quad |p| < |q| < 1, \quad \text{then} \]
\[ \sum_{n=-\infty}^{\infty} (bq^n, aq^{-n}; p)_\infty z^n q^{n(1-1/2)} = \frac{(-z, -q/z; q)_\infty}{\ln \frac{1}{q}} \int_0^\infty \frac{(bt, az/t; p)_\infty}{(-t, -q/t; q)_\infty} \frac{dt}{t}. \]

\[ \text{Proof.} \quad \text{First consider the case} \quad 0 < p < q < 1. \quad \text{The function} \quad f(a, b, z) \quad \text{from Lemma 2} \quad \text{can be written in the form} \]
\[ f(a, b, z) = (-z, -q/z; q)_\infty \int_{-\infty}^{\infty} \frac{(bq^\zeta z, azq^{-\zeta}; p)_\infty}{(-q^\zeta, -q^{-1}\zeta; q)_\infty} d\zeta. \]
According to Lemma 1, \( f(a, b, z) \) is a regular function of \( z \) in the region \( \text{Re} \, z > 0 \). As a result, \( f(a, b, z) \) has the Laurent series expansion

\[
f(a, b, z) = \sum_{n=-\infty}^{\infty} c_n(a, b)z^n, \quad \text{Re} \, z > 0.
\]

Functional equation (4) gives the following recursion relation for coefficients \( c_n(a, b) \)

\[
c_n(a, b) = (1 - bq^n)c_n(a, bp).
\]

This recursion means that

\[
c_n(a, b) = (bq^n; p)_{\infty}c_n(a, 0).
\]

The functional equation (5) gives

\[
c_n(a, b) = (1 - aq^{-n})c_n(a/p, b),
\]

from which one obtains

\[
c_n(a, b) = (aq^{-n}; p)_{\infty}c_n(0, b).
\]

By combining these equations one gets

\[
c_n(a, b) = (bq^n; p)_{\infty}c_n(a, 0) = (bq^n, aq^{-n}; p)_{\infty}c_n(0, 0).
\]

It is known that ([11], ex. 6.16)

\[
\int_{0}^{\infty} \frac{1}{(-t, -q/t; q)_{\infty}} \frac{dt}{t} = (q; q)_{\infty} \ln \frac{1}{q}.
\]

According to Jacobi triple product formula

\[
(q, -z, -q/z; q)_{\infty} = \sum_{n=-\infty}^{\infty} z^n q^{n(n-1)/2}
\]

this implies that \( c_n(0, 0) = z^n q^{n(n-1)/2} \), so finally

\[
c_n(a, b) = (bq^n, aq^{-n}; p)_{\infty}z^n q^{n(n-1)/2}.
\]

Now one needs to continue the result established for \( \text{Re} \, z > 0, 0 < p < q < 1 \) analytically to complex values of parameters \( z, p, q \) to complete the proof.

Series containing infinite products \( (bq^n, aq^{-n}; p)_{\infty} \) have been studied in [12]. It appears that the series in Theorem 2 have been first considered in the paper [17] which also contains a different representation for this sum in terms of an integral over a unit circle.

**Corollary 1.** The formula in Theorem 2 can be written in symmetric form

\[
\sum_{n=-\infty}^{\infty} \frac{(bq^n, aq^{-n}; p)_{\infty}}{(-zq^n, -q^{1-n}/z; q)_{\infty}} = \int_{-\infty}^{\infty} \frac{(bz^n, aq^{-x}; p)_{\infty}}{(-zq^n, -q^{1-x}/z; q)_{\infty}} \, dx,
\]

or in terms of \( q \)-binomial coefficients

\[
\sum_{n=-\infty}^{\infty} \left[ \frac{a}{b + \alpha n} \right]_p \frac{1}{(-zq^n, -q^{1-n}/z; q)_{\infty}} = \int_{-\infty}^{\infty} \left[ \frac{a}{b + \alpha x} \right]_p \frac{1}{(-zq^n, -q^{1-x}/z; q)_{\infty}} \, dx,
\]

where \( q = p^\alpha, 0 < \alpha < 1 \).
This gives an example of function for which sum equals integral. The case \(|p| = |q| < 1\), \(|b/a| < |z| < 1\) was known to Ramanujan. In this case, the series is Ramanujan’s \(1_1 \psi_1\) sum and the integral is Ramanujan’s \(q\)-beta integral \([11],\) chs. 5,6.

Now let \(z = e^{i\theta}, |\theta| < \pi\). Then

\[
\lim_{q \to 1^-} \frac{(-z, -q/z; q)_\infty}{(-2q^n, -q^1/z; q)_\infty} = (1 + z)^x(1 + 1/z)^{-x} = z^x.
\]

Let \(q \to 1\) with \(0 < \alpha < 1\) fixed in equation (6). Then formally

\[
\sum_{n=-\infty}^{\infty} \left( \frac{a}{b + \alpha n} \right) e^{i\theta n} = \int_{-\infty}^{\infty} \left( \frac{a}{b + \alpha x} \right) e^{i\theta x} dx, \quad 0 < \alpha < 1. \tag{7}
\]

The range of validity of (7) is \(-\pi \alpha < \theta < \pi \alpha\) as in (9), and not \(-\pi < \theta < \pi\). Continuing formal manipulations we obtain by using (7) and binomial theorem

\[
\int_{-\infty}^{\infty} \left( \frac{a}{b + \alpha x} \right) e^{i\theta x} dx = \frac{1}{\alpha} e^{-i\theta b/\alpha} \int_{-\infty}^{\infty} \left( \frac{a}{x} \right) e^{i\theta x/\alpha} dx
\]

\[
= \frac{1}{\alpha} e^{-i\theta b/\alpha} \sum_{n=-\infty}^{\infty} \left( \frac{a}{n} \right) e^{i\theta n/\alpha}
\]

\[
= \frac{1}{\alpha} e^{-i\theta b/\alpha} \sum_{n=0}^{\infty} \left( \frac{a}{n} \right) e^{i\theta n/\alpha}
\]

\[
= \frac{1}{\alpha} e^{-i\theta b/\alpha} (1 + e^{i\theta/\alpha})^a, \quad -\pi \alpha < \theta < \pi \alpha. \tag{8}
\]

Finally (7) and (8) imply

\[
\sum_{n=-\infty}^{\infty} \left( \frac{a}{b + \alpha n} \right) v^{b+\alpha n} = \frac{1}{\alpha} (1 + v)^a, \quad |v| = 1, \quad |\arg v| < \pi, \quad 0 < \alpha \leq 1, \tag{9}
\]

which is T. Osler’s generalization of binomial theorem \([18]\). According to Osler \([18]\), the special case \(\alpha = 1\) of (9) was first stated by Riemann \([24]\). It also follows from Ramanujan’s \(1_1 \psi_1\) sum in the limit \(q \to 1^-\).

It should be noted that while (9) has a closed form, the series in Theorem 2 does not. If \(p = q^2, z = 1, b = aq^2\), then one can prove that

\[
\sum_{n=-\infty}^{\infty} (bq^n, p/aq^n; p)_\infty z^n q^{n(n-1)/2} = 2 (qa, q/a; q^2) \sum_{n=-\infty}^{\infty} \frac{(-1/a)^n q^{n^2+n}}{1 - aq^{2n+1}}.
\]

The sum on the RHS is proportional to Appell-Lerch sum \(m(qa^2, q^2, q^2/a)\) in the notation of the paper \([19]\). In general Appell-Lerch sums do not have an infinite product representation. For example, by taking \(a = q^{-1/2}\) in \(m(qa^2, q^2, q^2/a)\) we get the sum of the type \(m(1, q^2, z)\) which is related to mock theta function of order 2 (see formula (4.2) in \([19]\)).

**Corollary 2.** The series

\[
\sum_{n=-\infty}^{\infty} \frac{(bq^n, p/aq^n; p)_\infty}{(-2q^n, -q/2q^n; q)_\infty}, \quad |p| < |q|
\]

with \(p\) and \(q\) fixed depends only on \(b/z\) and \(az\).
Theorem 3. 

\[ \int_{-\infty}^{\infty} \frac{(bq^x, aq^{-x}; p)_{\infty}}{(-q^x, -q^{1-x}; q)_{\infty}} e^{ixy} dx = \frac{2\pi i}{\log q} \left( -q, -q, e^{iy}, qe^{-iy}; q \right)_{\infty} \sum_{n=-\infty}^{\infty} \frac{(bq^n, aq^{-n}; p)_{\infty}}{(-q^n, -q^{1-n}; q)_{\infty}} e^{iny}. \]

Proof. Consider the contour integral

\[ \int_C \frac{(bq^z, aq^{-z}; p)_{\infty}}{(-q^z, -q^{1-z}; q)_{\infty}} e^{izy} dz \]

where \( C \) is rectangle with vertices at \((\pm R, 0), (\pm R, -2\pi i/\log q)\). In view of asymptotics found in the proof of Lemma \[\text{Lemma}\] integrals over the vertical segments vanish in the limit \( R \to +\infty \). Integrals over the horizontal segments are convergent and related by a factor of \(-e^{2\pi y/\log q}\). The integrand has simple poles at \( z = n - \pi i/\log q \) with residues

\[ -\frac{e^{\pi y/\log q}}{(q; q)_{\infty}^2 \log q} (-bq^n, -aq^{-n}; p)_{\infty} (-1)^n q^{n(n-1)/2} e^{iny}. \]

Application of the residue theorem yields

\[ \int_{-\infty}^{\infty} \frac{(bq^x, aq^{-x}; p)_{\infty}}{(-q^x, -q^{1-x}; q)_{\infty}} e^{ixy} dx = \frac{\pi i/\log q}{(q; q)_{\infty}^2 \sinh \frac{\pi y}{\log q}} \sum_{n=-\infty}^{\infty} (-bq^n, -aq^{-n}; p)_{\infty} (-1)^n q^{n(n-1)/2} e^{iny}. \]

According to Corollary \[\text{Corollary}\]

\[ \sum_{n=-\infty}^{\infty} (-bq^n, -aq^{-n}; p)_{\infty} (-1)^n q^{n(n-1)/2} e^{iny} = \frac{(e^{iy}, qe^{-iy}; q)_{\infty}}{(-e^{iy}, -qe^{-iy}; q)_{\infty}} \sum_{n=-\infty}^{\infty} (bq^n, aq^{-n}; p)_{\infty} q^{n(n-1)/2} e^{iny}. \]

To complete the proof observe that

\[ \sum_{n=-\infty}^{\infty} (bq^n, aq^{-n}; p)_{\infty} q^{n(n-1)/2} e^{iny} = (-1, -q; q)_{\infty} \sum_{n=-\infty}^{\infty} (bq^n, aq^{-n}; p)_{\infty} q^{n(n-1)/2} e^{iny} \]

and \((-1, -q; q)_{\infty} = 2(-q; q)_{\infty}^2\). \(\square\)

One can see from Theorem \[\text{Theorem}\] that the function

\[ g(x) = \frac{(bq^x, aq^{-x}; p)_{\infty}}{(-q^x, -q^{1-x}; q)_{\infty}} \]

is not band limited. However Fourier transform of \( g(x) \) vanishes at frequencies \( y = 2\pi m \), where \( m \neq 0 \) is an integer. Hence according to Poisson summation formula \[\text{Poisson}\]

\[ \sum_{n=-\infty}^{\infty} g(x) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)e^{-2\pi inx} dx = \int_{-\infty}^{\infty} g(x)dx \]

in agreement with Corollary \[\text{Corollary}\].

The fact that bilateral summation formulas in the theory of \( q \)-hypergeometric functions give examples of functions of the type \[\text{Type}\] has been recognized in the literature.
Corollary 3. Let $|p| < |q|$ and $m \in \mathbb{Z}$, then
\[
\int_{-\infty}^{\infty} \frac{(bq^n, aq^{-x}; p)}{(-q^n, -q^{1-x}; q)_{\infty}} q^{mx} \, dx = \sum_{n=-\infty}^{\infty} \frac{(bq^n, aq^{-n}; p)}{(-q^n, -q^{-1}; q)_{\infty}} q^{mn}.
\]

Proof. Resolve the $\frac{q}{z}$ ambiguity at the rhs of the formula of Theorem 2 using L'Hopital's Rule. \hfill \Box

Next we apply the method due to Bailey [22] to the identity in Theorem 2.

Theorem 4.
\[
\sum_{n=-\infty}^{\infty} \left( b_1q^n, b_2q^n, a_1q^{-n}, a_2q^{-n}; p \right) \propto z^n q^{n(n-1)} = \sum_{n=-\infty}^{\infty} \left( b_1q^n/z, b_2q^n/z, a_1zq^{-n}, a_2zq^{-n}; p \right) \propto z^{-n} q^{n(n-1)}.
\]

Proof. Multiplying the equations
\[
\sum_{n=-\infty}^{\infty} \left( b_1q^n, a_1q^{-n}; p \right) e^{i\theta n} q^{n(n-1)/2} = \frac{(-e^{i\theta}, -qe^{-i\theta}; q)_{\infty}}{\ln \frac{1}{q}} \int_{0}^{\infty} \frac{(b_1te^{-i\theta}, a_1e^{i\theta}/t; p)_{\infty}}{(-t, -q/t; q)_{\infty}} \, dt
\]
\[
\sum_{n=-\infty}^{\infty} \left( b_2q^n, a_2q^{-n}; p \right) e^{-i\theta n} z^n q^{n(n-1)/2} = \frac{(-ze^{-i\theta}, -qe^{i\theta}/z; q)_{\infty}}{\ln \frac{1}{q}} \int_{0}^{\infty} \frac{(b_2te^{i\theta}/z, a_2ze^{-i\theta}/t; p)_{\infty}}{(-t, -q/t; q)_{\infty}} \, dt
\]
and integrating with respect to $\theta$ one obtains
\[
\sum_{n=-\infty}^{\infty} \left( b_1q^n, b_2q^n, a_1q^{-n}, a_2q^{-n}; p \right) \propto z^n q^{n(n-1)}
\]
\[
= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{(-e^{i\theta}, -qe^{-i\theta}; q)_{\infty}}{\ln \frac{1}{q}} \int_{0}^{\infty} \frac{(b_1te^{-i\theta}, a_1e^{i\theta}/t_1; p)_{\infty}}{(-t_1, -q/t_1; q)_{\infty}} \, dt_1
\]
\[
\times \frac{(-ze^{-i\theta}, -qe^{i\theta}/z; q)_{\infty}}{\ln \frac{1}{q}} \int_{0}^{\infty} \frac{(b_2te^{i\theta}/z, a_2ze^{-i\theta}/t_2; p)_{\infty}}{(-t_2, -q/t_2; q)_{\infty}} \, dt_2
\]
\[
= z \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{(-e^{-i\theta}, -qe^{i\theta}; q)_{\infty}}{\ln \frac{1}{q}} \int_{0}^{\infty} \frac{(b_2te^{i\theta}/z, a_2ze^{-i\theta}/t_2; p)_{\infty}}{(-t_2, -q/t_2; q)_{\infty}} \, dt_2
\]
\[
\times \frac{(-e^{i\theta}/z, -qze^{-i\theta}; q)_{\infty}}{\ln \frac{1}{q}} \int_{0}^{\infty} \frac{(b_1te^{-i\theta}, a_1e^{i\theta}/t_1; p)_{\infty}}{(-t_1, -q/t_1; q)_{\infty}} \, dt_1
\]
\[
= z \sum_{n=-\infty}^{\infty} \left( b_1q^n/z, b_2q^n/z, a_1zq^{-n}, a_2zq^{-n}; p \right) \propto z^{-n} q^{n(n-1)}. \quad \Box
\]

Corollary 4. Let $0 < q < 1$ and $0 < \alpha < 1$, then
\[
\sum_{n=-\infty}^{\infty} \left[ \frac{a_1}{b_1 + \alpha n} \right]_p \left[ \frac{a_2}{b_2 + \alpha n} \right]_p p^{\alpha n(n-1)+\alpha n} = p^\alpha \sum_{n=-\infty}^{\infty} \left[ \frac{a_1}{b_1 - \theta + \alpha n} \right]_p \left[ \frac{a_2}{b_2 - \theta + \alpha n} \right]_p p^{\alpha n(n-1)+\alpha n}.
\]

Theorem 2 can be generalized.

Theorem 5. Let $q = p_1^{\alpha_1} = p_2^{\alpha_2}$ where $0 < \alpha_1 + \alpha_2 < 1$, then
\[ \sum_{n=-\infty}^{\infty} \left[ \frac{a_1}{b_1 + \alpha n} \right]_{p_1} \left[ \frac{a_2}{b_2 + \alpha n} \right]_{p_2} \frac{1}{(-zq^n, -q^{1-n}/z; q)_{\infty}} = \int_{-\infty}^{\infty} \left[ \frac{a_1}{b_1 + \alpha x} \right]_{p_1} \left[ \frac{a_2}{b_2 + \alpha x} \right]_{p_2} \frac{dx}{(-zq^x, -q^{1-x}/z; q)_{\infty}}. \]

[1] C. Störmer, *Om en generalisation af integralet* \( \int_{0}^{\infty} \frac{\sin ax}{x} \, dx = \frac{\pi}{2} \), Videnskaps-selskapet i Kristiania, 4, (1895).

[2] C. Störmer, *Sur une Généralisation de la formule* \( \frac{\pi}{2} = \frac{\sin x}{x} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} \ldots \), Acta math. 19, 341-350 (1895).


