Euler and Navier-Stokes Equations – From 2015-May-11 to 2019-December-15

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01 – Breakdown of Navier-Stokes Solutions

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Abstract – We have proved that there are initial velocities $u^0(x)$ and forces $F(x,t)$ such that there is no solution to the Navier-Stokes equations, which corresponds to the cases (C) and (D) of the problem relating to Navier-Stokes equations available on the website of the Clay Institute. First we study these cases at $t=0$ and then at $t \geq 0$.

Keywords – Navier-Stokes equations, Euler equations, continuity equation, breakdown, inexistence, existence, smoothness, solutions, uniqueness, gradient field, conservative field, velocity, pressure, external force, millenium problem.

Eureka! (Arquimedes)

1. Introdução

O fato de não ser possível resolver sempre o sistema $\frac{\partial p}{\partial x_i} = \phi_i$, $1 \leq i \leq 3$, nos leva a acreditar que não pode ser sempre possível encontrar solução para a Equação de Navier-Stokes em $n=3$ dimensões espaciais com força externa, ou seja,

(1) $$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i - \frac{\partial p}{\partial x_i} + F_i,\ 1 \leq i \leq 3,$$

para u_i, p, F_i funções da posição $x \in \mathbb{R}^3$ e do tempo $t \geq 0, t \in \mathbb{R}$. A constante $\nu \geq 0$ é o coeficiente de viscosidade, p representa a pressão e $u = (u_1, u_2, u_3)$ é a velocidade do fluido, medidas na posição x e tempo t. A função $F = (F_1, F_2, F_3)$ tem dimensão de aceleração ou força por unidade de massa, mas seguiremos denominando este vetor e suas componentes pelo nome genérico de força, tal como adotado em [1].

Sejam ϕ_i funções da posição $x \in \mathbb{R}^3$ e tempo $t \geq 0$ tal que não haja solução para o sistema $\frac{\partial p}{\partial x_i} = \phi_i$, $i = 1, 2, 3$, nossa hipótese.

Então, quaisquer que sejam u_i e os respectivos $u_i^0(x) = u_i(x, 0)$ é sempre possível encontrar forças F_i tais que

(2) $$\frac{\partial p}{\partial x_i} = \nu \nabla^2 u_i - \frac{\partial u_i}{\partial t} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} + F_i = \phi_i,$$

ou seja,

(3) $\quad F_i = \phi_i - \nu\nabla^2 u_i + \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}$,

para todo i tal que $1 \leq i \leq 3$, desde que as derivadas parciais dos campos vetoriais u_i existam.

Como não há uma solução possível para o sistema $\frac{\partial p}{\partial x_i} = \phi_i$, $1 \leq i \leq 3$, por hipótese, as funções F_i obtidas em (3) resultarão em não possibilidade de solução para o sistema (1), portanto é possível encontrar funções F_i para as componentes da força externa F tais que não haja solução para o sistema de equações diferenciais parciais (1), que são as equações de Navier-Stokes, para $i = 1, 2, 3$.

Verifica-se assim que existe a "quebra das soluções de Navier-Stokes" sobre $\mathbb{R}^3$ para específicas funções da força externa $F = (F_1, F_2, F_3)$, e então é possível solucionar este que é um dos mais difíceis problemas de Matemática em aberto.

A prova que fazemos é, em linhas gerais, bastante simples, por redução ao absurdo. Supomos por hipótese que não há equação de Navier-Stokes (1) sem solução (p, u) possível, ou seja, supomos que sempre existe solução para (1) dados $u^0(x)$ e $F(x, t)$, para $x \in \mathbb{R}^3$ e todo $t \geq 0$, e assim não existe quebra de soluções da equação de Navier-Stokes, supondo satisfeitas todas as demais condições que $p: \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}$ e $F, u: \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$ também devem obedecer neste problema, por exemplo, a equação da continuidade para densidade de massa constante (fluidos incompressíveis). Mas verificaremos que existem campos de velocidade $u(x, t) \in \mathbb{R}^3, x \in \mathbb{R}^3, t \geq 0$, os correspondentes $u^0(x) = u(x, 0)$ e campos $F(x, t) \in \mathbb{R}^3$ que obedecem a todas estas condições necessárias e tais que (1) não tem solução alguma em $t \geq 0$, para nenhuma pressão $p(x, t) \in \mathbb{R}$, seja periódica ou não, o que contradiz nossa hipótese inicial. Provaremos primeiro para $t = 0$, utilizando a condição inicial adicional $\frac{\partial u(x,t)}{\partial t}\Big|_{t=0}$, e a seguir para um tempo genérico $t \geq 0$.

Por outro lado, seguindo método similar ao aqui descrito, também é possível encontrar forças externas F tais que (1) tenha solução, inclusive para uma mesma velocidade inicial u^0 válida no caso de ocorrência de quebra de solução. E mesmo para os casos de existência de soluções, a solução de (1) não é única. Se $(p(x, t), u(x, t))$ é uma solução de (1), para $F(x, t)$ igual a zero ou não, haverá uma infinidade de outras soluções para (1), em especial as soluções da forma $(p(x, t) + \theta(t), u(x, t))$, já que a pressão na equação de Navier-Stokes aparece recebendo uma aplicação diferencial em relação ao espaço, $\nabla p = \left(\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_3}\right)$, sem proporcionar nenhuma influência no comportamento da velocidade u a soma da pressão com uma constante numérica ou função $\theta(t)$ diferenciável, dependente unicamente do tempo t, pois $\nabla p(x, t) = \nabla\big(p(x, t) + \theta(t)\big)$, para todo (x, t). Para obediência de condições iniciais, basta assumir $\theta(t=0)=0$, e assim $p(x,t) = p(x,t)+\theta(t)$ em $t=0$.

Chega-se assim à conclusão de que não há unicidade de soluções para as Equações de Navier-Stokes: seja $F=0$ (vetor nulo) ou não, seja $\nu=0$ ou não, se há alguma solução (p,u) para (1) então há infinitas outras soluções para (1), para os mesmos F e ν, dadas as mesmas condições iniciais, por exemplo,

$$(4) \qquad \begin{cases} u(x,0) = u^0(x) \\ \frac{\partial u(x,t)}{\partial t}\Big|_{t=0} = a^0(x) \\ p(x,0) = p^0(x) \end{cases}$$

devido à infinidade de soluções $\theta(t)$ possíveis de serem somadas à pressão $p(x,t)$, função da posição x e do tempo t. Mesmo o acréscimo de mais condições iniciais para p podem não resolver (1) de maneira única.

Assim como não há unicidade de soluções também não há unicidade em não soluções: se $\big(p(x,t),u(x,t)\big)$ não resolve (1) então $(p(x,t)+\theta(t),u(x,t))$ também não resolverá. Além disso, uma mesma velocidade inicial $u^0(x)$ e pressão inicial $p^0(x)$ podem corresponder tanto a casos de existência quanto de quebra de soluções, conforme a força externa $F(x,t)$, calculada em função de u e da derivada temporal de u, $\frac{\partial u(x,t)}{\partial t}$, implicar em um sistema de equações diferenciais parciais na incógnita p solúvel ou não.

Quer fixemos nossa análise unicamente ao tempo inicial $t=0$ ou não, para um mesmo valor das condições iniciais (4) é possível encontrar uma força F que implique em não solução para (1) e outra força G que implique em existência de soluções, conforme veremos na seção 3. Simbolicamente, dados $u(x,t), a^0(x) = \frac{\partial u(x,t)}{\partial t}\Big|_{t=0}$ e $u^0(x) = u(x,0)$ pode-se encontrar forças $F(x,t)$ e $G(x,t)$ tais que

$$(5) \qquad \forall u(x,t), \exists u^0(x), \exists a^0(x), \exists F(x,t), \nexists p \;/$$

$$\nabla p = \nu\nabla^2 u - \frac{\partial u}{\partial t} - (u\cdot\nabla)u + F,$$

$$(6) \qquad \forall u(x,t), \exists u^0(x), \exists a^0(x), \exists G(x,t), \exists p \;/$$

$$\nabla p = \nu\nabla^2 u - \frac{\partial u}{\partial t} - (u\cdot\nabla)u + G.$$

A condição inicial envolvendo $a^0(x)$ será utilizada nas seções 3 §3 e 4 §1, mas torna-se irrelevante na continuação destas seções 3 e 4, dando-se provas mais gerais em 3 §4 e 4 §2 para $t \geq 0$. A seção 3 §5 contém alguns esclarecimentos sobre as demonstrações utilizadas, no que diz respeito a usarmos a força F como uma função da velocidade u e suas derivadas (∂, ∇^2), e não apenas de u^0, x, t.

2. O Problema do Milênio

No famoso problema do milênio referente às equações de Navier-Stokes, descrito na página do Instituto Clay[1], das quatro possibilidades para sua solução as duas primeiras pedem uma prova de que existe uma solução para as funções da pressão $p(x,t)$ e velocidades $u_i(x,t)$ em $\mathbb{R}^3 \times [0,\infty)$, $1 \leq i \leq 3$, para o caso específico de $F(x,t) = (F_1, F_2, F_3)(x,t) = 0$ (ausência de força externa, vetor nulo 0) e $\nu > 0$. As duas últimas possibilidades pedem uma prova de que existem funções para a força externa $F(x,t) = (F_1, F_2, F_3)(x,t)$ e velocidade inicial $u^0(x)$ tais que não existe solução para as equações de Navier-Stokes com $\nu > 0$. O caso $\nu = 0$ resulta na chamada Equação de Euler, que também não tem solução geral conhecida para $n = 3$, mas esta não faz parte do problema do milênio.

Além de $\nu > 0$ e dimensão espacial $n = 3$ as quatro alternativas têm em comum a condição de divergente nulo para a velocidade, propriedade dos fluidos incompressíveis (densidade de massa constante na equação da continuidade),

(7) $$div\, u \equiv \nabla \cdot u = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0,$$ (fluidos incompressíveis)

e ser a velocidade inicial $u^0(x) = u(x,0)$ um campo vetorial C^∞ com divergente nulo ($\nabla \cdot u^0 = 0$) sobre $\mathbb{R}^3$. Para que uma solução (p,u) seja fisicamente razoável, se requer que $u(x,t)$ não cresça infinitamente para $|x| \to \infty$ e que

(8) $$p, u \in C^\infty(\mathbb{R}^3 \times [0,\infty))$$

e

(9) $$\int_{\mathbb{R}^3} |u(x,t)|^2 dx < C, \text{ para todo } t \geq 0,$$ (bounded energy)

satisfazendo (1) e (7).

Alternativamente, a condição (9) de energia (cinética) total limitada pode ser substituída pela condição de periodicidade espacial da velocidade e respectiva velocidade inicial, assim como pressão e força externa espacialmente periódicas, i.e.,

(10) $$u(x,t) = u(x + e_j, t),$$

(11) $$u^0(x) = u^0(x + e_j),$$

(12) $$p(x,t) = p(x + e_j, t)$$

e

(13) $$F(x,t) = F(x + e_j, t),$$

onde e_j é o j^{th} vetor unitário em $\mathbb{R}^3$, para $1 \leq j \leq 3$, igualdades válidas para u, p, F sobre $\mathbb{R}^3 \times [0, \infty)$ e u^0 sobre $\mathbb{R}^3$.

Neste artigo estamos tratando principalmente dos casos (C) e (D) descritos em [1], ou seja:

(C) Quebra das soluções da Equação de Navier-Stokes sobre $\mathbb{R}^3$. Para $\nu > 0$ e dimensão espacial $n = 3$ existem um campo vetorial suave e com divergência nula $u^0(x) = u(x, 0)$ sobre $\mathbb{R}^3$ e uma força externa suave $F(x, t)$ sobre $\mathbb{R}^3 \times [0, \infty)$ satisfazendo

(14) $|\partial_x^\alpha u^0(x)| \leq C_{\alpha k}(1 + |x|)^{-k}$ sobre $\mathbb{R}^3$, para quaisquer $\alpha \in \mathbb{N}_0^3$ e $k \geq 0$,

e

(15) $|\partial_x^\alpha \partial_t^m F(x, t)| \leq C_{\alpha m k}(1 + |x| + t)^{-k}$ sobre $\mathbb{R}^3 \times [0, \infty)$, para quaisquer $\alpha \in \mathbb{N}_0^3, m \in \mathbb{N}_0$ e $k \geq 0$,

tais que não existe solução (p, u) sobre $\mathbb{R}^3 \times [0, \infty)$ satisfazendo (1), (7), (8) e (9).

(D) Quebra das soluções da Equação de Navier-Stokes sobre $\mathbb{R}^3/\mathbb{Z}^3$. Para $\nu > 0$ e dimensão espacial $n = 3$ existem um campo vetorial suave e com divergência nula $u^0(x) = u(x, 0)$ sobre $\mathbb{R}^3$ e uma força externa suave $F(x, t)$ sobre $\mathbb{R}^3 \times [0, \infty)$ satisfazendo as condições de periodicidade espacial (11) e (13), e a condição

(16) $|\partial_x^\alpha \partial_t^m F(x, t)| \leq C_{\alpha m k}(1 + t)^{-k}$ sobre $\mathbb{R}^3 \times [0, \infty)$, para quaisquer $\alpha \in \mathbb{N}_0^3, m \in \mathbb{N}_0$ e $k \geq 0$,

tais que não existe solução (p, u) sobre $\mathbb{R}^3 \times [0, \infty)$ satisfazendo (1), (7), (8), (10) e (12).

Utilizamos $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$, o conjunto dos inteiros não negativos. Derivadas de ordem zero não alteram o valor da função.

Na seção 5 faremos alguns comentários sobre os casos (A) e (B), de existência de soluções.

3. O caso (C)

§ 1

Vamos encontrar primeiramente funções $u: \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$ que são soluções da equação diferencial parcial (equação da continuidade para densidade de massa constante)

(17) $\nabla \cdot u = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0,$

a condição de incompressibilidade (7).

Esta equação equivale à lei de Gauss para o campo magnético e para os campos elétrico e gravitacional no vácuo.

Soluções de (17) que correspondem a campos elétricos ou gravitacionais no vácuo, para uma única partícula na origem, fonte do campo (carga ou massa, respectivamente), são da forma

(18) $u = \frac{\alpha}{r^2}\hat{r} = \frac{\alpha}{r^3}\vec{r},$

onde $\vec{r} = (x, y, z)$ é o vetor posição, $\hat{r}$ seu versor, $r = \sqrt{x^2 + y^2 + z^2}$ o módulo de $\vec{r}$ e $\alpha \in \mathbb{R}$ o fator de proporcionalidade dependente do valor da carga ou massa, respectivamente.

Não fosse a condição (9) de energia total limitada, a existência de divergência na origem e sua derivabilidade nesse ponto as componentes dos campos elétricos e gravitacionais poderiam ser candidatas às funções u_i de componentes de velocidades, mas não às funções ϕ_i mencionadas na Introdução, tais que não exista solução para o sistema de equações diferenciais parciais

(19) $\frac{\partial p}{\partial x_i} = \phi_i,\ 1 \le i \le 3.$

Para satisfazer (9) e (14) vamos escolher para u campos vetoriais com decaimento exponencial em todas as três direções ortogonais tais que

(20) $\lim_{|x| \to \infty} u_i(x, t) = 0,\ 1 \le i \le 3,$

e que também devem obedecer (7), por exemplo,

(21) $u = ae^{-b(x_1^2 + x_2^2 + x_3^2)}(x_2 x_3, x_1 x_3, -2x_1 x_2),\ a \in \mathbb{R}^*, b \in (0,1],$

um campo de velocidades sem aceleração local, estacionário. Observo aqui que (9) bem poderia ser desprezada, ou pelo menos modificada, caso a força externa total aplicada $\int_V |F(x,t)|dx$ fosse infinita. Uma força total infinita corresponde normalmente a uma energia total também infinita, portanto, no que diz respeito aos fundamentos físicos, não haveria necessidade de limitar a uma constante C a integração do quadrado da velocidade sobre todo o espaço $\mathbb{R}^3$. Uma substituta mais natural para (9) seria, por exemplo,

(22) $\int_V |u(x,t)|^2 dx \le A + B \int_V |F(x,t)|dx,\ A > 0, B \ge 0,$

para todo $t \geq 0$, e em todo subconjunto $V \subseteq \mathbb{R}^3$ onde estiver sendo aplicada esta força.

Sabemos que a integração do sistema (19) só é possível no caso de campos conservativos e neste caso resulta em

(23) $$p = \int_L \phi \cdot dl + \theta(t),$$

com $\phi = (\phi_1, \phi_2, \phi_3)$ contínua e $\theta: [0, \infty) \to \mathbb{R}$ diferenciável, correspondendo no caso destes campos conservativos à função trabalho, ou variação da energia cinética (quando ϕ é a força elétrica ou gravitacional e $\theta(t) = 0$), igual à variação (negativa) da energia potencial. Nessa situação a integral sempre existe e, a menos da função $\theta(t)$, independe do caminho L entre os pontos $x_o \in \mathbb{R}^3$ e $x \in \mathbb{R}^3$, supondo que L seja contínuo por partes, de classe C^1 e não passe por nenhuma singularidade de ϕ. Diz-se que p é uma função potencial para ϕ.

Precisamos então buscar um campo vetorial $\phi = (\phi_1, \phi_2, \phi_3)$ que não seja gradiente, i.e., não deve existir uma função $p: \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}$ tal que exista solução para a equação

(24) $$\nabla p = \phi,$$

que equivale ao sistema (19) anterior.

Em muitos livros de Análise Matemática e Cálculo Diferencial e Integral pode-se encontrar a solução para este problema. Um dos grandes clássicos é o Apostol[2] (vol. II, cap. 10, Integrais de Linha), embora Courant, Elon Lages Lima, Guidorizzi, Kaplan, Piskunov, etc. sejam igualmente ótimas referências.

No teorema 10.6 de Apostol (seção 10.16) se prova que uma condição necessária para que um campo vetorial $f = (f_1, \dots, f_n)$ continuamente diferenciável em um conjunto aberto S de $\mathbb{R}^n$ seja um gradiente em S é que as derivadas parciais das componentes de f estejam ligadas pela relação

(25) $$D_i f_j(x) = D_j f_i(x),$$

para todo $i, j = 1,2, \dots, n$ e todo x de S. D_i é o operador diferencial $\frac{\partial}{\partial x_i}$.

No teorema 10.9 de Apostol (seção 10.21) se prova que a condição (25) também é uma condição suficiente se o conjunto S é um conjunto convexo aberto de $\mathbb{R}^n$.

Vamos então a seguir buscar um campo vetorial $\phi = (\phi_1, \phi_2, \phi_3)$, $\phi_i: \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}$, tal que

(26) $\frac{\partial \phi_i}{\partial x_j} \neq \frac{\partial \phi_j}{\partial x_i}, i \neq j,$

para algum par $(i, j), 1 \leq i, j \leq 3$, $x \in \mathbb{R}^3$ e tempos t não negativos. Adotaremos que nosso conjunto convexo aberto S é o próprio $\mathbb{R}^3$.

Além da condição (26) a condição (17) de incompressibilidade da velocidade também deve ser satisfeita, bem como as demais condições impostas neste problema do milênio, tais como (14) e (15).

Funções simples que obedecem (26) são, por exemplo,

1) $\big(ay, bx, c(x + y)\big)$, $\quad a \neq b \neq c$,

2) $(0, xzt, xyt)$, $\quad x, y, z \neq 0,\ t \geq 0$,

3) $(e^{-ayt}, e^{-bxt}, e^{-czt})$, $\quad a, b, c \neq 0,\ t \geq 0$,

onde usamos $x_1 \equiv x,\ x_2 \equiv y,\ x_3 \equiv z$, mas não podemos escolher arbitrariamente qualquer ϕ solução de (26).

Para que F e suas derivadas tendam a zero no infinito, e obedeçam (15), vamos escolher para ϕ uma função limitada, contínua, com limite zero no infinito, com todas as derivadas também contínuas (C^∞), limitadas e indo a zero no infinito, que obedeça (26) e que resulte numa função F, conforme (3), tal que seja possível provar (15).

Analisemos as três situações possíveis para φ.

Se o campo vetorial $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ definido por

(27) $\varphi_i = \nu \nabla^2 u_i - \frac{\partial u_i}{\partial t} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}, 1 \leq i \leq 3,$

não for um gradiente, i.e., for tal que

(28) $\frac{\partial \varphi_i}{\partial x_j} \neq \frac{\partial \varphi_j}{\partial x_i}$ para algum $i \neq j, 1 \leq i, j \leq 3$,

escolhemos $\phi_i = \varphi_i$, e então, conforme (3),

(29) $F_i = \phi_i - \varphi_i = 0$, para todo i tal que $1 \leq i \leq 3$.

Vê-se que é possível uma força nula obedecer às condições deste problema do milênio no caso de quebra de soluções. Assim, não me parece possível resolver em toda sua generalidade os casos (A) e (B) deste problema, embora não seja minha pretensão provar isto neste artigo.

Se φ for um gradiente devemos encontrar um campo vetorial $\omega = (\omega_1, \omega_2, \omega_3)$ que não seja gradiente, i.e., seja não conservativo, e assim o campo vetorial

(30) $\phi = \varphi + \omega$

também não será gradiente, será não conservativo, e

(31) $F_i = \phi_i - \varphi_i = \omega_i$, para todo i tal que $1 \leq i \leq 3$.

Um campo vetorial $\omega = F$ fácil de ser obtido é

(32) $\omega = (c_1\varphi_1, c_2\varphi_2, c_3\varphi_3)$,

para constantes reais $c_i \neq c_j \neq 0$, $i \neq j$.

Como neste caso φ é gradiente, i.e., conservativo, então (condição necessária)

(33) $\frac{\partial \varphi_i}{\partial x_j} = \frac{\partial \varphi_j}{\partial x_i}$ para todo i, j tais que $1 \leq i, j \leq 3$.

Mas se $\omega_i = c_i\varphi_i$ e as derivadas parciais $\frac{\partial \varphi_i}{\partial x_j}$ não são identicamente nulas então

(34) $\frac{\partial \omega_i}{\partial x_j} = c_i \frac{\partial \varphi_i}{\partial x_j} \neq c_j \frac{\partial \varphi_j}{\partial x_i} \neq 0$, para $c_i \neq c_j \neq 0$, $i \neq j$,

i.e.,

(35) $\frac{\partial \omega_i}{\partial x_j} \neq \frac{\partial \omega_j}{\partial x_i}$, $i \neq j$,

portanto ω não é gradiente e

(36) $F_i = \phi_i - \varphi_i = \omega_i = c_i\varphi_i$, para todo i tal que $1 \leq i \leq 3$.

O terceiro e último caso ocorre quando para todo $x \in \mathbb{R}^3, t \geq 0$,

(37) $\frac{\partial \varphi_i}{\partial x_j} = \frac{\partial \varphi_j}{\partial x_i} = 0$ para todo i, j tais que $1 \leq i, j \leq 3$,

indicando que φ é um campo conservativo e suas derivadas parciais de primeira ordem são iguais a zero.

Como buscamos algum par (i, j) tal que $\frac{\partial \phi_i}{\partial x_j} \neq \frac{\partial \phi_j}{\partial x_i}$ em geral e queremos alguma função F cujas sucessivas derivadas parciais mistas sejam da ordem de

$(1+|x|+t)^{-k}$ sobre $\mathbb{R}^3 \times [0,\infty)$ vamos escolher F tal que tenda a zero, assim como suas infinitas derivadas, em $|x| \to \infty$ e $t \to \infty$, conforme (15), i.e.,

$$(38)\quad \lim_{|x|\to\infty} \left|\frac{\partial^{p+q}}{\partial x^p \partial t^q} F(x,t)\right| = \lim_{t\to\infty} \left|\frac{\partial^{p+q}}{\partial x^p \partial t^q} F(x,t)\right| = 0,$$
$$p \geq 0,\ q \geq 0,\ \partial x^p \equiv \partial x_1^{p_1} \partial x_2^{p_2} \partial x_3^{p_3},\ p_1+p_2+p_3 = p,\ p_i \geq 0,$$

e seja F um campo não conservativo. Assim a soma $\varphi + F$, que deve ser igual a ∇p,

$$(39)\quad \varphi + F = \phi = \nabla p,$$

será igual a um campo ϕ não conservativo e portanto não haverá solução para (39), equivalente a (24) e (19). Usamos a propriedade de que a soma de um campo vetorial conservativo e um não conservativo é um campo vetorial não conservativo.

Escolhemos para F nesse caso um campo não conservativo que decresce exponencialmente em relação à posição e ao tempo em ao menos uma das coordenadas espaciais e pode ser igual a zero nas coordenadas restantes (se houver). Por exemplo,

$$(40)\quad F_i = a_i e^{-b_i(x_1^2+x_2^2+x_3^2)} \cdot e^{-c_i t},\ 1 \leq i \leq 3,$$

com $a_i \neq 0, b_i, c_i \in (0,1]$. As componentes F_i poderiam depender do tempo ou não, conforme (40), sem alterar a propriedade de ser a força externa F um campo não conservativo, mas a não dependência do tempo faria F desobedecer (15) para $m = 0$.

Para que F seja fisicamente consistente é necessário que a_i tenha a dimensão de aceleração ou força por unidade de massa, b_i tenha a dimensão de recíproco de comprimento ao quadrado e c_i dimensão de recíproco de tempo.

§ 2

Vistas as três situações possíveis para $\varphi = \phi - F$ vamos agora à demonstração com nosso exemplo específico. Suponhamos, por hipótese, que não há equação de Navier-Stokes sem solução (p,u) possível, ou seja, dados $u^0(x)$ e $F(x,t)$ para $x \in \mathbb{R}^3$ sempre há solução para (1), para todo instante $t \geq 0$, supondo ainda satisfeitas todas as condições que devem obedecer a pressão $p: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}$ e a velocidade $u: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$ neste problema do milênio, por exemplo, a condição de incompressibilidade (7).

Iniciemos ampliando a condição inicial $u(x,0) = u^0(x)$ do instante $t = 0$ para todo t do intervalo $0 \leq t \leq T,\ T \in \mathbb{R}_+$, ou seja, deve valer como condição de contorno

$$(41)\quad u(x,t) = u^T(x,t),\ 0 \leq t \leq T,$$

sendo $u^T: \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$.

A velocidade u escolhida em (21) independe do tempo t, assim deve valer, para todo t em $0 \leq t \leq T$,

(42) $$u(x,t) = u^T(x,t) = u^0(x) = ae^{-b\left(x_1^2+x_2^2+x_3^2\right)}(x_2x_3, x_1x_3, -2x_1x_2),$$
$a \in \mathbb{R}^*, b \in (0,1]$,

e então, no intervalo de tempo $0 \leq t < T$,

(43) $$\frac{\partial u}{\partial t} = 0,$$

o que corresponde a um fluido sem aceleração local, uma solução estacionária, cuja velocidade em um ponto não varia no tempo.

As outras derivadas parciais em $0 \leq t \leq T$, para u_1 e u_2, com $x, y, z \equiv x_1, x_2, x_3$ e $r^2 = \sqrt{x^2 + y^2 + z^2}$, são

(44) $$\frac{\partial u_i}{\partial x_i} = -2abxyze^{-br^2},$$

(45) $$\begin{cases} \frac{\partial u_1}{\partial y} = -aze^{-br^2}(2by^2 - 1) \\ \frac{\partial u_1}{\partial z} = -aye^{-br^2}(2bz^2 - 1) \\ \frac{\partial u_2}{\partial x} = -aze^{-br^2}(2bx^2 - 1) \\ \frac{\partial u_2}{\partial z} = -axe^{-br^2}(2bz^2 - 1) \end{cases}$$

(46) $$\begin{cases} \frac{\partial^2}{\partial x^2} u_1 = 2abyze^{-br^2}(2bx^2 - 1) \\ \frac{\partial^2}{\partial y^2} u_1 = 2abyze^{-br^2}(2by^2 - 3) \\ \frac{\partial^2}{\partial z^2} u_1 = 2abyze^{-br^2}(2bz^2 - 3) \\ \nabla^2 u_1 = \left(\sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}\right) u_1 = 2abyze^{-br^2}(2br^2 - 7) \end{cases}$$

(47) $$\begin{cases} \frac{\partial^2}{\partial x^2} u_2 = 2abxze^{-br^2}(2bx^2 - 3) \\ \frac{\partial^2}{\partial y^2} u_2 = 2abxze^{-br^2}(2by^2 - 1) \\ \frac{\partial^2}{\partial z^2} u_2 = 2abxze^{-br^2}(2bz^2 - 3) \\ \nabla^2 u_2 = \left(\sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}\right) u_2 = 2abxze^{-br^2}(2br^2 - 7) \end{cases}$$

$$(48) \quad \begin{cases} \sum_{j=1}^{3} u_j \frac{\partial u_1}{\partial x_j} = a^2 x e^{-2br^2}(z^2 - 2y^2) \\ \sum_{j=1}^{3} u_j \frac{\partial u_2}{\partial x_j} = a^2 y e^{-2br^2}(z^2 - 2x^2) \end{cases}$$

e assim, de (27),

$$(49) \quad \begin{cases} \varphi_i = \nu \nabla^2 u_i - \frac{\partial u_i}{\partial t} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} \\ \varphi_1 = \nu \cdot 2abyze^{-br^2}(2br^2 - 7) - a^2 x e^{-2br^2}(z^2 - 2y^2) \\ \varphi_2 = \nu \cdot 2abxze^{-br^2}(2br^2 - 7) - a^2 y e^{-2br^2}(z^2 - 2x^2) \end{cases}$$

Comparando estas duas derivadas,

$$(50) \quad \frac{\partial \varphi_1}{\partial y} = \nu \cdot 2abze^{-br^2}[(1 - 2by^2)(2br^2 - 7) + 4by^2] +$$

$$+4a^2 xye^{-2br^2}[b(z^2 - 2y^2) + 1]$$

e

$$(51) \quad \frac{\partial \varphi_2}{\partial x} = \nu \cdot 2abze^{-br^2}[(1 - 2bx^2)(2br^2 - 7) + 4bx^2] +$$

$$+4a^2 xye^{-2br^2}[b(z^2 - 2x^2) + 1],$$

temos $\frac{\partial \varphi_1}{\partial y} \neq \frac{\partial \varphi_2}{\partial x}$, em geral, então φ é um campo vetorial não conservativo. Conforme (29), escolhendo $\phi_i = \varphi_i$, para $i = 1, 2, 3$, chega-se a

$$(52) \quad F_i = \phi_i - \varphi_i = 0.$$

Descreveríamos assim o movimento de um fluido não acelerado (localmente) nas três direções ortogonais $(1, 0, 0), (0, 1, 0), (0, 0, 1)$, sem força externa, no intervalo de tempo $0 \leq t < T$, mas há o problema de não se encontrar a pressão do sistema.

Como φ é não conservativo e $\varphi = \phi$ então ϕ é não conservativo. Como deveria valer a equação (2)

$$(53) \quad \frac{\partial p}{\partial x_i} = \phi_i, \ 1 \leq i \leq 3,$$

para haver solução de (1), mas ϕ é um campo vetorial não conservativo, i.e., não gradiente, então o sistema acima não tem solução, e portanto encontramos uma velocidade inicial $u^0(x) = u^T(x, t) = u(x, t)$ e uma força externa $F(x, t) = 0$ tal que não há solução para a equação de Navier-Stokes (1) no intervalo de tempo $0 \leq t < T$. Como nossa hipótese inicial admite haver solução (p, u) em todo $t \geq 0$ chegamos a uma contradição, o que invalida nossa hipótese inicial.

§ 3

Para uma demonstração compatível ao problema do milênio é necessário que $T \to 0$. A condição inicial requerida para a velocidade é $u(x,0) = u^0(x)$, portanto não podemos prefixar $u(x,t) = u^T(x,t)$, para $0 \le t \le T$, $T \in \mathbb{R}_+$, em nossa demonstração final.

A equação de Navier-Stokes (1) e a condição de incompressibilidade (7) devem ser satisfeitas para todo instante $t \ge 0$, portanto também em $t = 0$.

Em $t = 0$, sendo u e u^0 de classe C^∞, temos

(54) $\frac{\partial u_i}{\partial x_j}|_{t=0} = \frac{\partial u_i^0}{\partial x_j}$,

(55) $\nabla^2 u_i|_{t=0} = \nabla^2 u_i^0$,

(56) $\sum_{j=1}^3 u_j \frac{\partial u_i}{\partial x_j}|_{t=0} = \sum_{j=1}^3 u_j^0 \frac{\partial u_i^0}{\partial x_j}$,

(57) $\frac{\partial p}{\partial x_j}|_{t=0} = \frac{\partial p(x,0)}{\partial x_j}$,

mas nem sempre vale

(58) $\frac{\partial u_i}{\partial t}|_{t=0} = \frac{\partial u_i^0}{\partial t}$,

pois $\frac{\partial u_i^0}{\partial t}$ é identicamente nulo, enquanto $\frac{\partial u_i}{\partial t}|_{t=0}$ pode ser nulo ou não nos movimentos acelerados em geral, independentemente do valor de $u_i^0(x)$.

A equação (1) em $t = 0$ pode então ser reescrita como

(59) $\frac{\partial u_i}{\partial t}|_{t=0} + \sum_{j=1}^3 u_j^0 \frac{\partial u_i^0}{\partial x_j} = \nu \nabla^2 u_i^0 - \frac{\partial p}{\partial x_i}(x,0) + F_i(x,0),\ 1 \le i \le 3$,

ou, definindo $p(x,0) = p^0(x)$ e $F_i(x,0) = F_i^0(x)$,

(60) $\frac{\partial p^0}{\partial x_i} = \nu \nabla^2 u_i^0 - \frac{\partial u_i}{\partial t}|_{t=0} - \sum_{j=1}^3 u_j^0 \frac{\partial u_i^0}{\partial x_j} + F_i^0 = \phi_i^0$,

equação similar a (2) para $t = 0$.

Já vimos que sistemas semelhantes a (60), para $1 \le i \le 3$, só terão solução se ϕ^0 for um campo vetorial gradiente, ou conservativo, qualquer que seja o valor de t, e obviamente para $t = 0$ esta exigência precisará também ser obedecida.

Tal como feito em (23), a solução de (60), no caso de ϕ^0 ser gradiente, é

(61) $p^0 = \int_L \phi^0 \cdot dl + \theta(t=0),$

e assim fica claro que $\frac{\partial u_i}{\partial t}|_{t=0} = a_i^0(x)$, por não ter seu valor univocamente determinado através de u_i^0 e u^0, em geral, nem de F_i^0 e F^0, proporcionará um valor para a pressão inicial p^0 que dependerá deste valor de $\frac{\partial u_i}{\partial t}|_{t=0}$, a variação temporal inicial da componente u_i da velocidade.

Também podemos encontrar combinações de $\frac{\partial u_i}{\partial t}|_{t=0}$ e F_i^0, $1 \le i \le 3$, tais que o sistema (60) tenha solução ou não, conforme resultem em campos vetoriais ϕ^0 gradientes ou não, seguindo método semelhante ao indicado anteriormente nesta seção, por isso $u^0(x)$ e $F^0(x)$ não determinam de maneira única a quebra ou não das soluções das equações de Navier-Stokes em $t=0$. Consequentemente, $u^0(x)$ e $F^0(x)$ não determinam de maneira única a quebra ou não das soluções das equações de Navier-Stokes sobre $\mathbb{R}^3 \times [0,\infty)$ (ver equações (5) e (6)).

Usando o exemplo de velocidade inicial utilizado na seção 3 §2, em (42),

(62) $u^0(x) = ae^{-b(x_1^2+x_2^2+x_3^2)}(x_2x_3, x_1x_3, -2x_1x_2), a \in \mathbb{R}^*, b \in (0,1],$

para $0 \le t \le T$, façamos $T \to 0$, $F=0$ e escolhamos $\frac{\partial u}{\partial t}|_{t=0} = a^0(x) = 0$ como mais uma condição inicial, tal qual (43). Isso fará com que não haja solução para p em $t=0$, conforme os cálculos da seção 3 §2, e assim mostramos um exemplo de quebra de soluções da equação de Navier-Stokes em $t=0$ pelo acréscimo da condição inicial adicional $a^0(x)=0$. Sendo assim, não houve solução para (1) em todo $t \ge 0$, o que contraria nossa hipótese inicial. Tal exemplo satisfaz a todos os requisitos que devem ser obedecidos por $u^0(x)$ e $F(x,t)$.

§ 4

Faremos agora uma demonstração genérica para a quebra de soluções de Navier-Stokes sem utilizarmos nenhuma condição de contorno adicional, e para todo $t \ge 0$. Assemelha-se ao que já foi feito na Introdução, com uma descrição mais apropriada para o domínio, imagem e condições das variáveis. De fato as seções 3 §2 e 3 §3 poderiam ser excluídas do presente trabalho, não são de leitura obrigatória, uma vez que a prova mais abrangente é a deste §4. Optei por preservá-las porque correspondem a uma sequência de pensamentos que pode apoiar o entendimento completo deste problema.

Para um tempo real $t \ge 0$ qualquer, dada uma velocidade $u(x,t): \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$ que obedeça a todas as condições deste problema, descritas na seção 2, e tal que

(63) $u^0(x) = u(x,0)$

seja a velocidade inicial escolhida no nosso problema, para que haja solução de Navier-Stokes deve valer

(64) $\frac{\partial p}{\partial x_i} = \nu\nabla^2 u_i - \frac{\partial u_i}{\partial t} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} + F_i = \varphi_i + F_i = \phi_i, 1 \leq i \leq 3,$

com

(65) $\varphi_i = \nu\nabla^2 u_i - \frac{\partial u_i}{\partial t} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}, 1 \leq i \leq 3,$

onde se supõe que $u^0(x)$ por nós escolhido também obedece a todas as condições necessárias, em especial (14). Por convenção, escolhamos sempre $u^0(x)$ não gradiente, i.e., não conservativo.

Para este campo vetorial de velocidades $u(x,t)$ é possível calcular $\varphi(x,t)$, de (65), escolher um campo ϕ não conservativo com as mesmas propriedades razoáveis que devem obedecer u, p e F para o caso (C), e calcular

(66) $F = \phi - \varphi = \phi - \nu\nabla^2 u + \frac{\partial u}{\partial t} + (u \cdot \nabla)u.$

Escolhamos, por exemplo,

(67) $\phi(x,t) = u^0(x),$

que é um campo não conservativo pela nossa convenção e independente do tempo t. Suponhamos que a compatibilidade dimensional física entre ϕ e u^0 seja feita pela multiplicação do fator 1, cuja dimensão compatibiliza ambos os campos.

O valor para as componentes de $F(x,t)$ que obtemos de (66) é então

(68) $F_i = u_i^0 - \nu\nabla^2 u_i + \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}, 1 \leq i \leq 3,$

que deve satisfazer às mencionadas condições da seção 2 e depende de $\frac{\partial u_i}{\partial t}$, qualquer que seja o valor de $t \geq 0$. Evidentemente, se os F_i obtidos em (68) não obedecerem aos requisitos esperados escolhe-se outros $u^0(x)$ e $u(x,t)$ e repete-se o processo até que se obtenham componentes F_i adequadas, em especial que pertençam a C^∞ e obedeçam (15).

A força $F = (F_1, F_2, F_3)$ calculada pelo método acima e a velocidade inicial u^0 escolhida convenientemente em (63) garantem que chegue-se a um valor impossível de ser obtido para a pressão p, pois $\phi = u^0$ é não conservativo, segundo nossa escolha, o que prova a ocorrência de quebra (inexistência) de soluções para as equações de Navier-Stokes, conforme queríamos. Nossa hipótese inicial de que sempre há solução (p, u) possível para (1) foi então violada: dado p talvez possamos sempre encontrar u, mas dado u (e respectivo u^0) podemos não encontrar p, quando ϕ é não gradiente.

Claro que este raciocínio também pode levar a encontrar uma solução para p, trocando-se as funções não gradientes por gradientes. Devemos encontrar como resultado de (64) uma função ϕ gradiente, e para isso as escolhas preferenciais a serem feitas são combinações de F, u^0, u e φ também gradientes. É o que formulamos resumida e simbolicamente nas equações (5) e (6) da Introdução.

§ 5

Neste parágrafo explica-se melhor a prova do § 4 anterior.

Substituindo (68) em (64) obtemos

$$(69) \quad \frac{\partial p}{\partial x_i} = u_i^0, 1 \leq i \leq 3,$$

que não possui solução por ser u^0 não gradiente, pela nossa definição, e assim encontramos $u^0(x)$, $u(x,t)$ e $F(x,t) = H(u^0(x),\ u(x,t), \partial, \nabla^2, x, t)$ que levam à quebra (inexistência) das soluções de Navier-Stokes. Transformamos então a equação original (1) nesta equação (69).

Talvez seja difícil (ou até muito difícil) entender como é possível fazer com que $F(x,t) = H(u^0(x),\ u(x,t), \partial, \nabla^2, x, t)$ possa ser calculado e usado na demonstração. Pode-se pensar que devemos apenas encontrar "de alguma maneira" velocidades iniciais $u^0(x)$ e forças externas $F(x,t)$ únicas, fixas, tais que (1) não tenha solução alguma, para qualquer par de variáveis (p, u) que possam existir. Mais exatamente, parece que não podemos dar como exemplo uma força que depende da velocidade e suas derivadas (∂, ∇^2) em $t \geq 0$.

Vejamos então.

(I) Se u resolve (1) então u é uma função de F e u^0, suponhamos $u = f(F, u^0, \partial, \nabla^2, x, t) = g(x,t)$.

(II) Se u é uma função de F e u^0 então F é uma função ou uma relação de u e u^0, mesmo que tal relação não seja unívoca, i.e., $F = f^{-1}(u, u^0, \partial, \nabla^2, x, t) = g^{-1}(x,t)$.

(III) Se F pode ser expressa como função (ou relação) de u e u^0 a equação (68) que utilizamos pode ser aceita, no que diz respeito a ser F dependente de u e u^0. Vejam também que a definição do problema não proíbe que F seja função de u e suas derivadas, o que nos dá liberdade para que seja parte da estratégia de nossa solução.

A segunda objeção que pode ser feita é o fato de prefixarmos u, e não apenas u^0, de tal modo que escolhemos F dado por (68) igual a

(70) $$F = u^0 - \nu\nabla^2 u + \frac{\partial u}{\partial t} + (u \cdot \nabla)u.$$

Mas qual o significado de não existir (p, u), na definição do problema dado em [1]? Simbolicamente, usando Lógica, a não existência do par de variáveis (p, u) equivale à seguinte sentença:

(71) $$\nexists(p, u) \leftrightarrow ((\exists p \wedge \nexists u) \vee (\exists u \wedge \nexists p) \vee (\nexists p \wedge \nexists u)).$$

A opção que adotamos dentre as três possibilidades acima foi a existência de u com a não existência de p, i.e.,

(72) $$(\exists u \wedge \nexists p) \rightarrow \nexists(p, u).$$

Acredito que com estas explicações as dúvidas sobre a validade das demonstrações anteriores sejam eliminadas. A seguir um resumo da definição do problema para o caso (C), onde se acrescentou um novo requisito referente à existência de u (destacado na cor azul), mantendo-se a de não existência de (p, u), equivalente a $\nexists p$, não existência de p. Os números entre asteriscos (*) referem-se à numeração original das respectivas equações em [1].

$\nu > 0, n = 3$

$\exists u^0(x): \mathbb{R}^3$ smooth (C^∞), divergence-free ($\nabla \cdot u^0 = 0$) (ver nota A)

$\exists F(x, t): \mathbb{R}^3 \times [0, \infty)$ smooth (C^∞)

(*4*) $|\partial_x^\alpha u^0(x)| \le C_{\alpha k}(1 + |x|)^{-k}: \mathbb{R}^3,\ \forall \alpha, k$

(*5*) $|\partial_x^\alpha \partial_t^m F(x, t)| \le C_{\alpha m k}(1 + |x| + t)^{-k}: \mathbb{R}^3 \times [0, \infty)$, $\forall \alpha, m, k$

$\exists u(x, t): \mathbb{R}^3 \times [0, \infty)$ smooth (C^∞)

$\nexists(p, u): \mathbb{R}^3 \times [0, \infty)$ /

(*1*) $\frac{\partial u_i}{\partial t} + \sum_{j=1}^3 u_j \frac{\partial u_i}{\partial x_j} = \nu\nabla^2 u_i - \frac{\partial p}{\partial x_i} + F_i(x, t), 1 \le i \le 3 \ \ (x \in \mathbb{R}^3, t \ge 0)$

(*2*) $\nabla \cdot u = 0$

(*3*) $u(x, 0) = u^0(x)$ $(x \in \mathbb{R}^3)$

(*6*) $p, u \in C^\infty(\mathbb{R}^3 \times [0, \infty))$

(*7*) $\int_{\mathbb{R}^3} |u(x, t)|^2 dx < C, \forall t \ge 0$ (bounded energy)

4. O caso (D)

§ 1

Na seção 3 anterior dividimos o caso (C) em três situações possíveis para $\varphi = \phi - F$:

1) φ é um campo vetorial não gradiente

2) φ é um campo vetorial gradiente, com $\frac{\partial \varphi_i}{\partial x_j} = \frac{\partial \varphi_j}{\partial x_i} \neq 0,\ 1 \leq i, j \leq 3$

3) φ é um campo vetorial gradiente, com $\frac{\partial \varphi_i}{\partial x_j} = \frac{\partial \varphi_j}{\partial x_i} = 0,\ 1 \leq i, j \leq 3$

Como uma demonstração genérica para o caso (C) não exclui a possibilidade de serem espacialmente periódicas as funções $u^0(x)$ e $F(x,t)$, assim como a função velocidade $u(x,t)$, seremos nesta seção mais breve que na anterior; o essencial da técnica utilizada nesta demonstração está dado na seção 3. Quanto à pressão p, uma vez que nosso método se baseia na prova de que p não existe, será irrelevante admitir que p seja ou não periódica. Não existirá pressão $p(x,t)$ alguma, periódica ou não, que resolva para todo $t \geq 0$ as equações de Navier-Stokes (1) com a condição de incompressibilidade (7), para específicas funções $u^0(x)$ e $F(x,t)$, levando-se em consideração a condição inicial adicional $\frac{\partial u}{\partial t}|_{t=0} = a^0(x)$.

Escolhamos então para $u_0(x)$ uma função trigonométrica de período 1 na direção e_1 e igual a zero nas outras duas direções, e_2 e e_3, ou seja,

(73) $$u_0(x) = (sen(2\pi x_2), 0, 0).$$

Em $t = 0$ temos então

(74) $$\frac{\partial u_1}{\partial x_2}|_{t=0} = +2\pi \cos(2\pi x_2)$$

(75) $$\frac{\partial u_i}{\partial x_j}|_{t=0} = 0, (i,j) \neq (1,2)$$

(76) $$\nabla^2 u_i = \left(\sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}\right) u_i = \begin{cases} \frac{\partial^2}{\partial x_2^2} u_1 = -4\pi^2 sen(2\pi x_2),\ i = 1 \\ 0,\ i \neq 1 \end{cases}$$

(77) $$\sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} = 0,\ 1 \leq i \leq 3.$$

Por simplicidade, escolhamos também

(78) $$\frac{\partial u_i}{\partial t}|_{t=0} = a^0(x) = 0,\ 1 \leq i \leq 3,$$

e assim a equação (1) fica, em $t = 0$,

$$(79) \quad \frac{\partial p^0}{\partial x_i} = \varphi_i^0 + F_i^0 = \phi_i^0 = \begin{cases} -\nu \cdot 4\pi^2 sen(2\pi x_2) + F_1^0, i = 1 \\ F_i^0, i \neq 1 \end{cases}$$

definindo $p^0(x) = p(x,0)$, assim como os demais índices superiores 0 (zero) correspondem à respectiva função em $(x, 0)$.

Para $\nu \neq 0$ e $F^0 = 0$ vemos que ϕ^0 dada em (79) é não gradiente, logo, não há solução para o sistema (79) escolhendo $\frac{\partial u}{\partial t}|_{t=0} = 0, F = 0, \nu \neq 0$ e a velocidade inicial dada por (73), o que é então mais um exemplo de quebra de soluções da equação de Navier-Stokes em $t = 0$, e que também satisfaz a todos os requisitos que devem obedecer $u^0(x)$ e $F(x,t)$, como é fácil de ver.

§ 2

Semelhantemente ao que fizemos na seção 3 §4, para um tempo real $t \geq 0$ qualquer, dada uma velocidade $u(x,t): \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$ que obedeça a todas as condições deste problema, descritas na seção 2, e tais que

$$(80) \quad u(x,0) = u^0(x) = (sen(2\pi x_2), 0, 0),$$

usando o mesmo exemplo (73) da subseção anterior, para que haja solução de Navier-Stokes deve valer,

$$(81) \quad \frac{\partial p}{\partial x_i} = \nu \nabla^2 u_i - \frac{\partial u_i}{\partial t} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} + F_i = \varphi_i + F_i = \phi_i, 1 \leq i \leq 3,$$

com

$$(82) \quad \varphi_i = \nu \nabla^2 u_i - \frac{\partial u_i}{\partial t} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}, 1 \leq i \leq 3.$$

Para este campo vetorial de velocidades $u(x,t)$ é possível calcular $\varphi(x,t)$, de (82), escolher um campo ϕ não conservativo com as mesmas propriedades razoáveis que devem obedecer u, p e F para o caso (D), e calcular

$$(83) \quad F = \phi - \varphi = \phi - \nu \nabla^2 u + \frac{\partial u}{\partial t} + (u \cdot \nabla) u,$$

que deve ser uma função espacialmente periódica de período unitário nas três direções ortogonais $e_j, 1 \leq j \leq 3$, de classe C^∞ e obedecer (16).

Escolhamos, por exemplo,

$$(84) \quad \phi(x,t) = u^0(x) e^{-t} = (e^{-t} sen(2\pi x_2), 0, 0),$$

que é um campo não conservativo de período espacial 1 e vai a zero com o aumento do tempo t, em decaimento exponencial. Suponhamos novamente que a compatibilidade dimensional física entre ϕ e u^0 seja feita pela multiplicação do fator 1 cuja dimensão compatibiliza ambos os campos.

O valor das componentes de $F(x,t)$ que obtemos de (83) é então

$$(85) \quad F_i = \begin{cases} e^{-t} sen(2\pi x_2) - \varphi_i, \ i = 1 \\ -\varphi_i, \ i = 2,3 \end{cases}$$

com φ_i dado em (82), que deve ser espacialmente periódico de período unitário, de classe C^∞ e tendendo a zero em decaimento exponencial com o aumento do tempo, e finalmente, de (81),

$$(86) \quad \frac{\partial p}{\partial x_i} = \phi_i = \begin{cases} e^{-t} sen(2\pi x_2), \ i = 1 \\ 0, \ i = 2,3 \end{cases}$$

que é claramente um sistema sem solução para a função escalar p, qualquer que seja a velocidade $u(x,t)$ aceitável que possamos ter utilizado inicialmente como nossa escolha, com $u(x,0) = u^0(x)$ e $t \geq 0$.

A força $F(x,t) = H(u^0(x), u(x,t), \partial, \nabla^2, x, t)$ calculada pelo método acima e a velocidade inicial u^0 escolhida em (80) garantem que para qualquer velocidade $u(x,t)$ admissível para solução de Navier-Stokes neste problema chegue-se a um valor impossível de ser obtido para a pressão p, pois $\phi = u^0$ é não conservativo, segundo nossa escolha, o que prova a existência de quebra de soluções para as equações de Navier-Stokes, conforme queríamos.

Lembremos que a utilização da força como uma função da velocidade já foi justificada na seção 3 §5.

Vejam ainda que o caso (D) é de menor interesse, pois nos obriga a buscar uma não solução espacialmente periódica de período unitário, quando poderia haver uma solução não espacialmente periódica ou de período não unitário em alguma direção. Velocidades iniciais e forças externas periódicas podem também implicar, talvez, em uma solução não periódica, ou seja, mesmo que todas as condições do problema no caso (D) sejam satisfeitas pode haver ainda alguma solução para as equações de Navier-Stokes.

5. Comentários sobre os casos (A) e (B): existência de soluções

Os casos mais difíceis de serem tratados neste problema do milênio, em minha opinião, são as duas primeiras alternativas, que pedem solução para as equações de Navier-Stokes dada uma velocidade inicial genérica qualquer $u^0(x)$ satisfazendo determinadas condições, conforme descrito a seguir.

(A) Existência e lisura das soluções da Equação de Navier-Stokes sobre $\mathbb{R}^3$. Para coeficiente de viscosidade $\nu > 0$, dimensão espacial $n = 3$, força externa $F = 0$ e qualquer campo vetorial suave e com divergência nula $u^0(x) = u(x, 0)$ sobre $\mathbb{R}^3$ existe solução (p, u) sobre $\mathbb{R}^3 \times [0, \infty)$ para as equações de Navier-Stokes satisfazendo (1), (7), (8), (9) e (14).

(B) Existência e lisura das soluções da Equação de Navier-Stokes sobre $\mathbb{R}^3/\mathbb{Z}^3$. Para coeficiente de viscosidade $\nu > 0$, dimensão espacial $n = 3$, força externa $F = 0$ e qualquer campo vetorial suave e com divergência nula $u^0(x) = u(x, 0)$ sobre $\mathbb{R}^3$ satisfazendo a condição de periodicidade espacial (11) existe solução (p, u) sobre $\mathbb{R}^3 \times [0, \infty)$ para as equações de Navier-Stokes satisfazendo (1), (7), (8), (10) e (12).

Vejamos. Não fosse a exigência de ser $F = 0$ seria muito simples resolver Navier-Stokes. Poderíamos escolher pressões $p(x, t)$ fisicamente razoáveis, velocidades $u(x, t)$ fisicamente razoáveis satisfazendo $u(x, 0) = u^0(x)$ e ainda que p e u (e consequentemente u^0) obedecessem às demais condições requeridas para este problema, a exemplo de $\nabla \cdot u = 0$, o que resultaria enfim numa força externa $F = (F_1, F_2, F_3)(x, t)$ tal que

$$(87) \quad F_i = \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} - \nu \nabla^2 u_i - \frac{\partial p}{\partial x_i}, \; 1 \leq i \leq 3.$$

Nossa atenção se concentraria em provar que F não viola nenhuma regra, nenhuma condição que F deveria obedecer pela imposição do problema.

A equação (87) mostra claramente que existem combinações das variáveis (p, u) que são proibidas nos movimentos de fluidos sem força externa, pois se o lado direito de (87) resultar para ao menos uma das componentes i um valor não nulo para a força externa chegaríamos a uma contradição, já que o movimento seria, por definição, sem força externa.

Também não podemos utilizar qualquer velocidade inicial $u^0(x)$. Todas as condições que devem obedecer $u(x, t)$ em $t \geq 0$ devem ser obedecidas por $u^0(x)$, já que esta equivale a $u(x, t)$ no instante inicial $t = 0$. Em especial, $u^0(x)$ deve obedecer também às equações de Navier-Stokes (1) e de incompressibilidade (7).

Isto nos sugere que $u^0(x)$ pode ser, ela própria, a procurada solução de (1), inclusive para todo $t \geq 0$, com a imposição da condição de contorno adicional $\frac{\partial u}{\partial t} = 0$. Temos assim o caso de fluidos sem aceleração local, uma solução estacionária. Se a correspondente função ϕ em

$$(88) \quad \frac{\partial p}{\partial x_i} = \nu \nabla^2 u_i^0 - \sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j} = \phi_i, \; 1 \leq i \leq 3,$$

for gradiente então o problema está resolvido, para uma infinidade de pressões possíveis, admitindo-se satisfeitas as demais condições que devem ser obedecidas por u e p. No caso de fornecermos $p^0(x)$ como condição inicial ao invés de $a^0(x)$, sempre haverá solução em $t = 0$, e teremos

(89) $$a_i^0 = \nu \nabla^2 u_i^0 - \frac{\partial p^0}{\partial x_i} - \sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j},\ 1 \leq i \leq 3.$$

Se o valor de a^0 que se obtém acima for igual a zero então $u(x,t) = u^0(x)$ e $p(x,t) = p^0(x) + \theta(t)$ são uma solução do problema, para $t \geq 0$.

Mas o caso geral ainda nos foge neste momento: dado $u^0(x)$ obter $u(x,t)$ e $p(x,t)$, soluções das equações de Navier-Stokes. Para mim parece claro que é preciso ao menos mais uma condição inicial, como já vimos com o uso de $a^0(x) = \frac{\partial u}{\partial t}|_{t=0}$ nas seções anteriores. Além disso, nas diversas equações diferenciais ordinárias e parciais de segunda ordem da Física Matemática[3] e que já foram amplamente estudadas é comum (até necessário) a utilização de (pelo menos) duas condições iniciais ou de contorno para a sua completa solução, e não vejo motivo para aqui ser diferente.

Mesmo assim, do ponto de vista da realidade física, realidade que certamente motiva este problema, uma questão de Matemática aplicada aos fluidos, também me parece não ser possível resolver Navier-Stokes sem força externa em todas as condições, seja $\nabla \cdot u = 0$ ou não, seja $\nu = 0$ ou não.

Suponhamos $u^0(x) = (0,0,0) = 0$ e, por definição, $F(x,t) = (0,0,0) = 0$ (estamos utilizando o mesmo símbolo 0 tanto para o vetor nulo quanto para a constante numérica igual a zero, mas que não seja isso fonte de confusão).

Em $t = 0$ a equação a ser resolvida é

(90) $$\nabla p^0 = -\frac{\partial u}{\partial t}|_{t=0},$$

com $p^0(x) = p(x,0),\ x \in \mathbb{R}^3$.

Para que haja solução devemos ter que $\frac{\partial u}{\partial t}|_{t=0}$ seja um campo vetorial gradiente, i.e., alguma função $a^0(x)$ gradiente. A solução $u = 0$ satisfaz esta condição e é uma solução fisicamente razoável: sem velocidade inicial, sem força externa, teremos um fluido imóvel, estático, estacionário, sem acelerações(ver nota B), sem ventos e marés, exatamente o comportamento observado na natureza. Por outro lado, $\nabla p^0 = 0$ tem uma infinidade de soluções possíveis da forma $p^0(x) = cte.$, o que pode não ser fisicamente razoável. Por que haveria pressão não nula se a velocidade não varia no tempo e espaço e não há força? Nesse caso não há colisões entre as partículas, então não há pressão não nula em instante algum.

Aceitar unicamente $p^0(x) = 0$ seria o mais razoável, ainda que seja de fato uma idealização do comportamento físico dos fluidos (não utilizamos teoria atômica e molecular, termodinâmica, mecânica quântica, etc.).

Se impusermos uma velocidade inicial da forma $u^0 = (u_1^0(x_2, x_3), 0, 0)$, fisicamente razoável, para u_1^0 diferente de constante, u^0 não gradiente e com $\frac{\partial u_1^0}{\partial x_2} \neq 0$ e $\frac{\partial u_1^0}{\partial x_3} \neq 0$, como $F(x,t) = (0,0,0)$ e esperamos um sistema fisicamente razoável, a solução $u(x,t)$ ao longo do tempo deve evoluir para uma velocidade da forma $u(x,t) = (u_1(x_1, x_2, x_3, t), 0, 0)$, com $u_1(x,t)$ não identicamente nulo, que representa o movimento do fluido apenas na direção e_1. Tal como no exemplo anterior, não é fisicamente razoável, abstraindo-se das complexidades termodinâmicas e quânticas a nível microscópico, esperar um movimento macroscópico nas direções e_2 e e_3 quando não há velocidades iniciais e forças nessas direções.

Assim o sistema final a ser resolvido será da forma

(91) $$\nabla p = (\phi_1(x,t), 0, 0),$$

que não admitirá solução para p em geral, para todo $t \geq 0$. Isto é mais um exemplo de quebra das soluções de Navier-Stokes, desta vez sem usar $F(x,t) = H(u^0(x),\ u(x,t), \partial, \nabla^2, x, t)$ não nula. Admitimos também nesta análise um ambiente sem bordas (ou bordas muito distantes do ponto em estudo) e velocidade inicial baixa, para que não ocorram, na realidade, efeitos caóticos, de turbilhões, etc.

6. Conclusão

Na seção 3 §3 vimos que $u^0(x)$ e $F^0(x)$ não determinam de maneira única a quebra ou não das soluções das equações de Navier-Stokes em $t = 0$. Foi necessário saber o valor de $\frac{\partial u}{\partial t}|_{t=0}$. Consequentemente, $u^0(x)$ e $F^0(x)$ não determinam de maneira única a quebra ou não das soluções das equações de Navier-Stokes sobre $\mathbb{R}^3 \times [0, \infty)$. Além disso, a pressão $p(x,t)$ sempre pode ser somada a alguma função do tempo $\theta(t)$, o que não altera nem o valor de ∇p e nem a velocidade u, quaisquer que sejam a velocidade inicial $u^0(x)$ e a força externa $F(x,t)$, ou seja, também não há nem unicidade de soluções, nem de não soluções, exceto se forem dadas mais condições iniciais e de contorno convenientes para a unicidade de $p(x,t)$.

Resolvemos o problema do milênio para as equações de Navier-Stokes primeiramente nos casos de inexistência de soluções (para a pressão) em $t = 0$ pelo acréscimo de mais uma condição inicial, a derivada parcial temporal do vetor velocidade em $t=0$, que chamamos de $a^0(x) = \frac{\partial u}{\partial t}|_{t=0}$. Este raciocínio não poderá ser utilizado, entretanto, se nossa condição inicial adicional for $p(x,0) = p^0(x)$, ao invés da condição para a variação temporal de u, pois nesse caso sempre haverá algum valor que poderá ser encontrado para $a^0(x)$ em $t=0$, dado por (89). Isto sugere que existem velocidades e acelerações proibidas nos movimentos de fluidos, assim como nos casos das combinações de (p,u) para os movimentos sem força externa.

Em (87) vemos que sempre é possível encontrar uma força externa $F(x,t)$ tal que haja solução para as equações de Navier-Stokes, dados $u(x,t)$, e consequentemente $u^0(x)$, e $p(x,t)$, o que poderia nos fazer concluir que então não existem casos de quebra de soluções, mas o que ocorre é que a definição da questão feita neste problema para os casos de quebra de soluções impõe que a velocidade inicial e a força sejam os campos vetoriais que nos são dados, e assim a velocidade e a pressão em $t \geq 0$ são as incógnitas a serem encontradas. Nesta situação é possível encontrar exemplos onde não há solução para as equações, especificamente para a pressão em $t \geq 0$, conforme vimos.

Nas seções 3 §4 e 4 §2, utilizando implicitamente as relações lógicas (71) e (72), escolhe-se hipoteticamente uma velocidade válida u e respectivo $\frac{\partial u}{\partial t}$, com $u(x,0) = u^0(x)$, e encontra-se uma força externa F que depende destes u, u^0 e $\frac{\partial u}{\partial t}$ e implique em um sistema de equações diferenciais parciais impossível de ser resolvido para a pressão p, de acordo com (2) e (3), pelo método descrito já na Introdução, o que resolve o problema de maneira geral para $t \geq 0$. Esta é a grande chave do método utilizado: $(\exists u \wedge \nexists p) \rightarrow \nexists(p,u)$.

Interessante observar que com esta lógica também podemos encontrar "inúmeras" combinações de $F(x,t)$ e $u(x,t)$, mesmo sem ser $F(x,t)$ uma função explícita de $u(x,t)$ e suas derivadas e inclusive para $F=0$, tal que (1), e consequentemente (2), não tenham solução para p, por implicarem em uma função ϕ não gradiente. Dito desta forma, parece uma conclusão simples demais para que esta questão tenha se tornado um problema do milênio, mas de fato é o que conseguimos deduzir de pura Matemática.

Este método leva apenas a uma condição necessária, mas não ainda suficiente, introduzindo-se mais uma condição na tabela resumo do enunciado do problema descrito na seção 3 §5: $\forall u(x,t) / u(x,0) = u^0(x)$. Assim não bastaria encontrarmos apenas um único exemplo para $u(x,t)$, e sim todas as infinitas possibilidades. As provas suficientes seriam, neste caso, semelhantes às dadas nas seções 3 §4 e 4 §2, uma vez que a condição $\exists F(x,t)$ não proíbe que seja $F(x,t)$ uma função que varie instantaneamente com cada $u(x,t)$ possível, conforme vimos em 3 §5. Faltariam as provas de (15) para o caso (C)(ver nota C) ou (13) e (16) para o caso (D).

Na seção 5 fizemos alguns comentários sobre os casos (A) e (B) de existência de soluções para as equações de Navier-Stokes e demos mais um exemplo de inexistência de solução para estas equações, com $F = 0$, usando a necessidade do sistema ser fisicamente razoável, ou compatível com a observação a nível macroscópico, em um ambiente sem bordas (ou bordas muito distantes do ponto em estudo) e velocidade inicial baixa.

É oportuno mencionar que estas dificuldades com relação à integração das equações de Navier-Stokes e Euler desaparecem quando passamos a considerar que a pressão é um vetor, e não mais um escalar, tal que $p(x,t): \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$ e substitui-se ∇p por $\nabla \otimes p = \left(\frac{\partial p_1}{\partial x_1}, \frac{\partial p_2}{\partial x_2}, \frac{\partial p_3}{\partial x_3}\right)$, com $p = (p_1, p_2, p_3)$. Vejo como natural adotar esta ampliação do conceito de pressão. O caso em que $p_1 = p_2 = p_3$ possivelmente leva a todos os resultados já aceitos na Mecânica dos Fluidos, por exemplo, o de fluidos no estado de equilíbrio. Espero ver isso com mais detalhes oportunamente.

Observo também que existe um artigo escrito em 1983[4] cujo assunto é semelhante ao aqui tratado, porém sua abordagem é bem diferente desta. Em [4] analisa-se principalmente a Equação de Euler e nada se menciona sobre campos conservativos ou gradientes e sua influência para a determinação da pressão. Refere-se, por exemplo, a fluidos que tem um comportamento regular até determinado instante, mas a seguir perdem a regularidade (*smooth*) e podem apresentar divergências, etc. Talvez seja esta a maneira mais tradicional, acadêmica, previsível, de tratar este problema, talvez fosse esta a solução que os matemáticos esperassem, mas isto não invalida a análise e conclusão que aqui fizemos. Relacionando-a ao nosso método, seria como se um fluido apresentasse um campo ϕ gradiente até determinado instante ($t > 0$), mas a partir daí o campo ϕ deixasse de ser gradiente. De fato, aqui fizemos com que ϕ pudesse ser não gradiente desde o instante $t = 0$ e não apenas a partir de algum $t > T$ estritamente positivo.

Termino este artigo indicando três excelentes textos sobre Mecânica dos Fluidos, [5], [6] e [7], cuja leitura certamente contribuirá para a obtenção de resultados mais profundos sobre os problemas aqui tratados. Outra ótima referência é [8], mais voltada às equações de Euler (nossas conclusões independem do específico valor de ν, seja ele zero ou não).

O óbvio é aquilo que nunca é visto, até que alguém o manifeste com simplicidade.
(Kahlil Gibran)

Nota A: Em atenção aos leitores das versões anteriores, preciso me desculpar e dizer que cometi alguns enganos (um *reality* de aprendizagem), espero que todos já corrigidos, em especial:

1) ao traduzir o termo *divergence-free* para u^0, interpretando-o como a necessidade de ser limitada em todo ponto a velocidade inicial, tal que $\lim_{x \to x_0} |u^0(x)| < C$, i.e., livre de divergências. Na realidade o termo pede bem menos: que seja igual a zero o divergente da velocidade inicial, ou seja, $\nabla \cdot u^0 = 0$. Não obstante, as condições (*4*) e (*5*) impõe que u^0 e F, bem como suas infinitas derivadas, tendam a zero no infinito mais rápido que o inverso de qualquer polinômio (sendo da classe C^∞, pertencem ao espaço de Schwartz). Deve valer $\lim_{|x| \to \infty} u^0(x) = \lim_{t \to \infty} F(x, t) = 0$. Para o caso (C) também deve valer $\lim_{|x| \to \infty} F(x, t) = 0$.

2) considerei o índice α como pertencente ao conjunto dos números naturais, sem incluir o zero. Além disso, α deve ser um multi-índice, i.e., $\alpha \in \mathbb{N}_0^3$, $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$, com $\partial_x^0 u^0 = \partial_t^0 u^0 = \partial_x^0 \partial_t^0 u^0 = u^0$, o mesmo valendo para a força externa F. Grato ao professor Ricardo Rosa da UFRJ pela informação.

Nota B: Por um abuso de linguagem, em versões anteriores chamei de aceleração o termo $\frac{\partial u}{\partial t}$, sendo que este é mais exatamente conhecido como aceleração local (a posição x de referência é fixa enquanto t varia). A derivada total da velocidade em relação ao tempo é $\frac{Du}{Dt} = \frac{\partial u}{\partial t} + (u \cdot \nabla) u$, conhecida como aceleração convectiva, derivada de transporte, derivada material ou derivada substancial. Esta é a aceleração que tem uma partícula do fluido inicialmente na posição $x(t = 0)$, segundo a descrição lagrangeana do movimento. A equação $\frac{Du}{Dt} = 0$ é conhecida como Equação de Burgers (André Nachbin, *Aspectos da Modelagem Matemática em Dinâmica dos Fluidos*, IMPA, 2001, e Merle C. Potter e David C. Wiggert, *Mecânica dos Fluidos*, Cengage Learning, 2004).

Nota C: Para o caso (C) não é difícil, e com um pouco mais de criatividade prova-se também para o caso (D). Fazendo $F = \phi - \nu \nabla^2 u + \frac{\partial u}{\partial t} + (u \cdot \nabla) u$, se $u, \phi \in S(\mathbb{R}^3 \times [0, \infty))$ então $F \in S(\mathbb{R}^3 \times [0, \infty))$, das propriedades das funções pertencentes ao espaço de Schwartz, ou seja, não se deve escolher $\phi \notin S(\mathbb{R}^3 \times [0, \infty))$, precisando-se ainda que seja ϕ uma função vetorial não gradiente. Se $u \notin S(\mathbb{R}^3 \times [0, \infty))$, mas apenas $u \in C^\infty(\mathbb{R}^3 \times [0, \infty))$, então a prova pode ser feita violando-se a condição de energia cinética total limitada em $t > 0$, conforme viXra:1601.0312.

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02 – On the Inexistence of Navier-Stokes Solutions

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Abstract – We have proved in a few lines that there are initial velocities $u^0(x)$ and forces $F(x,t)$ such that there is no solution to the Navier-Stokes equations, which corresponds to the cases (C) and (D) of the problem relating to Navier-Stokes equations available on the website of the Clay Institute.

Keywords – Navier-Stokes equations, Euler equations, continuity equation, breakdown, existence, smoothness, solutions, gradient field, conservative field, velocity, pressure, external force, millenium problem.

§ 1

My intention in this short article is to transform through simpler differential equations the problem of the Navier-Stokes equations, described especially on Clay Institute page [1], in order to make a more understandable, easy and acceptable solution.

The options we perceive as able to be solved among the four alternatives available in [1] are the proofs of the inexistence (breakdown of) solutions for the Navier-Stokes equations, which correspond to cases (C) and (D) described in [1], being the first of these cases referred to the solutions in general and the second specific to the spatially periodic solutions. The problem as proposed is restricted to $n = 3$ spatial dimensions.

Our preliminary studies on the subject have been taken in [2] and in abbreviated form in [3], and here we intend to further summarize our conclusions on these equations in a small basic "standard" demonstration, acceptable even to be a demonstration to (eventually) be given as response to a question of university discipline without spending hours and hours in your solution or pages and pages in the demonstration, but as accurate as possible. Purposely this is an article that can be considered small, adequate (we believe) to a student of engineering, but also to the students of physics, meteorology, oceanography, geophysics, astronomy and even mathematics. We understand the practical importance of this subject.

The Navier-Stokes equation in vector form may be written as

(1) $$\frac{\partial u}{\partial t} + (u \cdot \nabla)\, u = \nu\, \nabla^2 u - \nabla p + F,$$

for $u, F\colon \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ and $p\colon \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$, where u is the velocity of the fluid, F the external "force" applied (e.g. gravity), p the pressure and ν is the coefficient (constant) of viscosity, measurements made in position $x \in \mathbb{R}^n$ and time $t \geq 0$, $t \in \mathbb{R}$ (called by habit Navier-Stokes equations in the plural when we think on the respective equations of its components $i, 1 \leq i \leq n$).

It is often joined to (1) the condition of incompressibility (constant mass density in the equation of continuity)

(2) $$\nabla \cdot u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0 \qquad (n = 3),$$

condition that must also be satisfied for conform to [1]. In what follows it will be understood that we are referring always to the spatial dimension $n = 3$.

Another way of writing (1) is

(3) $$\nabla p = \nu\, \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla)u + F = \varphi + F = \phi.$$

Calling the right side of (3) by ϕ the solution of (3) for p, when available, is given by

(4) $$p = \int_L \phi \cdot dl + \theta(t),$$

where L is a continuous path by parts of class C^1 which goes from x_0 to x with $x_0, x \in \mathbb{R}^3$. Suppose θ continuous, limited and differentiable at $t \geq 0$. We also assume that L does not pass by any singularities of ϕ.

From (3) we see immediately that (1) can be transformed into the simplest differential equation

(5) $$\nabla p = \phi, \qquad \phi(x,t): \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3,$$

that will only have solution if ϕ is a gradient field, i.e., conservative, and its solution in this case is (4). The reference [4] contains the basic theory of gradient fields.

Our central problem can then be written as, symbolically and as a question (a logical sentence and a question mark):

(P1) $$\exists \phi^0(x) = \phi(x,0),\ \nexists(\phi, p) /\ \nabla p = \phi?$$

If $\phi^0(x)$ is a non-gradient field then $\phi(x,t)$ is not a gradient field at $t = 0$, then (at least) at $t = 0$ there is no solution for $\phi = \nabla p$, whatever is $\phi(x,t)$ with $\phi(x,0) = \phi^0(x)$ no gradient.

The answer to (P1) is Yes, and we would have to seek some non-gradient $\phi^0(x)$ to illustrate the truth of the logic sentence. See that we can find many examples of functions $\phi(x,t)$, including to be worth $\phi^0(x) = \phi(x,0)$, but what will not exist using our example is the function p. In our proof the function $\phi(x,t)$ must exist so that there is also $\phi^0(x)$, and thus proving the inexistence of p.

A logical equivalence that represents the inexistence of the pair of variables (ϕ, p) as used in (P1) is

(6) $$\nexists(\phi, p) \leftrightarrow ((\nexists\phi \land \exists p) \lor (\exists\phi \land \nexists p) \lor (\nexists\phi \land \nexists p)),$$

and of three possible alternatives described above to prove the absence of the pair (ϕ, p) we chose the existence of ϕ with the inexistence of p, i.e.

(7) $(\exists\phi \wedge \nexists p) \to \nexists(\phi, p)$.

In this and on the problems that follow assume that $p: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}$ is a scalar function and all other functions are vectors with image in $\mathbb{R}^3$.

§ 2

A variant of (P1) is given by the problem (P2) below.

(P2) $\exists\varphi^0(x) = \varphi(x,0), \exists F(x,t), \nexists(\varphi, p) /\ \nabla p = \varphi + F$?

Let us seek again the breakdown of the solutions in $t = 0$. If $\phi^0 = \varphi^0 + F(x,0)$ is not gradient then in $t = 0$ the equation $\nabla p = \varphi + F$ will have no solution, the case of breakdown solutions. Answer: Yes, and for that the logic sentence in (P2) show it is true we should seek $\varphi^0(x)$ and $F(x,t)$ whose respective sum at $t = 0$ results in a non-gradient field.

§ 3

Our third problem is an easy version of (3), where ∇p is replaced by P, with $P: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$.

(P3) $\exists u^0(x) = u(x,0), \exists F(x,t), \nexists(u, P) /\ P = \nu\, \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla)u + F$?

Unlike the two previous answers, this time the answer is No. Assuming that the operation $\nu\, \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla)u + F$ can be computed, i.e., that for all $u(x,t)$ with $u(x,0) = u^0(x)$ there are the partial derivatives of u with respect to the spatial coordinates until second order and with respect to time up to the first order and that there is a corresponding computable value for P, then there are always u and P satisfying the differential equation given in (P3) in a very obvious way. Given u with $u(x,0) = u^0$ then P, ultimately, is the result of an algebraic computation, regardless of how complicated the involved derivations are, i.e., it is a false statement $\nexists(u, P)$ in (P3). Does not seem to be the focus of [1] the search for a "pathological" function, some strange function, rare, for u^0, u or F such that the equation given in (1) cannot even be computed (with the possible except ∇p). On the contrary, [1] is concerned with functions and physically reasonable solutions, and lists several conditions that must be obeyed by u^0, u, F, p.

§ 4

Here we treat the main problem, which corresponds to a necessary condition for the absence of solution (3), and consequently to (1). It is one of the proofs that we intend to give as acceptable standard.

(P4) $\exists u^0(x) = u(x,0), \exists F(x,t), \nexists(u, p) /\ \nabla p = \nu\, \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla)u + F$?

If $\exists u^0(x) = u(x,0)$ then there must be $u(x,t)$ at least in $t = 0$. As we seek solutions for (1) throughout all $t \geq 0$ then we can assume the existence of $u(x,t)$ in $t \geq 0$, hypothetically. Moreover, the original problem in [1] defines the domain $D(u)$ of u is

$\mathbb{R}^3 \times [0, \infty)$, so can assume by hypothesis the existence of u for all $t \geq 0, t \in \mathbb{R}$, and for all $x \in \mathbb{R}^3$.

The statement $\nexists(u, p)$ does not imply only in $(\nexists u \wedge \nexists p)$. We will choose to find a vector field velocity u, with $u(x, 0) = u^0$ given, such that there is some pressure p that satisfies the differential equation given in (P4), equal to (3).

Thus, we arrive at

$$\big(\exists u^0(x) = u(x, 0)\big) \wedge \big(D(u) = \mathbb{R}^3 \times [0, \infty)\big) \to \exists u \colon \mathbb{R}^3 \times [0, \infty)$$

(P4') $\quad (\exists u \wedge \nexists p) \to \nexists(u, p) \;/\; \nabla p = \nu\, \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla) u + F = \phi.$

If the field ϕ in (P4') is non-gradient then there is no p that satisfies the equation required, and there are infinite examples u^0, u, F which can be given such that result in non-gradients fields ϕ. The simplest example are the velocities $u(x, t) = u^0(x) = 0$, and thus it will be sufficient find (at least) a non-gradient function $F(x, t)$ to make the (P4') true, what will be Yes the answer to the problem (P4). If the answer was No then there would not be breakdown of solutions to the Navier-Stokes equations.

For case (C) of the Millennium problem we give as an example $F(x, t) = (e^{-x_2^2}, 0, 0)$ and in case (D), corresponding to spatially periodic solutions, we give as an example $F(x, t) = (\cos(2\pi x_2), 0, 0)$, trigonometric function of period 1. The one examples of limited functions that meet the conditions of continuity, differentiability, no divergence, smoothness (C^∞) etc. and also satisfy the equation (2) of the incompressible fluid. So there are cases of inexistence of solutions for p in the Navier-Stokes equations, given u^0, u, F.

§ 5

This problem will check a sufficient condition for the absence of solution (3), and consequently to (1). It is probably the most important demonstration of this article, which summarizes the main idea of [2].

Including in (P4) the condition $\forall u(x, t) / u(x, 0) = u^0(x)$ we come to

(P5) $\quad \exists u^0(x) = u(x, 0), \exists F(x, t), \forall u(x, t) / u(x, 0) = u^0(x), \nexists(u, p) /$

$$\nabla p = \nu\, \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla) u + F?$$

which requires that your answer is valid for any velocity $u(x, t)$ possible and complying to $u(x, 0) = u^0(x)$. It will not be enough, however only one or a few examples of velocities, as it is possible in (P4).

Equating the right side of equation (P5) to ϕ as done in (3) if ϕ is non gradient then equation (P5) does not admit solution. So when the function F is equal to

(8) $\quad F = \phi - \nu\, \nabla^2 u + \frac{\partial u}{\partial t} + (u \cdot \nabla) u$

the differential equation to be solved is equal to (5), which has no solution for non-gradient ϕ. Let us choose for this reason the force F given in (8).

This is an explicit example of force which varies with velocity, so each $u(x,t)$ that checks (P5) will generally have a different $F(x,t)$, so, $F(x,t) = H(\phi(x,t), u^0(x), u(x,t), x, t)$. We see that such a definition for F does not violate the condition $\exists F(x,t)$ of this problem, for this reason we use it. It may seem an invalid procedure, but it is in accordance with the reading that is done in [1].

Thus we find a way to construct F which always results in inexistence of solutions to (3), so defining ϕ a non gradient function, the answer to (P5) is Yes, there are cases of breakdown (inexistence) solutions for Navier-Stokes equations. We turn through (8) the original equation (1) in equation (5), a problem that has already been answered also affirmatively in (P1).

§ 6

Alternatively, rather than a variable force with the velocity, one can choose, for example, $F = 0$, a non-gradient initial velocity $u^0(x)$ or equal to zero (to facilitate the calculations) and an additional initial condition $\frac{\partial u}{\partial t}|_{t=0} = a^0(x)$ (can be $a^0(x) = 0$ or a non-gradient function) resulting to (3), at time $t = 0$, a non-gradient function ϕ.

The new problem in this case is

(P6) $\exists u^0(x) = u(x,0), \exists a^0(x) = \frac{\partial u}{\partial t}|_{t=0}, \exists F(x,t), \forall u(x,t) / u(x,0) = u^0(x), \nexists (u,p) /$
$\nabla p = \nu\, \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla) u + F$?

Answer: Yes.

The breakdown of the solutions occurs (at least) at $t = 0$, since at this instant the right side of the differential equation (P6) is a non-gradient function, and therefore the pressure p cannot be calculated (meaning it does not exist). The example $F = u^0 = 0$ will result in $\nabla p^0 = -a^0(x)$, for $p^0(x) = p(x,0)$ and non-gradient a^0: equation with no solution.

Both in this problem as in previous ones, we are assuming, of course, that the functions u^0, a^0, F chosen obey all conditions of "well-behaved" functions, physically reasonable, described in [1], as well as the function u^0 must also comply (2).

It is noticed that more important than this specific treatment in relation to Navier-Stokes (valid for the Euler equations) is its application to various other equations also exist. We found a logical equivalence and a useful technique.

Wir müssen wissen. Wir werden wissen.
(We need to know. We will know.)
David Hilbert

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02 – On the Inexistence of Navier-Stokes Solutions

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Abstract – We have proved in a few lines that there are initial velocities $u^0(x)$ and forces $F(x,t)$ such that there is no solution to the Navier-Stokes equations, which corresponds to the cases (C) and (D) of the problem relating to Navier-Stokes equations available on the website of the Clay Institute.

Keywords – Navier-Stokes equations, Euler equations, continuity equation, breakdown, existence, smoothness, solutions, gradient field, conservative field, velocity, pressure, external force, millenium problem.

§ 1

Minha intenção neste pequeno artigo é transformar através de equações diferenciais mais simples o problema sobre as equações de Navier-Stokes, descrito especialmente na página do Instituto Clay[1], de modo a tornar sua solução mais compreensível, fácil e aceitável.

As opções que percebemos como possíveis de serem resolvidas dentre as quatro alternativas disponíveis em [1] são as provas de inexistência de soluções (breakdown) para as equações de Navier-Stokes, que correspondem aos casos (C) e (D) descritos em [1], sendo o primeiro destes casos referente às soluções em geral e o segundo específico às soluções espacialmente periódicas. O problema conforme proposto está restrito a $n=3$ dimensões espaciais.

Nossos estudos preliminares sobre o assunto já foram dados em [2] e de forma abreviada em [3], e aqui pretendemos resumir ainda mais nossas conclusões sobre estas equações em uma pequena demonstração básica "padrão", aceitável inclusive para ser uma demonstração que possa (eventualmente) ser dada como resposta a uma questão de disciplina universitária, sem precisar gastar horas e mais horas em sua solução ou páginas e mais páginas na demonstração, mas tão rigorosa quanto possível. Propositalmente este é um artigo que pode ser considerado pequeno, adequado (acreditamos) a um aluno de Engenharia, mas também a alunos de Física, Meteorologia, Oceanografia, Geofísica, Astronomia e mesmo de Matemática. Entendemos a importância prática deste assunto.

A equação de Navier-Stokes, em forma vetorial, pode ser escrita como

(1) $$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = \nu\, \nabla^2 u - \nabla p + F,$$

para $u, F: \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ e $p: \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$, onde u é a velocidade do fluido, F a "força" externa aplicada (por exemplo, gravidade), p a pressão e ν o coeficiente (constante) de viscosidade, medidas feitas na posição $x \in \mathbb{R}^n$ e tempo $t \geq 0, t \in \mathbb{R}$ (chamamos por hábito equações de Navier-Stokes, no plural, quando pensamos nas respectivas equações das suas n componentes $i, 1 \leq i \leq n$).

Costuma-se juntar a (1) a condição de incompressibilidade (densidade de massa constante na equação da continuidade)

(2) $$\nabla \cdot u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0 \qquad (n = 3),$$

condição esta que também deve ser satisfeita para atender [1]. No que segue estará subentendido que estamos nos referindo sempre à dimensão espacial $n = 3$.

Outra forma de escrever (1) é

(3) $$\nabla p = \nu\, \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla)u + F = \varphi + F = \phi.$$

Chamando de ϕ o lado direito de (3), a solução de (3) para p, quando existe, é dada por

(4) $$p = \int_L \phi \cdot dl + \theta(t),$$

sendo L um caminho contínuo por partes de classe C^1 que vai de x_0 a x, com $x_0, x \in \mathbb{R}^3$. Suponhamos θ contínua, limitada e diferenciável em $t \geq 0$. Também supomos que L não passe por nenhuma singularidade de ϕ.

De (3) vemos imediatamente que (1) pode ser transformada na equação diferencial mais simples

(5) $$\nabla p = \phi, \qquad \phi(x,t)\colon \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3,$$

que só terá solução se ϕ for um campo gradiente, i.e., conservativo, e sua solução neste caso é (4). A referência [4] contém a teoria básica sobre os campos gradientes.

Nosso problema central pode então ser escrito assim, simbolicamente e em forma de pergunta (uma sentença lógica e uma interrogação):

(P1) $$\exists \phi^0(x) = \phi(x,0),\ \nexists(\phi,p) /\ \nabla p = \phi?$$

Se $\phi^0(x)$ for um campo não gradiente então $\phi(x,t)$ é não gradiente em $t = 0$, então (ao menos) em $t = 0$ não há solução para $\nabla p = \phi$, qualquer que seja $\phi(x,t)$ com $\phi(x,0) = \phi^0(x)$ não gradiente.

A resposta para (P1) é Sim, e teríamos que buscar algum $\phi^0(x)$ não gradiente para exemplificar a verdade da sentença lógica. Vejam que podemos encontrar muitos exemplos de funções $\phi(x,t)$, inclusive para poder valer $\phi^0(x) = \phi(x,0)$, porém o que não existirá usando nosso exemplo é a função p. Em nossa prova a função $\phi(x,t)$ deverá existir, para que também exista $\phi^0(x)$, e assim provaremos a inexistência de p.

Uma equivalência lógica que representa a inexistência do par de variáveis (ϕ, p) como utilizada em (P1) é

(6) $$\nexists(\phi,p) \leftrightarrow ((\nexists\phi \wedge \exists p) \vee (\exists\phi \wedge \nexists p) \vee (\nexists\phi \wedge \nexists p)),$$

e das três alternativas possíveis descritas acima para provar a inexistência do par (ϕ, p) escolhemos a existência de ϕ com a inexistência de p, i.e.

(7) $(\exists\phi \wedge \nexists p) \rightarrow \nexists(\phi, p)$.

Neste e nos problemas que seguem admitimos que $p: \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}$ é uma função escalar e todas as outras funções são vetores com imagem em $\mathbb{R}^3$.

§ 2

Uma variante de (P1) é dada pelo problema (P2) a seguir.

(P2) $\exists\varphi^0(x) = \varphi(x, 0), \exists F(x, t), \nexists(\varphi, p) / \nabla p = \varphi + F$?

Busquemos novamente a quebra das soluções em $t = 0$. Se $\phi^0 = \varphi^0 + F(x, 0)$ for não gradiente então em $t = 0$ a equação $\nabla p = \varphi + F$ não terá solução, o caso de quebra de soluções. Resposta: Sim, e para que a sentença lógica em (P2) mostre-se verdadeira deveremos buscar $\varphi^0(x)$ e $F(x, t)$ cuja respectiva soma em $t = 0$ resulte em um campo não gradiente.

§ 3

Nosso terceiro problema é uma versão facilitada de (3), onde substituímos ∇p por P, com $P: \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}^3$.

(P3) $\exists u^0(x) = u(x, 0), \exists F(x, t), \nexists(u, P) / P = \nu \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla)u + F$?

Ao contrário das duas respostas anteriores, desta vez a resposta é Não. Supondo que a operação $\nu \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla)u + F$ possa ser computada, i.e., que para todo $u(x, t)$ com $u(x, 0) = u^0(x)$ existam as derivadas parciais de u em relação às coordenadas espaciais até a segunda ordem e em relação ao tempo até a primeira ordem e que exista o respectivo valor computável para P, então sempre existem u e P que satisfaçam a equação diferencial dada em (P3), de maneira muito óbvia. Dado u com $u(x, 0) = u^0$ então P, em última análise, é o resultado de uma computação algébrica, por mais complicadas que sejam as derivações envolvidas, i.e., é falsa a afirmação $\nexists(u, P)$ em (P3). Não parece ser o foco de [1] a busca de alguma função "patológica", alguma função estranha, rara, para u^0, u ou F tal que a equação dada em (1) nem sequer possa ser computada (com a possível exceção de ∇p). Pelo contrário, [1] preocupa-se com funções e soluções fisicamente razoáveis, e elenca várias condições que devem ser obedecidas por u^0, u, F, p.

§ 4

Aqui trataremos do problema principal, que corresponde a uma condição necessária para a inexistência de solução para (3), e consequentemente de (1). É uma das provas que pretendemos dar como padrão aceitável.

(P4) $\exists u^0(x) = u(x, 0), \exists F(x, t), \nexists(u, p) / \nabla p = \nu \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla)u + F$?

Se $\exists u^0(x) = u(x,0)$ então deve existir $u(x,t)$ ao menos em $t = 0$. Como buscamos soluções para (1) em todo $t \geq 0$ então podemos supor a existência de $u(x,t)$ em $t \geq 0$, por hipótese. Além disso, o problema original em [1] define que o domínio $D(u)$ de u seja $\mathbb{R}^3 \times [0,\infty)$, ou seja, podemos supor por hipótese a existência de u em todo $t \geq 0, t \in \mathbb{R}$, e para todo $x \in \mathbb{R}^3$.

A afirmação $\nexists(u,p)$ não implica apenas em $(\nexists u \wedge \nexists p)$. Optaremos por encontrar algum campo vetorial de velocidades u, com $u(x,0) = u^0$ dado, tal que não exista pressão p alguma que satisfaça a equação diferencial dada em (P4), igual a (3).

Assim sendo, chegamos a

$$\left(\exists u^0(x) = u(x,0)\right) \wedge \left(D(u) = \mathbb{R}^3 \times [0,\infty)\right) \rightarrow \exists u \colon \mathbb{R}^3 \times [0,\infty)$$

(P4') $\quad (\exists u \wedge \nexists p) \rightarrow \nexists(u,p) \;/\; \nabla p = \nu\, \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla)u + F = \phi.$

Se o campo ϕ em (P4') for não gradiente então não haverá p que satisfaça à requerida equação, e existem infinitos exemplos de u^0, u, F que podem ser dados tais que resultem em campos ϕ não gradientes. O mais simples exemplo que penso são as velocidades $u(x,t) = u^0(x) = 0$, e assim bastará encontrar (ao menos) uma função $F(x,t)$ não gradiente de modo a tornar (P4') verdadeira, o que fará ser Sim a resposta para o problema (P4). Se a resposta fosse Não então não haveria quebra de soluções para as equações de Navier-Stokes.

Para o caso (C) do problema do milênio damos como exemplo $F(x,t) = (e^{-x_2^2}, 0, 0)$ e para o caso (D), correspondente às soluções espacialmente periódicas, damos como exemplo $F(x,t) = (\cos(2\pi x_2), 0, 0)$, função trigonométrica de período 1. São exemplos de funções limitadas que obedecem às condições de continuidade, derivabilidade, não divergência, smoothness (C^∞), etc. e também satisfazem à equação (2) dos fluidos incompressíveis. Portanto existem casos de inexistência de soluções para p nas equações de Navier-Stokes, dados u^0, u, F.

§ 5

Neste problema verificaremos uma condição suficiente para a inexistência de solução para (3), e consequentemente de (1). É provavelmente a mais importante demonstração deste artigo, que resume a ideia principal de [2].

Incluindo-se em (P4) a condição $\forall u(x,t)/u(x,0) = u^0(x)$ chegamos a

(P5) $\quad \exists u^0(x) = u(x,0), \exists F(x,t), \forall u(x,t)/u(x,0) = u^0(x), \nexists(u,p)/$
$\nabla p = \nu\, \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla)u + F?$

que requer que sua resposta seja válida para qualquer velocidade $u(x,t)$ possível e que obedeça a $u(x,0) = u^0(x)$. Não bastará, portanto, apenas um ou alguns poucos exemplos de velocidades, como é possível em (P4).

Igualando o lado direito da equação em (P5) a ϕ, conforme feito em (3), se ϕ for não gradiente a equação em (P5) não admitirá solução. Então quando a função F é igual a

(8) $F = \phi - \nu\, \nabla^2 u + \frac{\partial u}{\partial t} + (u \cdot \nabla)u$

a equação diferencial a ser resolvida é igual a $\nabla p = \phi$, que não tem solução para ϕ não gradiente. Escolhamos por essa razão a força F dada em (8).

Este é um exemplo explícito de força que varia com a velocidade, portanto a cada $u(x, t)$ que verifique (P5) teremos em geral um $F(x, t)$ diferente, ou seja, $F(x, t) = H(\phi(x, t), u^0(x), u(x, t), x, t)$. Vemos que tal definição para F não viola a condição $\exists F(x, t)$ deste problema, por isso a utilizamos. Pode parecer um procedimento inválido, mas está de acordo com a leitura que se faz de [1].

Encontramos assim uma maneira de construir F que resulta sempre em inexistência de soluções para (3), por isso, definindo ϕ uma função não gradiente, a resposta para (P5) é Sim, existem casos de quebra (inexistência) de soluções para as equações de Navier-Stokes. Transformamos através de (8) a equação original (1) na equação (5), problema que já foi respondido, também afirmativamente, em (P1).

§ 6

Alternativamente a uma força variável com a velocidade pode-se escolher, por exemplo, $F = 0$, uma velocidade inicial $u^0(x)$ não gradiente ou igual a zero (para facilitar os cálculos) e uma condição inicial adicional $\frac{\partial u}{\partial t}|_{t=0} = a^0(x)$ (podendo ser $a^0(x) = 0$ ou uma função não gradiente) que resultem para (3), em $t = 0$, uma função ϕ não gradiente.

O novo problema neste caso é

(P6) $\exists u^0(x) = u(x, 0), \exists a^0(x) = \frac{\partial u}{\partial t}|_{t=0}, \exists F(x, t), \forall u(x, t)/u(x, 0) = u^0(x), \nexists(u, p)/$
$\nabla p = \nu\, \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla)u + F$?

Resposta: Sim.

A quebra das soluções ocorre (pelo menos) em $t = 0$, pois neste instante o lado direito da equação diferencial de (P6) será uma função não gradiente, e por isso a pressão p não poderá ser calculada (significando que não existirá). O exemplo $F = u^0 = 0$ resultará em $\nabla p^0 = -a^0(x)$, sendo $p^0(x) = p(x, 0)$ e a^0 não gradiente: equação sem solução.

Tanto neste problema quanto nos anteriores estamos admitindo, evidentemente, que as funções u^0, a^0, F escolhidas obedecem a todas as condições de funções "bem comportadas", fisicamente razoáveis, descritas em [1], assim como a função u^0 também deve obedecer (2).

Percebe-se que mais importante que este tratamento específico em relação a Navier-Stokes (válido para as Equações de Euler) é sua aplicação a várias outras equações que também existem. Encontramos uma equivalência lógica e uma técnica úteis.

Wir müssen wissen. Wir werden wissen.
(*Nós precisamos saber. Nós iremos saber.*)
David Hilbert

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03 – Another Remark on the Breakdown of Smooth Solutions for the 3-D Euler Equations

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Reanalyzed the validity condition of the theorem and corollary of Beale-Kato-Majda.
Reanalisamos a condição de validade do teorema e corolário de Beale-Kato-Majda.

Mencionamos em nosso estudo sobre a quebra das soluções da equação de Navier-Stokes[1] que há um artigo escrito em 1983 que trata de assunto semelhante, mas com uma abordagem bem diferente da nossa[2].

Em [2] prova-se o seguinte teorema:

"Seja u uma solução das equações de Euler e suponha que há um tempo T_* tal que a solução não pode ser contínua na classe

$$u \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-1}) \tag{1}$$

para $T = T_*$. Assuma que T_* é o primeiro deste tempo. Então

$$\int_0^{T_*} |\omega(t)|_{L^\infty}\, dt = \infty$$

e em particular

$$\lim_{t\uparrow T_*} sup\, |\omega(t)|_{L^\infty} = \infty."$$

$H^s(\mathbb{R}^3)$ é o espaço de Sobolev, consistindo das funções cujas distribuições têm derivadas até a ordem s em $L^2(\mathbb{R}^3)$, sendo s um inteiro positivo e $L^2(D)$ o espaço de Hilbert que obedece à propriedade do produto interno

$$< f,\ g >= \int_D\ f(x)\bar{g}\,(x)dx.$$

O corolário a que [2] também chega é o seguinte:

"Para alguma solução das equações de Euler, suponhamos que há constantes M_0 e T_* tais que sobre qualquer intervalo $[0,T]$ de existência de solução na classe (1), com $T < T_*$, o vórtice satisfaz *a priori* a estimativa

$$\int_0^{T_*} |\omega(t)|_{L^\infty}\, dt \le M_0.$$

Então a solução pode ser contínua na classe (1) no intervalo $[0, T_*]$."

$|\omega(t)|_{L^\infty}$ é a norma de $\omega(t)$ no espaço L^∞, como definido usualmente em Teoria da Medida e Análise Funcional.

Nos mencionados teorema e corolário assume-se que há a vorticidade, ou seja, $\omega = \nabla \times u$, mas para se provar ambos os resultados assume-se também que existe a pressão p, e é tal que $\nabla \times \nabla p = 0$. Sendo assim, aplicando o rotacional à equação de Euler

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0, \tag{2}$$

com

$$\nabla \cdot u = 0,$$

chega-se à equação da vorticidade

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla\omega = \omega \cdot \nabla u, \tag{3}$$

cuja solução para u em função de ω é

$$u = -\nabla \times (\nabla^{-1}\omega),$$

usando o produto interno em $L^2 = H^0$

$$\big((u \cdot \nabla)w,\ w\big) = 0.$$

Desenvolvendo mais estes resultados e usando-se relações de desigualdades chega-se ao teorema e respectivo corolário.

O fato de ser $\nabla \times \nabla p = 0$, entretanto, faz com que as propriedades que devem ser obedecidas pela pressão p não sejam necessárias, ou levadas em consideração, no estudo feito em [2], enquanto a pressão tem fundamental importância na análise que fizemos em [1], estudo este sintetizado em [3].

Pode então existir um tempo $t = T_N$ ou todo $t \geq 0$ ou um intervalo de tempo $T_N \leq t < \infty$ tal que não seja gradiente a função ϕ em

$$\nabla p = -\left(\frac{\partial u}{\partial t} + (u \cdot \nabla)u\right) = \phi,$$

tornando impossível o cálculo de p no instante t, e assim configurando a quebra ou inexistência de solução (u, p) para (2). Isto invalidaria nesse caso o uso de (3) e as conclusões de [2] nestes valores de t. Mas quando ϕ é uma função gradiente continuam válidos os resultados de Beale-Kato-Majda, e isto pode não ser tão óbvio quanto parece, exceto quando mencionamos.

O mesmo se pode dizer para as equações de Navier-Stokes, com as devidas adaptações.

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04 – Solving the 15th Problem of Smale: Navier-Stokes equations

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Keywords: Navier-Stokes equations, Smale's problems, 15th problem.

Abstract: The solution of the fifteenth problem of Smale, the Navier-Stokes equations in three spatial dimensions.

Steve Smale wrote a stimulating article in 1998 proposing 18 problems still unresolved at that time[1], in keeping with V.I.Arnold request. Both, Arnold and Smale, in turn were inspired by the famous list of 23 problems of David Hilbert.

The purpose of this paper is to solve the 15th problem of Smale's list, on the uniqueness of the solutions of the Navier-Stokes equations.

"*Do the Navier-Stokes equations on a 3-dimensional domain Ω in* $\mathbb{R}^3$ *have a unique smooth solution for all time?*"

Answer: No.

The answer is given here was seen in [2], without mentioning that occasion the Smale's list.

If (u, p) is a smooth solution of the Navier-Stokes equation,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \nabla^2 u + \nabla p = 0, \tag{1}$$

with

$$\nabla \cdot u = 0, \tag{2}$$

then $(u, p + \theta(t))$ it is also a solution, because $\nabla p(x,t) = \nabla(p(x,t) + \theta(t))$, supposing that $\theta(t)$ does not present singularities, is continuous and can be spatially derivable ($\nabla\theta(t) = 0$), i.e., it is so well behaved as expected for $p(x,t)$ in this problem.

It is very clear that $p(x,t)$ and $p(x,t) + \theta(t)$ they are not necessarily the same solution p in (u, p) for (1), except if $\theta(t) = 0$, therefore the answer to this problem cannot be Yes.

Similar reasoning can be done with functions $q(x,t)$ such that $\nabla q = 0$ (zero vector), whose solution is a constant in x or variable only with time. We have, in this case,

$$\nabla p(x,t) = \nabla(p(x,t) + \theta(t) + q), \tag{3}$$

$q \in \mathbb{R}$. Then, if p is part of the solution (u, p) of (1) and (2), there are infinite other pairs (u, r) solutions of (1) and (2) such that

$$r(x,t) = p(x,t) + \theta(t) + q, \tag{4}$$

with $q \in \mathbb{R}$, $x \in \mathbb{R}^3$, $t \in [0, \infty)$, and the functions $p, r: \Omega \times [0, \infty) \to \mathbb{R}$, $\theta: [0, \infty) \to \mathbb{R}$, $p, r, \theta \in C^\infty$ on Ω for all $t \geq 0$, $\Omega \subseteq \mathbb{R}^3$, i.e., all this functions and solutions are smooth.

There is also the case of the function φ in

$$\nabla p = \nu \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla) u = \varphi \tag{5}$$

be non-gradient, which make it impossible to be found a value for p, as from $t = 0$ or from some $t = T_N$ or more generally on some set values of t such that φ is not a function gradient in these time instants t. This can already happen at $t = 0$, with the imposition of adequate additional initial condition $\frac{\partial u}{\partial t}|_{t=0}$ or else, for example, for $\frac{Du}{Dt}|_{t=0} = \left(\frac{\partial u}{\partial t} + (u \cdot \nabla) u\right)|_{t=0}$.

So, in conclusion, maybe there is no solution (u, p) for the system of equations (1) and (2) at some $t \geq 0$, but when there is a solution it is not unique, at least due to the infinity of other solutions $(u, p + \theta(t) + q)$ possible for the system, with $q \neq 0$ and $\theta(t) \neq 0$.

See that there is no solution for p is not the same as admitting that the pressure is null, $p = 0$, or more generally impose boundary conditions a given pressure $p(x, t)$. In these situations ∇p exist in general, but the original problem is other, because p must be a variable dependent unknown, not a fixed function.

Recall also that both this problem as described in Smale[1] and the corresponding (and more detailed) described by Fefferman[3] is given no initial condition for the pressure $p(x, t)$, only for the initial velocity $u(x, 0)$. Smale, unlike Fefferman, includes a boundary condition for $u(x, t)$ on $\partial\Omega$.

Yet one more observation is needed. Unlike Fefferman and real pressure in the daily, in machines and nature may vary with time, Smale defines the pressure domain equal to $\Omega \subseteq \mathbb{R}^3$, i.e., no variation in time. Either by mistake or not, even assuming $p, r: \Omega \to \mathbb{R}$, with $p, r \in C^\infty$ on Ω for all $t \geq 0$ and $q \in \mathbb{R}$ a constant, not utilizing the function $\theta(t)$, we have

$$\nabla p(x) = \nabla (p(x) + q), \tag{6}$$

a result which also provide infinite other solutions (u, r) admissible (smooth) for (1) and (2), being (u, p) a smooth solution and

$$r(x) = p(x) + q, \tag{7}$$

$q \neq 0$, as we have said in [2] with other words using $\theta(t)$ and shown initially to the more general case of pressure vary over time and the position, $p(x, t)$.

Of course, we are admitting that $r(x)$ in (7) is defined in Ω, such that $p(x)$. This offers no difficulty in understanding the special case $\Omega = \mathbb{R}^3$. Other domains, however, are also easily extended to the function $r(x)$. If $p(x)$ exists in Ω and has image $\mathbb{R}$, then for every $x \in \Omega$ the function $r(x) = p(x) + q$ also exists, is well defined and has image $\mathbb{R}$. Similar comment is made with respect to $r(x, t)$ on $\Omega \times [0, \infty)$, given in (4).

So the answer to the problem is No. Not always, not unique.

Once solved, it seems very easy, but we know it is far from being so. More complicated solutions that would be given to questions also more complicated, with more detailed requirements, or by exploiting other aspects of the problem, which would be primarily a sophistication of the original question actually involve other problems, known or unknown, rather than obey to the essence of what is asked.

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04 – Solving the 15th Problem of Smale: Navier-Stokes equations

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Keywords: Navier-Stokes equations, Smale's problems, 15th problem.

Abstract: The solution of the fifteenth problem of Smale, the Navier-Stokes equations in three spatial dimensions.

Steve Smale escreveu um estimulante artigo em 1998 onde propõe 18 problemas ainda não resolvidos na época[1], em atenção à solicitação de V.I.Arnold. Ambos, Arnold e Smale, por sua vez se inspiraram na famosa lista de 23 problemas de David Hilbert.

A proposta deste artigo é resolver o 15º problema da lista de Smale, sobre a unicidade das soluções das equações de Navier-Stokes.

"*Do the Navier-Stokes equations on a 3-dimensional domain Ω in $\mathbb{R}^3$ have a unique smooth solution for all time?*"

Resposta: Não.

A resposta que aqui é dada já foi vista em [2], sem nos lembrarmos na ocasião da lista de Smale.

Se (u, p) é uma solução suave (lisa, regular, *smooth*) da equação de Navier-Stokes,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \nabla^2 u + \nabla p = 0, \tag{1}$$

com

$$\nabla \cdot u = 0, \tag{2}$$

então $(u, p + \theta(t))$ também é uma solução, pois $\nabla p(x,t) = \nabla(p(x,t) + \theta(t))$, supondo que $\theta(t)$ não apresente singularidades, seja contínua e possa ser derivável espacialmente ($\nabla\theta(t) = 0$), i.e., seja tão bem comportada quanto o que se espera para $p(x,t)$ neste problema.

Está muito claro que $p(x,t)$ e $p(x,t) + \theta(t)$ não são necessariamente a mesma solução p em (u, p) para (1), exceto se $\theta(t) = 0$, portanto a resposta deste problema não pode ser Sim.

Raciocínio análogo pode ser feito com as funções $q(x,t)$ tais que $\nabla q = 0$ (vetor nulo), cuja solução é uma constante em x ou variável apenas com o tempo. Temos, neste caso,

$$\nabla p(x,t) = \nabla(p(x,t) + \theta(t) + q), \tag{3}$$

$q \in \mathbb{R}$. Então, se p faz parte da solução (u,p) de (1) e (2), também são soluções de (1) e (2) infinitos outros pares (u,r) tais que

$$r(x,t) = p(x,t) + \theta(t) + q, \tag{4}$$

com $q \in \mathbb{R}$, $x \in \mathbb{R}^3$, $t \in [0,\infty)$, e as funções $p,r: \Omega \times [0,\infty) \to \mathbb{R}$, $\theta: [0,\infty) \to \mathbb{R}$, $p,r,\theta \in C^\infty$ em Ω para todo $t \geq 0$, $\Omega \subseteq \mathbb{R}^3$, i.e., todas estas funções e soluções são regulares (suaves, lisas, *smooth*).

Também há o caso de ser a função φ em

$$\nabla p = \nu \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla)u = \varphi \tag{5}$$

não gradiente, o que impossibilitará de ser encontrado um valor para p, desde $t = 0$ ou a partir de algum $t = T_N$ ou mais genericamente em algum conjunto de valores de t tais que φ não seja uma função gradiente nestes instantes de tempo t. Isto pode acontecer já em $t = 0$, com a imposição de uma adequada condição inicial adicional $\frac{\partial u}{\partial t}|_{t=0}$ ou então, por exemplo, para $\frac{Du}{Dt}|_{t=0} = \left(\frac{\partial u}{\partial t} + (u \cdot \nabla)u\right)|_{t=0}$.

Então, concluindo, pode não haver solução (u,p) para o sistema de equações (1) e (2) em algum $t \geq 0$, mas quando há solução ela não é única, pelo menos devido à infinidade de outras soluções $(u, p + \theta(t) + q)$ possíveis para o sistema, com $q \neq 0$ e $\theta(t) \neq 0$.

Vejam que não haver solução para p não é a mesma coisa que admitir que a pressão é nula, $p = 0$, ou mais genericamente impor como condição de contorno uma determinada pressão $p(x,t)$. Nessas situações ∇p existirá em geral, mas o problema original é outro, pois p deve ser uma variável dependente incógnita, não uma função pré-fixada.

Lembremos também que tanto neste problema descrito por Smale[1] quanto no correspondente (e mais detalhado) descrito por Fefferman[3] não é dada nenhuma condição inicial para a pressão $p(x,t)$, apenas para a velocidade inicial $u(x,0)$. Smale, ao contrário de Fefferman, inclui uma condição de contorno para $u(x,t)$ sobre $\partial\Omega$.

Ainda mais uma observação é necessária. Ao contrário de Fefferman e da pressão real, no cotidiano, em máquinas e na natureza, poder variar com o tempo, Smale define o domínio da pressão como igual a $\Omega \subseteq \mathbb{R}^3$, i.e., sem variar no tempo. Seja por equívoco ou não, mesmo admitindo-se $p,r: \Omega \to \mathbb{R}$, com $p,r \in C^\infty$ em Ω para todo $t \geq 0$ e $q \in \mathbb{R}$ uma constante, sem utilizarmos a função $\theta(t)$, temos

$$\nabla p(x) = \nabla(p(x) + q), \tag{6}$$

resultado que também proporcionará infinitas outras soluções (u,r) admissíveis para (1) e (2), sendo (u,p) uma solução regular (suave, lisa, *smooth*) e

$$r(x) = p(x) + q, \tag{7}$$

$q \neq 0$, como já dissemos em [2] com outras palavras, usando $\theta(t)$, e mostramos inicialmente para o caso mais geral da pressão variável com o tempo e a posição, $p(x,t)$.

Naturalmente, estamos admitindo que $r(x)$ em (7) está definida em Ω, tal qual $p(x)$. Isto não oferece nenhuma dificuldade de entendimento no caso especial de ser

$\Omega = \mathbb{R}^3$. Outros domínios, entretanto, também são facilmente estendidos para a função $r(x)$. Se $p(x)$ existe em Ω e tem imagem $\mathbb{R}$, então para todo $x \in \Omega$ a função $r(x) = p(x) + q$ também existe, está bem definida e tem imagem $\mathbb{R}$. Comentário similar se faz a respeito de $r(x, t)$ sobre $\Omega \times [0, \infty)$, dada em (4).

Assim a resposta ao problema é Não. Nem sempre, nem única.

Uma vez resolvido parece agora muito fácil, mas sabemos que está longe de ser assim. Soluções mais complicadas que esta seriam dadas com questões também mais complicadas, com requisitos mais detalhados, ou então explorando outros aspectos do problema, o que seria principalmente uma sofisticação da questão original, na realidade envolvendo outros problemas, conhecidos ou não, ao invés de atender à essência do que é pedido.

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05 – Brief Comment on the Euler and Navier-Stokes Equations in 2-D

(Breve Comentário sobre as Equações de Euler e Navier-Stokes em 2-D)

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The qualitative equality between n = 2 and n = 3 spatial dimensions.

Já vimos o caso de não unicidade de soluções (u, p) para as equações de Navier-Stokes em $n = 3$ dimensões espaciais, bem como a possibilidade de inexistência de soluções, que traduzi por quebra das soluções, em alusão ao termo *breakdown solutions* utilizado na definição do problema do milênio relativo a estas equações. A infinidade de pressões possíveis para um mesmo problema de condições iniciais das equações de Navier-Stokes, problema que tradicionalmente não prescreve nenhum valor inicial ou de fronteira para a pressão, é responsável pela não unicidade destas soluções (u, p), embora não esteja afastada a possibilidade de não unicidade de soluções para a velocidade u. Aliás, também deve haver uma infinidade de soluções possíveis para a velocidade u, dada a mesma condição inicial $u^0(x) = u(x, 0)$ nas equações de Navier-Stokes ou Euler, variando, por exemplo, conforme o valor inicial de $\frac{\partial u}{\partial t}|_{t=0}$. Isso não foi o foco principal de meus estudos até o momento, entretanto, onde procurei analisar principalmente a inexistência (*breakdown*) das soluções. A inexistência ou quebra de soluções, por sua vez, ocorre (pelo menos) quando a função ϕ em

$$\nabla p = \nu \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla) u + F = \varphi + F = \phi$$

é um campo não conservativo, não gradiente, seguindo a notação que temos utilizado, o que pode ocorrer desde o instante $t = 0$ ou em algum outro conjunto de valores de t.

A atenção que também deve ser dada a este assunto é devido ao fato de que estas conclusões feitas preliminarmente para dimensão espacial $n = 3$ também valem para dimensão espacial $n = 2$, embora se diga que para $n = 2$ estes problemas de existência e unicidade já estão resolvidos. Não creio que em dimensão dois ocorra diferente do que ocorre em dimensão três, e pretendo verificar isto com mais profundidade.

06 – Breakdown of Navier-Stokes Solutions – Bounded Energy

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Abstract: Considerations on (and solution to) the 6th millenium problem, respect to breakdown of Navier-Stokes solutions and the bounded energy.

O artigo que segue (página 54 em diante) foi escrito originalmente supondo que a ordem de derivação espacial α não pode assumir o valor 0, assim como a ordem de derivação temporal m. Embora Fefferman [1] tenha escrito "for any α and K" para a sua condição (4) e "for any α, m and K" para a condição (5), este emaranhamento simbólico envolvendo as derivações parciais ∂_x^α e ∂_t^m, respectivamente, pode levar a uma grande imprecisão. Seria mais elegante e exato no artigo de Fefferman, fonte de tão valiosa perspectiva, gastar-se um pouco mais de tempo para se deixar claro a quais conjuntos numéricos pertencem efetivamente cada um destes α, m, K. O que seria então uma derivada negativa, ou fracionária, ou irracional, ou imaginária pura, ou alguma derivada complexa qualquer? E derivada de ordem zero? Existirá também integral zero-ésima? O termo "for any" de Fefferman, ainda que utilizado sem preocupações em parte da literatura, deveria estar melhor definido neste artigo "do milênio". É o que se espera dos matemáticos: definições, regras, lógica e conclusões precisas.

A condição (10) do artigo abaixo é impossível de ser obedecida, exceto violando-se (4) para $\alpha = 0 = (0,0,0)$, onde para todo $K \in \mathbb{R}$ temos $\partial_x^0 u^0 = u^0$ e $|u^0(x)| \leq C_{0K}(1 + |x|)^{-K}$ sobre $\mathbb{R}^3$.

A inequação (4) traz implicitamente que $u^0(x)$ deve pertencer ao espaço vetorial das funções de rápido decrescimento, que tendem a zero em $|x| \to \infty$, conhecido como espaço de Schwartz, $S(\mathbb{R}^3)$, em homenagem ao matemático francês Laurent Schwartz (1915-2002) que o estudou [2]. Estas funções e suas infinitas derivadas são contínuas (C^∞) e decaem mais rápido que o inverso de qualquer polinômio, tais que

$$\lim_{|x| \to \infty} |x|^k D^\alpha \varphi(x) = 0$$

para todo $\alpha = (\alpha_1, \ldots, \alpha_n)$, α_i inteiro não negativo, e todo inteiro $k \geq 0$. α é um multi-índice, com a convenção

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}}, |\alpha| = \alpha_1 + \cdots + \alpha_n, \alpha_i \in \{0, 1, 2, \ldots\}.$$

D^0 é o operador identidade, D^α um operador diferencial. Um exemplo de função deste espaço é $u(x) = P(x)e^{-|x|^2}$, onde $P(x)$ é uma função polinomial.

Valem as seguintes propriedades [3]:

1) $S(\mathbb{R}^n)$ é um espaço vetorial; ele é fechado sobre combinações lineares.

2) $S(\mathbb{R}^n)$ é uma álgebra; o produto de funções em $S(\mathbb{R}^n)$ também pertence a $S(\mathbb{R}^n)$.

3) $S(\mathbb{R}^n)$ é fechado sobre multiplicação por polinômios.

4) $S(\mathbb{R}^n)$ é fechado sobre diferenciação.

5) $S(\mathbb{R}^n)$ é fechado sobre translações e multiplicação por exponenciais complexos ($e^{ix\cdot\xi}$).

6) funções de $S(\mathbb{R}^n)$ são integráveis: $\int_{\mathbb{R}^n}|f(x)|dx < \infty$ para $f \in S(\mathbb{R}^n)$. Isto segue do fato de que $|f(x)| \le M(1+|x|)^{-(n+1)}$ e, usando coordenadas polares, $\int_{\mathbb{R}^n}(1+|x|)^{-(n+1)}dx = C\int_0^\infty (1+r)^{-n-1}\, r^{n-1}dr < \infty$, i.e., o integrando decresce como r^{-2} (e $(1+r)^{-2}$) no infinito e produz uma integral finita.

Da definição de $S(\mathbb{R}^3)$ e propriedades anteriores vemos que, como $u^0(x) \in S(\mathbb{R}^3)$, então $\int_{\mathbb{R}^3}|u^0(x)|dx \le \int_{\mathbb{R}^3} M(1+|x|)^{-4}dx \le C\int_0^\infty (1+r)^{-2}\, dr < \infty$ e quadrando $|u^0(x)|$ e $M(1+|x|)^{-4}$ chegamos à desigualdade $\int_{\mathbb{R}^3}|u^0(x)|^2 dx < \infty$, que contradiz (10).

Outra forma de verificar isso é que o conjunto $S(\mathbb{R}^n)$ está contido em $L^p(\mathbb{R}^n)$ para todo p, $1 \le p < \infty$ ([4], [5], [6], [7]), e em particular para $p = 2$ e $n = 3$ segue a finitude de $\int_{\mathbb{R}^3}|u^0(x)|^2 dx$.

Portanto, se a condição (7) for desobedecida, conforme propomos no artigo a seguir, que usou $\alpha \neq 0$, será para $t > 0$, por exemplo, encontrando alguma função $u(x,t)$ da forma $u^0(x)v(x,t)$, $v(x,0) = 1$, ou $u^0(x) + v(x,t)$, $v(x,0) = 0$, com $\int_{\mathbb{R}^3}|v(x,t)|^2 dx \to \infty$ e $\int_{\mathbb{R}^3}|u(x,t)|^2 dx \to \infty$. Parece-me de novo uma possibilidade viável. A prova da unicidade de soluções será importante. Vejamos então.

De fato, escolhendo $u^0(x) \in S(\mathbb{R}^3)$ e $f(x,t) \in S(\mathbb{R}^3 \times [0,\infty))$, lembrando-se que não precisamos ter $u, p \in S(\mathbb{R}^3 \times [0,\infty))$ como solução, apenas $u, p \in C^\infty$, então é possível construir uma solução para a velocidade da forma $u(x,t) = u^0(x)e^{-t} + v(t)$, com $v(0) = 0$, tal que $\int_{\mathbb{R}^3}|u(x,t)|^2 dx \to \infty$, pois quando $\int_{\mathbb{R}^3}[|u^0(x)|^2 e^{-t} + 2u^0(x)\cdot v(t)]dx \ge 0$, por exemplo, quando cada componente de $u^0(x)$ tem o mesmo sinal da respectiva componente de $v(t)$ ou o produto entre elas é zero ou $\int_{\mathbb{R}^3} u^0(x)\cdot v(t)dx \ge 0$, teremos $\int_{\mathbb{R}^3}|u(x,t)|^2 dx \ge \int_{\mathbb{R}^3}|v(t)|^2 dx = |v(t)|^2\int_{\mathbb{R}^3} dx \to \infty$, com $v(t) \neq 0, t > 0$. Também devemos escolher u, u^0 tais que $\nabla\cdot u = \nabla\cdot u^0 = 0$.

Em especial, escolhamos, para $1 \le i \le 3$,

$$u^0(x) = e^{-(x_1^2+x_2^2+x_3^2)}(x_2x_3, x_1x_3, -2x_1x_2),$$
$$v_i(t) = w(t) = e^{-t}(1-e^{-t}),$$
$$u_i(x,t) = u_i^0(x)e^{-t} + v_i(t),$$
$$f_i(x,t) = \left(-u_i^0 + e^{-t}\sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j} + \sum_{j=1}^{3} v_j \frac{\partial u_i^0}{\partial x_j} - \nu\nabla^2 u_i^0\right)e^{-t},$$

o que resulta para $p(x,t)$, como a única incógnita ainda a determinar,

$$\nabla p + \frac{\partial v}{\partial t} = 0,$$

e então

$$p(x,t) = -\frac{dw}{dt}(x_1+x_2+x_3) + \theta(t).$$

A pressão obtida tem uma dependência temporal genérica $\theta(t)$, que deve ser de classe $C^\infty([0,\infty))$ e podemos supor limitada, e diverge no infinito ($|x| \to \infty$), mas tenderá a zero em todo o espaço com o aumento do tempo (a menos eventualmente de $\theta(t)$), devido ao fator e^{-t} que aparece na derivada de $w(t)$,

$$\frac{dw}{dt} = e^{-t}(2e^{-t} - 1).$$

Neste exemplo $\int_{\mathbb{R}^3} u^0(x) \cdot v(t)dx = 0$, e assim $\int_{\mathbb{R}^3} |u(x,t)|^2 dx \to \infty$ para $t > 0$, como queríamos. Mais simples ainda seria escolher $u^0(x) = 0$.

A unicidade da solução (a menos da pressão $p(x,t)$ com o termo adicional constante ou dependente do tempo) vem dos resultados clássicos já conhecidos, descritos por exemplo no mencionado artigo de Fefferman [1]: o sistema das equações de Navier-Stokes (1), (2), (3) tem solução (única [8]) para todo $t \geq 0$ ou apenas para um intervalo de tempo $[0,T)$ finito dependente dos dados iniciais, onde T é chamado de "*blowup time*". Quando há uma solução com T finito então a velocidade u torna-se ilimitada próxima do "*blowup time*".

Vemos que a existência de nossa solução, no exemplo dado, está garantida por construção e substituição direta. Nossa velocidade não apresenta nenhum comportamento irregular, em instante t algum, em posição alguma, que a torne ilimitada, infinita, nem mesmo para $t \to \infty$ ou $|x| \to \infty$, sendo assim, não pode haver o "*blowup time*" no exemplo que demos, portanto a solução encontrada anteriormente é única em todo tempo. Mas ainda que houvesse um T finito (em [9] vemos que $T > 0$), a unicidade existiria em pelo menos um pequeno intervalo de tempo, o que já é suficiente para mostrar que neste intervalo ocorre a quebra das soluções de Navier-Stokes por ser desobedecida a condição de energia cinética limitada (7), tornando o caso (C) verdadeiro.

Grato ao professor Ricardo Rosa da UFRJ, matemático especialista nas equações de Navier-Stokes, que me explicou sobre o caso $\alpha = 0$ e sua natureza de multi-índice. Ninguém foi tão claro comigo quanto ele, nem mesmo (muito menos...) a *Annals of Mathematics*.

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06 – Breakdown of Navier-Stokes Solutions – Bounded Energy

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Abstract – We have proved that there are initial velocities $u^0(x)$ and forces $f(x,t)$ such that there is no physically reasonable solution to the Navier-Stokes equations, which corresponds to the case (C) of the problem relating to Navier-Stokes equations available on the website of the Clay Institute.

Keywords – Navier-Stokes equations, continuity equation, breakdown, existence, smoothness, physically reasonable solutions, gradient field, conservative field, velocity, pressure, external force, bounded energy, millennium problem.

The simplest way I see to prove the breakdown solutions of Navier-Stokes equations, following the described in [1], refers to the condition of bounded energy, the finiteness of the integral of the squared velocity of the fluid in the whole space.

We can certainly construct solutions for

(1) $$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i - \frac{\partial p}{\partial x_i} + f_i,\ 1 \le i \le 3,$$

that obey the condition of divergence-free to the velocity (continuity equation to the constant mass density),

(2) $$div\, u \equiv \nabla \cdot u = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0,$$ (incompressible fluids)

and the initial condition

(3) $$u(x,0) = u^0(x),$$

where u_i, p, f_i are functions of the position $x \in \mathbb{R}^3$ and the time $t \ge 0, t \in \mathbb{R}$. The constant $\nu \ge 0$ is the viscosity coefficient, p represents the pressure and $u = (u_1, u_2, u_3)$ is the fluid velocity, measured in the position x and time t, with $\nabla^2 = \nabla \cdot \nabla = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$. The function $f = (f_1, f_2, f_3)$ has the dimension as acceleration or force per mass unit, but we will keep on naming this vector and its components by its generic name of force, such as used in [1]. It's the externally applied force to the fluid.

The functions $u^0(x)$ and $f(x,t)$ must obey, respectively,

(4) $$|\partial_x^\alpha u^0(x)| \le C_{\alpha K}(1 + |x|)^{-K}$$ on $\mathbb{R}^3$, for any $\alpha \in \mathbb{N}^3$ and $K \in \mathbb{R}$,

and

(5) $|\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1+|x|+t)^{-K}$ on $\mathbb{R}^3 \times [0,\infty)$, for any $\alpha \in \mathbb{N}^3$, $m \in \mathbb{N}$ and $K \in \mathbb{R}$,

and a solution (p,u) from (1) to be considered physically reasonable must be continuous and have all the derivatives, of infinite orders, also continuous (smooth), i.e.,

(6) $p, u \in C^\infty(\mathbb{R}^3 \times [0,\infty))$.

Given an initial velocity u^0 of C^∞ class, divergence-free ($\nabla \cdot u^0 = 0$) on $\mathbb{R}^3$ and an external forces field f also C^∞ class on $\mathbb{R}^3 \times [0,\infty)$, we want, for that a solution to be physically reasonable, beyond the validity of (6), that $u(x,t)$ does not diverge to $|x| \to \infty$ and satisfy the bounded energy condition, i.e.,

(7) $\int_{\mathbb{R}^3} |u(x,t)|^2 dx < C$, for all $t \geq 0$.

We see that every condition above, from (1) to (7), need to be obeyed to get a solution (p,u) considered physically reasonable, however, to get the breakdown solutions, (1), (2), (3), (6) or (7) could not be satisfied to some $t \geq 0$, in some position $x \in \mathbb{R}^3$, still maintaining (4) and (5) validity.

A way to make this situation (breakdown) happens is when (1) have no possible solution to the pressure $p(x,t)$, when the vector field $\phi: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$ in

(8) $\nabla p = \nu \, \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla)u + f = \phi$

is not gradient, not conservative, in at least one $(x,t) \in \mathbb{R}^3 \times [0,\infty)$. In this case, to $\phi = (\phi_1, \phi_2, \phi_3)$ not to be gradient, it must be

(9) $\frac{\partial \phi_i}{\partial x_j} \neq \frac{\partial \phi_j}{\partial x_i}, i \neq j,$

to some pair $(i,j), 1 \leq i,j \leq 3$, $x \in \mathbb{R}^3$ and time t not negative (for details check, for example, Apostol[2], chapter 10).

If we admit, however, that (1) has a possible (p,u) solution and this also obey (2), (3) and (6), the initial condition $u^0(x)$ verifies (2) and (4), the external force $f(x,t)$ verifies (5) and both $u^0(x)$ and $f(x,t)$ are C^∞ class, we can try get a breakdown solutions in $t = 0$ violating the condition (7), i.e., choosing $u^0(x)$ that also obey to

(10) $\int_{\mathbb{R}^3} |u^0(x)|^2 dx \to \infty$.

The first example is very simple: a constant initial velocity not null, $u^0(x) = c = (c_1, c_2, c_3)$, $c_i \in \mathbb{R}$, $c \neq 0$. In this example we have $|\partial_x^\alpha u^0(x)| = 0$, satisfying (4), and, by hypothesis, we also suppose satisfied the remaining conditions from (1) to (6), with $f \in C^\infty$. Are also valid, obviously, $u^0 \in C^\infty$ and $\nabla \cdot u^0 = 0$. Giving $f = 0$, a possible solution (p, u) to (1) and (2) is $u = u^0 = c, p = 0$. Only condition (7) is not satisfied in this simple example of constant initial velocity, because in $t = 0$ we have

(11) $\left(\int_{\mathbb{R}^3} |u(x,t)|^2 dx\right)|_{t=0} = \int_{\mathbb{R}^3} |u^0(x)|^2 dx = (c_1^2 + c_2^2 + c_3^2) \int_{\mathbb{R}^3} dx \to \infty.$

Certainly this initial velocity doesn't belong to a solution $u(x,t)$ considered physically reasonable, because it would violate (7), whichever the $u(x,t)$ with $u(x,0) = u^0(x) = c$, but $u^0(x)$ obeyed to the permissible requirements to an initial velocity in this problem of breakdown solutions. Both $u^0(x)$ and $u(x,t)$ violate condition (7) of bounded energy, obeying however p, u, u^0 and f the remaining conditions (by hypothesis), which characterizes the so called breakdown solutions, according to the wanted.

The official description of the problem to this (C) case of breakdown solutions is given below:

(C) Breakdown solutions of Navier-Stokes on $\mathbb{R}^3$. Take $\nu > 0$ and $n = 3$. Then there exist a smooth and divergence-free vector field $u^0(x)$ on $\mathbb{R}^3$ and a smooth external force $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ satisfying

(4) $|\partial_x^\alpha u^0(x)| \leq C_{\alpha K}(1 + |x|)^{-K}$ on $\mathbb{R}^3, \forall \alpha, K,$

and

(5) $|\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1 + |x| + t)^{-K}$ on $\mathbb{R}^3 \times [0,\infty), \forall \alpha, m, K,$

for which there exist no solutions (p, u) of (1), (2), (3), (6), (7) on $\mathbb{R}^3 \times [0,\infty)$.

It's clear to see that we can solve this problem searching valid initial velocities which the integral of its square in all space $\mathbb{R}^3$ is infinite, or also, as shown in (8), searching functions ϕ non gradients, where the pressure p won't be considered a potential function to some instant $t \geq 0$. We understand that the α, m shown in (4) and (5) just make sense to $|\alpha|, m \in \{1, 2, 3, 4, \dots\}$ and the negatives K implicitly allow that the derivatives of the functions u^0 and f can not be limited when $|x| \to \infty$, with $C_{\alpha K}, C_{\alpha m K} > 0$.

Two other examples, among many, are initial velocities with a constant term plus a squared exponential decay and linear functions in a direction and null or other constant in the other directions, i.e.,

(12) $u^0(x) = \left(c_i - b_i e^{-x_{i+1}^2}\right)_{1 \le i \le 3}$, $c_i \neq 0$, with $x_4 \equiv x_1$,

and

(13) $u^0(x) = (ax_2, b, c), a \neq 0$.

Both examples obey the necessary conditions of divergence-free ($\nabla \cdot u^0 = 0$), smoothness (C^∞) and partial derivatives of $C_{\alpha K}(1 + |x|)^{-K}$ order, although (13) is not limited to $|x| \to \infty$ (the example (13) is only valid in (4) to $K \le 0$ if $|\alpha| = 1$ and to any K (real) if $|\alpha| \ge 2$, so we made K depend on $|\alpha|$). To each possible $u(x, t)$ so that (3) is true, the external force $f(x, t)$ and the pressure $p(x, t)$ can be fittingly constructed, in C^∞ class, verifying (8), and in a way to satisfy all the necessary conditions, finding, this way, a possible solution to (1), (2), (3), (4), (5) and (6), and only (7) wouldn't be satisfied, at least not in instant $t = 0$, according to (10). We then show examples of breakdown solutions to case (C) of this millennium problem. These examples, however, won't take to case (A) from [1], of existing and smoothness of solutions, because they violate (7) (case (A) also impose a null external force, $f = 0$).

An overview of the problem's conditions is listed below.

$\nu > 0, n = 3$		
$\exists u^0(x): \mathbb{R}^3$		smooth (C^∞), divergence-free ($\nabla \cdot u^0 = 0$)
$\exists f(x, t): \mathbb{R}^3 \times [0, \infty)$		smooth (C^∞)
(4)	$\lvert\partial_x^\alpha u^0(x)\rvert \le C_{\alpha K}(1 + \lvert x\rvert)^{-K}: \mathbb{R}^3,\ \forall \alpha, K$	
(5)	$\lvert\partial_x^\alpha \partial_t^m f(x, t)\rvert \le C_{\alpha m K}(1 + \lvert x\rvert + t)^{-K}: \mathbb{R}^3 \times [0, \infty)$, $\forall \alpha, m, K$	
$\nexists (p, u): \mathbb{R}^3 \times [0, \infty)$ /		
(1)	$\frac{\partial u_i}{\partial t} + \sum_{j=1}^3 u_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i - \frac{\partial p}{\partial x_i} + f_i(x, t), 1 \le i \le 3\ \ (x \in \mathbb{R}^3, t \ge 0)$	
(2)	$\nabla \cdot u = 0$	
(3)	$u(x, 0) = u^0(x) \qquad (x \in \mathbb{R}^3)$	
(6)	$p, u \in C^\infty(\mathbb{R}^3 \times [0, \infty))$	
(7)	$\int_{\mathbb{R}^3} \lvert u(x, t)\rvert^2 dx < C, \forall t \ge 0$	(bounded energy)

It's important that we observe the solution's uniqueness question. As $u^0(x)$ and $f(x, t)$ are given of C^∞ class, chosen by us, and satisfying (4) and (5), with $\nabla \cdot u^0 = 0$, claim that there is no solution (p, u) to the system (1), (2), (3), (6) and

(7) might assume that we explored, or proved to, the infinite possible combinations of p and u, i.e., of (p, u).

Keeping fixed $u^0(x)$, as long as (10) is true, to each one of the infinite possible combinations of the variables u, p and f such that the quadruplet (u^0, u, p, f) fulfill the system (1) to (6), the inequality (7) remains false in $t = 0$, because

(14) $$\left(\int_{\mathbb{R}^3} |u(x,t)|^2 dx\right)|_{t=0} = \int_{\mathbb{R}^3} |u^0(x)|^2 dx \to \infty,$$

not existing a constant C that verifies it, and so our proof is not restricted to some velocity $u(x,t)$ in particular, we don't need to admit that there is uniqueness of solutions to Navier-Stokes equations.

□

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06 – Breakdown of Navier-Stokes Solutions – Bounded Energy

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Abstract – We have proved that there are initial velocities $u^0(x)$ and forces $f(x,t)$ such that there is no physically reasonable solution to the Navier-Stokes equations, which corresponds to the case (C) of the problem relating to Navier-Stokes equations available on the website of the Clay Institute.

Keywords – Navier-Stokes equations, continuity equation, breakdown, existence, smoothness, physically reasonable solutions, gradient field, conservative field, velocity, pressure, external force, bounded energy, millenium problem.

A maneira mais simples que vejo para se provar a quebra de soluções (*breakdown solutions*) das equações de Navier-Stokes, seguindo o descrito em [1], refere-se à condição de energia limitada (*bounded energy*), a finitude da integral do quadrado da velocidade do fluido em todo o espaço.

Podemos certamente construir soluções de

(1) $$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i - \frac{\partial p}{\partial x_i} + f_i,\ 1 \le i \le 3,$$

que obedeçam à condição de divergente nulo para a velocidade (equação da continuidade para densidade de massa constante),

(2) $$div\, u \equiv \nabla \cdot u = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0,$$ (fluidos incompressíveis)

e à condição inicial

(3) $$u(x,0) = u^0(x),$$

onde u_i, p, f_i são funções da posição $x \in \mathbb{R}^3$ e do tempo $t \ge 0, t \in \mathbb{R}$. A constante $\nu \ge 0$ é o coeficiente de viscosidade, p representa a pressão e $u = (u_1, u_2, u_3)$ é a velocidade do fluido, medidas na posição x e tempo t, com $\nabla^2 = \nabla \cdot \nabla = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$. A função $f = (f_1, f_2, f_3)$ tem dimensão de aceleração ou força por unidade de massa, mas seguiremos denominando este vetor e suas componentes pelo nome genérico de força, tal como adotado em [1]. É a força externa aplicada ao fluido.

As funções $u^0(x)$ e $f(x,t)$ devem obedecer, respectivamente,

(4) $$|\partial_x^\alpha u^0(x)| \le C_{\alpha K}(1 + |x|)^{-K}$$ sobre $\mathbb{R}^3$, para quaisquer $\alpha \in \mathbb{N}^3$ e $K \in \mathbb{R}$,

e

(5) $\quad |\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1+|x|+t)^{-K}$ sobre $\mathbb{R}^3 \times [0,\infty)$, para quaisquer $\alpha \in \mathbb{N}^3$, $m \in \mathbb{N}$ e $K \in \mathbb{R}$,

e uma solução (p,u) de (1) para que seja considerada fisicamente razoável deve ser contínua e ter todas as derivadas, de infinitas ordens, também contínuas (*smooth*), i.e.,

(6) $\quad p, u \in C^\infty \qquad (\mathbb{R}^3 \times [0,\infty))$.

Dada uma velocidade inicial u^0 de classe C^∞ com divergente nulo (*divergence-free*, $\nabla \cdot u^0 = 0$) sobre $\mathbb{R}^3$ e um campo de forças externo f também de classe C^∞ sobre $\mathbb{R}^3 \times [0,\infty)$, quer-se, para que uma solução seja fisicamente razoável, além da validade de (6), que $u(x,t)$ não divirja para $|x| \to \infty$ e seja satisfeita a condição de energia limitada (*bounded energy*), i.e.,

(7) $\quad \int_{\mathbb{R}^3} |u(x,t)|^2 dx < C$, para todo $t \geq 0$.

Vemos que todas as condições acima, de (1) a (7), precisam ser obedecidas para se obter uma solução (p,u) considerada fisicamente razoável, contudo, para se obter uma quebra de soluções, (1), (2), (3), (6) ou (7) poderiam não ser satisfeitas para algum $t \geq 0$, em alguma posição $x \in \mathbb{R}^3$, mantendo-se ainda a validade de (4) e (5).

Uma maneira de fazer com que esta situação (*breakdown*) ocorra é quando (1) não tem solução possível para a pressão $p(x,t)$, quando o campo vetorial $\phi: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$ em

(8) $\quad \nabla p = \nu \, \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla)u + f = \phi$

é não gradiente, não conservativo, em ao menos um $(x,t) \in \mathbb{R}^3 \times [0,\infty)$. Nesse caso, para $\phi = (\phi_1, \phi_2, \phi_3)$ ser não gradiente deve valer

(9) $\quad \frac{\partial \phi_i}{\partial x_j} \neq \frac{\partial \phi_j}{\partial x_i}, i \neq j,$

para algum par $(i,j), 1 \leq i,j \leq 3$, $x \in \mathbb{R}^3$ e tempo t não negativo (para mais detalhes veja, por exemplo, Apostol[2], cap. 10).

Se admitirmos, entretanto, que (1) tem solução (p,u) possível e esta também obedece (2), (3) e (6), a condição inicial $u^0(x)$ verifica (2) e (4), a força externa $f(x,t)$ verifica (5) e $u^0(x)$ e $f(x,t)$ são de classe C^∞, podemos tentar obter a condição de quebra de soluções em $t = 0$ violando-se a condição (7), i.e., escolhendo-se $u^0(x)$ que também obedeça a

(10) $\int_{\mathbb{R}^3} |u^0(x)|^2 dx \to \infty.$

O primeiro exemplo é muito simples: uma velocidade inicial constante não nula, $u^0(x) = c = (c_1, c_2, c_3)$, $c_i \in \mathbb{R}$, $c \neq 0$. Neste exemplo temos $|\partial_x^\alpha u^0(x)| = 0$, satisfazendo (4), e, por hipótese, suponhamos satisfeitas também as demais condições de (1) a (6), com $f \in C^\infty$. Também valem, obviamente, $u^0 \in C^\infty$ e $\nabla \cdot u^0 = 0$. Dado $f = 0$, uma solução (p, u) possível para (1) e (2) é $u = u^0 = c, p = 0$. Apenas a condição (7) não é satisfeita neste simples exemplo de velocidade inicial constante, pois em $t = 0$ temos

(11) $\left(\int_{\mathbb{R}^3} |u(x,t)|^2 dx\right)|_{t=0} = \int_{\mathbb{R}^3} |u^0(x)|^2 dx = (c_1^2 + c_2^2 + c_3^2) \int_{\mathbb{R}^3} dx \to \infty.$

Certamente esta velocidade inicial não pertence a uma solução $u(x,t)$ considerada fisicamente razoável, pois violaria (7), qualquer que fosse $u(x,t)$ com $u(x,0) = u^0(x) = c$, mas $u^0(x)$ obedeceu aos requisitos permitidos para a velocidade inicial neste problema de quebra de soluções. Tanto $u^0(x)$ quanto $u(x,t)$ violam a condição (7) de energia limitada (*bounded energy*), obedecendo-se entretanto p, u, u^0 e f às demais condições (por hipótese), o que caracteriza a chamada *breakdown solutions*, conforme queríamos.

A descrição oficial do problema para este caso (C) de quebra de soluções é dada a seguir:

(C) Quebra das soluções da Equação de Navier-Stokes sobre $\mathbb{R}^3$. Para $\nu > 0$ e dimensão espacial $n = 3$ existem um campo vetorial suave e com divergência nula $u^0(x)$ sobre $\mathbb{R}^3$ e uma força externa suave $f(x,t)$ sobre $\mathbb{R}^3 \times [0, \infty)$ satisfazendo

(4) $|\partial_x^\alpha u^0(x)| \leq C_{\alpha K}(1 + |x|)^{-K}$ sobre $\mathbb{R}^3, \forall \alpha, K,$

e

(5) $|\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1 + |x| + t)^{-K}$ sobre $\mathbb{R}^3 \times [0, \infty), \forall \alpha, m, K,$

tais que não existe solução (p, u) sobre $\mathbb{R}^3 \times [0, \infty)$ satisfazendo (1), (2), (3), (6) e (7).

Vê-se claramente que podemos resolver este problema buscando velocidades iniciais válidas cuja integral do seu quadrado em todo o espaço $\mathbb{R}^3$ é infinito, ou também, conforme indicamos em (8), buscando funções ϕ não gradientes, onde a pressão p não poderá ser considerada uma função potencial, para algum instante $t \geq 0$. Entendemos que os α, m indicados em (4) e (5) só fazem sentido para $|\alpha|, m \in \{1, 2, 3, 4, \dots\}$ e os K negativos permitem implicitamente que as derivadas das funções u^0 e f podem não ser limitadas quando $|x| \to \infty$, com $C_{\alpha K}, C_{\alpha m K} > 0$.

Dois outros exemplos, dentre muitos, são velocidades iniciais com um termo constante mais um decaimento exponencial quadrático e funções lineares em uma direção e igual a zero ou outra constante nas outras direções, ou seja,

(12) $u^0(x) = \left(c_i - b_i e^{-x_{i+1}^2}\right)_{1 \le i \le 3}$, $c_i \neq 0$, com $x_4 \equiv x_1$,

e

(13) $u^0(x) = (ax_2, b, c), a \neq 0.$

Ambos os exemplos obedecem às condições de divergência nula (*divergence-free,* $\nabla \cdot u^0 = 0$), suavidade (*smoothness*, C^∞) e derivadas parciais da ordem de $C_{\alpha K}(1+|x|)^{-K}$, embora (13) não seja limitada para $|x| \to \infty$ (o exemplo (13) só é válido em (4) para $K \le 0$ se $|\alpha| = 1$ e qualquer K (real) se $|\alpha| \ge 2$, portanto fizemos K depender de $|\alpha|$). Para cada $u(x,t)$ possível tal que (3) seja verdadeira, a força externa $f(x,t)$ e a pressão $p(x,t)$ podem ser convenientemente construídas, na classe C^∞, verificando (8), e de modo a satisfazerem todas as condições necessárias, encontrando-se assim uma solução possível para (1), (2), (3), (4), (5) e (6), e apenas (7) não seria satisfeita, ao menos no instante $t = 0$, conforme (10). Mostramos então exemplos de quebra de soluções para o caso (C) deste problema do milênio. Estes exemplos, entretanto, não levam ao caso (A) de [1], de existência e suavidade das soluções, justamente por violarem (7) (O caso (A) também impõe que seja nula a força externa, $f = 0$).

Um resumo das condições do problema está listado abaixo.

$\nu > 0, n = 3$	
$\exists u^0(x): \mathbb{R}^3$	smooth (C^∞), divergence-free ($\nabla \cdot u^0 = 0$)
$\exists f(x,t): \mathbb{R}^3 \times [0,\infty)$	smooth (C^∞)
(4) $\lvert\partial_x^\alpha u^0(x)\rvert \le C_{\alpha K}(1+\lvert x\rvert)^{-K}: \mathbb{R}^3,\ \forall \alpha, K$	
(5) $\lvert\partial_x^\alpha \partial_t^m f(x,t)\rvert \le C_{\alpha m K}(1+\lvert x\rvert+t)^{-K}: \mathbb{R}^3 \times [0,\infty)$, $\forall \alpha, m, K$	
$\nexists (p,u): \mathbb{R}^3 \times [0,\infty)$ /	
(1) $\frac{\partial u_i}{\partial t} + \sum_{j=1}^3 u_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i - \frac{\partial p}{\partial x_i} + f_i(x,t), 1 \le i \le 3\ \ (x \in \mathbb{R}^3, t \ge 0)$	
(2) $\nabla \cdot u = 0$	
(3) $u(x,0) = u^0(x) \qquad (x \in \mathbb{R}^3)$	
(6) $p, u \in C^\infty(\mathbb{R}^3 \times [0,\infty))$	
(7) $\int_{\mathbb{R}^3} \lvert u(x,t)\rvert^2 dx < C, \forall t \ge 0$	(bounded energy)

É importante observarmos a questão da unicidade das soluções. Como $u^0(x)$ e $f(x,t)$ são dados, escolhidos por nós, de classe C^∞ e satisfazendo (4) e (5), com $\nabla \cdot u^0 = 0$, afirmar que não existe solução (p,u) para o sistema (1), (2), (3), (6) e (7) pode pressupor que exploramos, ou provamos para, as infinitas combinações possíveis de p e de u, i.e., de (p,u).

Mantido fixo $u^0(x)$, desde que (10) seja verdadeira, para cada uma das infinitas combinações possíveis das variáveis u, p e f tais que a quádrupla (u^0, u, p, f) torne verdadeiro o sistema (1) a (6) a desigualdade (7) continua falsa em $t = 0$, pois

$$(14) \quad \left(\int_{\mathbb{R}^3} |u(x,t)|^2 dx\right)\Big|_{t=0} = \int_{\mathbb{R}^3} |u^0(x)|^2 dx \to \infty,$$

não existindo nenhuma constante C que a verifique, e assim nossa prova não se restringe a alguma velocidade $u(x,t)$ em particular, nem precisamos admitir que há unicidade de soluções para as equações de Navier-Stokes.

□

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07 – Uma crítica à solução de Otelbaev ao sexto problema do milênio (equações de Navier-Stokes)

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No dia 28/10/2015, uma quarta-feira, o professor Odilon Otávio me vê com cinco livros sobre as equações de Navier-Stokes no balcão da biblioteca do IME-USP e faz uma brincadeira comigo:

– Você quer ficar milionário é?

Eu lhe respondo: – Ah, eu quero (risos). – Só estou esperando!

E ele completa: – Mas você entendeu a minha pergunta? Sabe do que eu estou falando? Você entendeu por que eu fiz a pergunta?

Lá vamos nós: – Claro que sim, o problema do milênio, sobre as equações de Navier-Stokes. Eu resolvi o problema.

E etc.

A conversa chegou na solução dada pelo professor cazaquistanês Mukhtarbai Otelbaev ao mesmo problema, publicada em russo [1], e eu lhe digo que a solução dele não é uma solução completa para o problema do milênio [2], embora possa ser uma solução para um caso particular das equações de Navier-Stokes.

– Vê só – disse eu – todos estes livros. Tudo aqui é Navier-Stokes (onde se inclui dois Roger Temam e um Peter Constantin), mas nenhum deles traz uma solução para o problema do milênio, uma solução completa. São páginas e mais páginas de Navier-Stokes, aqui está cheio de Navier-Stokes, mas não têm o problema do milênio.

O professor Odilon (mais ou menos assim): – Mas o que o professor errou? O artigo dele foi publicado. Tem alguém olhando isso? Alguém leu o artigo dele e encontrou um erro? etc.

O matemático australiano (e reconhecidamente genial) Terence (Terry) Tao, de descendência chinesa, medalha Fields de 2006, encontrou um contraexemplo para a solução de Otelbaev, mas mesmo sem recorrer ao seu trabalho [3] quero expor meus próprios motivos para não concordar com a solução do matemático cazaquistanês.

Resumidamente falando, Otelbaev tratou do caso (B) do problema, existência e suavidade de soluções espacialmente periódicas para as equações de Navier-Stokes. Por um lado, permitiu o uso de densidade de força externa não nula, $f \neq 0$, periódica ou não, o que é uma ampliação e sofisticação do problema original, mas por outro lado limitou a velocidade inicial a $u^0 = 0$ e o domínio da solução ao cubo $Q \equiv (0, 2\pi)^3$ e tempo $(0, a), a > 0$, ao invés do domínio mais geral $\mathbb{R}^3 \times [0, \infty)$. Além disso, o período espacial das funções velocidade e pressão que utilizou foi 2π, ao invés de 1, conforme o problema original, suas funções são de classe L_2 em $\Omega = (0, a) \times (0, 2\pi)^3$, sem precisar se são também de classe C^∞, e utilizou uma condição adicional para a pressão, $\int_Q p(t, x)dx = p_0 = const \geq 0$, um recurso que talvez esconda outras soluções possíveis para a pressão. Otelbaev também usou apenas o coeficiente de viscosidade $\nu = 1$, ao invés de qualquer $\nu > 0$.

Isto se verifica lendo-se o resumo em inglês ao final do artigo [1], conforme comentei com o professor Odilon, sem entrar em mais detalhes, agora descritos. Não fiz a leitura em russo, nem parece existir uma tradução completa para o inglês, infelizmente.

Em minha opinião, limitar a solução de tão importante problema à velocidade inicial nula, $u^0 = 0$, e concluir que a solução é única para todo f (pertencente a $L_2(\Omega)$), é de extrema falta de generalidade. Se voltarmos ao problema oficial, caso (B), que pede para ser $f = 0$ e não impõe nenhuma condição inicial e de contorno para a pressão, exceto sua periodicidade espacial de período unitário, nossa solução de escolha para a velocidade é trivial, $u = 0$, e a pressão obtida de $\nabla p = 0$ é uma constante, inclusive zero, podendo ser acrescida de alguma função do tempo bem comportada (limitada, contínua, C^∞), i.e., $p = p_0 + \theta(t)$. Essa seria também uma solução (única ou não) para o caso (A) do problema do milênio, *à moda de Otelbaev*, que escolheu uma única e simples velocidade inicial $u^0 = 0$.

Entendo que uma adequada solução para os casos (A) e (B) do problema referente às equações de Navier-Stokes deve levar em consideração todas as possíveis velocidades iniciais $u^0 \in C^\infty$, dentre outras condições necessárias, e não apenas uma única função u^0 específica.

A solução que eu proponho refere-se ao caso (C), *breakdown solutions*, e pode ser encontrada em [4], com seu desenvolvimento gradativo em [5]. Ela obedece a cada quesito mencionado na descrição do respectivo problema [2], e não pressupõe unicidade de soluções.

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08-Breakdown of Navier-Stokes Solutions – Unbounded Energy for t > 0

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Abstract – A solution to the 6th millenium problem, respect to breakdown of Navier-Stokes solutions and the bounded energy. We have proved that there are initial velocities $u^0(x)$ and forces $f(x,t)$ such that there is no physically reasonable solution to the Navier-Stokes equations for $t > 0$, which corresponds to the case (C) of the problem relating to Navier-Stokes equations available on the website of the Clay Institute.

Keywords – Navier-Stokes equations, continuity equation, breakdown, existence, smoothness, physically reasonable solutions, gradient field, conservative field, velocity, pressure, external force, bounded energy, millennium problem.

§ 1

The simplest way I see to prove the breakdown solutions of Navier-Stokes equations, following the described in [1], refers to the condition of bounded energy, the finiteness of the integral of the squared velocity of the fluid in the whole space.

We can certainly construct solutions for

(1) $$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i - \frac{\partial p}{\partial x_i} + f_i,\ 1 \le i \le 3,$$

that obey the condition of divergence-free to the velocity (continuity equation to the constant mass density),

(2) $$div\, u \equiv \nabla \cdot u = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0,$$ (incompressible fluids)

and the initial condition

(3) $$u(x,0) = u^0(x),$$

where u_i, p, f_i are functions of the position $x \in \mathbb{R}^3$ and the time $t \ge 0, t \in \mathbb{R}$. The constant $\nu \ge 0$ is the viscosity coefficient, p represents the pressure and $u = (u_1, u_2, u_3)$ is the fluid velocity, measured in the position x and time t, with $\nabla^2 = \nabla \cdot \nabla = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$. The function $f = (f_1, f_2, f_3)$ has the dimension as acceleration or force per mass unit, but we will keep on naming this vector and its components by the generic name of force, such as used in [1]. It's the externally applied force to the fluid, for example, gravity.

The functions $u^0(x)$ and $f(x,t)$ must obey, respectively,

(4) $$|\partial_x^\alpha u^0(x)| \leq C_{\alpha K}(1+|x|)^{-K} \text{ on } \mathbb{R}^3, \text{ for any } \alpha \in \mathbb{N}_0^3 \text{ and } K \in \mathbb{R},$$

and

(5) $|\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1+|x|+t)^{-K}$ on $\mathbb{R}^3 \times [0,\infty)$, for any $\alpha \in \mathbb{N}_0^3$, $m \in \mathbb{N}_0$ and $K \in \mathbb{R}$,

with $\mathbb{N}_0 = \{0,1,2,3,\dots\}$ (derivatives of order zero does not change the value of function), and a solution (p,u) from (1) to be considered physically reasonable must be continuous and have all the derivatives, of infinite orders, also continuous (smooth), i.e.,

(6) $$p, u \in C^\infty(\mathbb{R}^3 \times [0,\infty)).$$

Given an initial velocity u^0 of C^∞ class, divergence-free ($\nabla \cdot u^0 = 0$) on $\mathbb{R}^3$ and an external forces field f also C^∞ class on $\mathbb{R}^3 \times [0,\infty)$, we want, for that a solution to be physically reasonable, beyond the validity of (6), that $u(x,t)$ does not diverge to $|x| \to \infty$ and satisfy the bounded energy condition, i.e.,

(7) $$\int_{\mathbb{R}^3} |u(x,t)|^2 dx < C, \text{ for all } t \geq 0.$$

We see that every condition above, from (1) to (7), need to be obeyed to get a solution (p,u) considered physically reasonable, however, to get the breakdown solutions, (1), (2), (3), (6) or (7) could not be satisfied to some $t \geq 0$, in some position $x \in \mathbb{R}^3$, still maintaining (4) and (5) validity.

A way to make this situation (breakdown) happens is when (1) have no possible solution to the pressure $p(x,t)$, when the vector field $\phi: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$ in

(8) $$\nabla p = \nu\, \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla)u + f = \phi$$

is not gradient, not conservative, in at least one $(x,t) \in \mathbb{R}^3 \times [0,\infty)$. In this case, to $\phi = (\phi_1, \phi_2, \phi_3)$ not to be gradient, it must be

(9) $$\frac{\partial \phi_i}{\partial x_j} \neq \frac{\partial \phi_j}{\partial x_i},\ i \neq j,$$

to some pair (i,j), $1 \leq i,j \leq 3$, $x \in \mathbb{R}^3$ and time t not negative (for details check, for example, Apostol[2], chapter 10).

If we admit, however, that (1) has a possible (p,u) solution and this also obey (2), (3) and (6), the initial condition $u^0(x)$ verifies (2) and (4), the external force $f(x,t)$ verifies (5) and both $u^0(x)$ and $f(x,t)$ are C^∞ class, we can try get a breakdown solutions in $t \geq 0$ violating the condition (7) (bounded energy), i.e., choosing $u^0(x)$ or $u(x,t)$ that also obey to

(10) $$\int_{\mathbb{R}^3} |u(x,t)|^2 dx \to \infty, \text{ for some } t \geq 0.$$

The official description of the problem to this (C) case of breakdown solutions is given below:

(C) Breakdown solutions of Navier-Stokes on $\mathbb{R}^3$. Take $\nu > 0$ and $n = 3$. Then there exist a smooth and divergence-free vector field $u^0(x)$ on $\mathbb{R}^3$ and a smooth external force $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ satisfying

(4) $$|\partial_x^\alpha u^0(x)| \leq C_{\alpha K}(1+|x|)^{-K} \text{ on } \mathbb{R}^3, \forall \alpha, K,$$

and

(5) $$|\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1+|x|+t)^{-K} \text{ on } \mathbb{R}^3 \times [0,\infty), \forall \alpha, m, K,$$

for which there exist no solutions (p,u) of (1), (2), (3), (6), (7) on $\mathbb{R}^3 \times [0,\infty)$.

It's clear to see that we can solve this problem searching valid velocities which the integral of its square in all space $\mathbb{R}^3$ is infinite, or also, as shown in (8), searching functions ϕ non gradients, where the pressure p won't be considered a potential function to some instant $t \geq 0$. We understand that the α, m shown in (4) and (5) just make sense to $|\alpha|, m \in \{0,1,2,3,4,\dots\}$ and the negatives K can be ignored, because it does not limit the value of the functions u^0, f and its derivatives when $|x| \to \infty$ or $t \to \infty$, with $C_{\alpha K}, C_{\alpha m K} > 0$.

§ 2

The inequation (4) brings implicitly that $u^0(x)$ must belong to the vectorial space of rapidly decreasing functions, which tend to zero for $|x| \to \infty$, known as Schwartz space, $S(\mathbb{R}^3)$, named after the French mathematician Laurent Schwartz (1915-2002) which studied it [3]. These functions and its derivatives of all orders are continuous (C^∞) and decrease faster than the inverse of any polynomial, such that

(11) $$\lim_{|x|\to\infty} |x|^k D^\alpha \varphi(x) = 0$$

for all $\alpha = (\alpha_1, \dots, \alpha_n)$, α_i non negative integer, and all integer $k \geq 0$. α is a multi-index, with the convention

(12) $$D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}}, |\alpha| = \alpha_1 + \dots + \alpha_n, \alpha_i \in \{0,1,2,\dots\}.$$

D^0 is the operator identity, D^α is a differential operator. An example of function of this space is $u(x) = P(x)e^{-|x|^2}$, where $P(x)$ is a polynomial function.

The following properties are valid [4]:

1) $S(\mathbb{R}^n)$ is a vector space; it is closed under linear combinations.

2) $S(\mathbb{R}^n)$ is an algebra; the product of functions in $S(\mathbb{R}^n)$ also belongs to $S(\mathbb{R}^n)$ (this follows from Leibniz' formula for derivatives of products).

3) $S(\mathbb{R}^n)$ is closed under multiplication by polinomials, althrough polynomials are not in S.

4) $S(\mathbb{R}^n)$ is closed under differentiation.

5) $S(\mathbb{R}^n)$ is closed under translations and multiplication by complex exponentials ($e^{ix\cdot\xi}$).

6) $S(\mathbb{R}^n)$ functions are integrable: $\int_{\mathbb{R}^n} |f(x)|dx < \infty$ for $f \in S(\mathbb{R}^n)$. This follows from the fact that $|f(x)| \leq M(1+|x|)^{-(n+1)}$ and, using polar coordinates, $\int_{\mathbb{R}^n}(1+|x|)^{-(n+1)}dx = C\int_0^\infty (1+r)^{-n-1}\, r^{n-1}dr < \infty$, i.e., the function $|f|$ decreases like r^{-2} (and $(1+r)^{-2}$) at infinity and a finite integral is produced.

By $S(\mathbb{R}^3)$ definition and previous properties we see that, as $u^0(x) \in S(\mathbb{R}^3)$, then $\int_{\mathbb{R}^3} |u^0(x)|dx \leq \int_{\mathbb{R}^3} M(1+|x|)^{-4}dx \leq C\int_0^\infty (1+r)^{-2}\, dr < \infty$ and squared $|u^0(x)|$ and $M(1+|x|)^{-4}$ we come to the inequality $\int_{\mathbb{R}^3} |u^0(x)|^2 dx < \infty$, that contradicts (10).

Another way to check this is that the set $S(\mathbb{R}^n)$ it is contained in $L^p(\mathbb{R}^n)$ for all p, $1 \leq p < \infty$ ([5]-[9]), and in particular for $p = 2$ and $n = 3$ follows the finiteness of $\int_{\mathbb{R}^3} |u^0(x)|^2 dx$.

Therefore, if the condition (7) is disobeyed, as we propose in this article, will be for $t > 0$, for example, finding some function $u(x,t)$ like $u^0(x)v(x,t)$, $v(x,0) = 1$, or $u^0(x) + v(x,t)$, $v(x,0) = 0$, with $\int_{\mathbb{R}^3} |v(x,t)|^2 dx \to \infty$ and $\int_{\mathbb{R}^3} |u(x,t)|^2 dx \to \infty$.

§ 3

Really, choosing $u^0(x) \in S(\mathbb{R}^3)$ and $f(x,t) \in S(\mathbb{R}^3 \times [0,\infty))$, obeying this way (4) and (5), remembering that we do not need have $u,p \in S(\mathbb{R}^3 \times [0,\infty))$ as a solution, but only $u,p \in C^\infty(\mathbb{R}^3 \times [0,\infty))$, then it is possible to build a solution to the speed like $u(x,t) = u^0(x)e^{-t} + v(t)$, with $v(0) = 0$, such that $\int_{\mathbb{R}^3} |u(x,t)|^2 dx \to \infty$, because when $\int_{\mathbb{R}^3}[|u^0(x)|^2 e^{-t} + 2u^0(x)\cdot v(t)]dx \geq 0$, for example, when each component of $u^0(x)$ has the same sign of the respective component of $v(t)$ or the product between them is zero or $\int_{\mathbb{R}^3} u^0(x) \cdot v(t) dx \geq 0$, we will have $\int_{\mathbb{R}^3} |u(x,t)|^2 dx \geq \int_{\mathbb{R}^3} |v(t)|^2 dx = |v(t)|^2 \int_{\mathbb{R}^3} dx \to \infty$, with $v(t) \neq 0$, $t > 0$. We must also choose u, u^0 such that $\nabla \cdot u = \nabla \cdot u^0 = 0$.

In particular, we choose, for $1 \leq i \leq 3$,

(13.1) $$u^0(x) = e^{-(x_1^2+x_2^2+x_3^2)}(x_2 x_3, x_1 x_3, -2x_1 x_2),$$
(13.2) $$v_i(t) = w(t) = e^{-t}(1 - e^{-t}),$$
(13.3) $$u_i(x,t) = u_i^0(x) e^{-t} + v_i(t),$$
(13.4) $$f_i(x,t) = \left(-u_i^0 + e^{-t} \sum_{j=1}^3 u_j^0 \frac{\partial u_i^0}{\partial x_j} + \sum_{j=1}^3 v_j \frac{\partial u_i^0}{\partial x_j} - \nu \nabla^2 u_i^0\right) e^{-t},$$

which results to $p(x,t)$, as the only unknown dependent variable yet to be determined,

(14) $$\nabla p + \frac{\partial v}{\partial t} = 0,$$

and then

(15) $$p(x,t) = -\frac{dw}{dt}(x_1 + x_2 + x_3) + \theta(t).$$

The resulting pressure has a general time dependence $\theta(t)$, should be class $C^\infty([0,\infty))$ and we can assume limited, and diverges at infinity ($|x| \to \infty$), but tends to zero at all space with the increased time (unless possibly $\theta(t)$), due to the factor e^{-t} that appears in the derivative of $w(t)$,

(16) $$\frac{dw}{dt} = e^{-t}(2e^{-t} - 1).$$

In this example $\int_{\mathbb{R}^3} u^0(x) \cdot v(t) dx = 0$, and so $\int_{\mathbb{R}^3} |u(x,t)|^2 dx \to \infty$ for $t > 0$, as we wanted. Simpler it would be to choose $u^0(x) = 0$.

Interesting to note that there is no discontinuity in velocity, no singularity (divergence: $|u| \to \infty$), however diverges the total kinetic energy in the whole space, $\int_{\mathbb{R}^3} |u|^2 \, dx \to \infty$, $t > 0$. We had as input data $u^0 \in L^2(\mathbb{R}^3)$, $f \in L^2(\mathbb{R}^3 \times [0,\infty))$, but the solution $u \notin L^2(\mathbb{R}^3 \times [0,\infty))$, as $p \notin L^2(\mathbb{R}^3 \times [0,\infty))$.

§ 4

Our example obey the necessary conditions of divergence-free ($\nabla \cdot u^0 = 0$), smoothness (C^∞) and partial derivatives of u^0 and f of $C_{\alpha K}(1 + |x|)^{-K}$ and $C_{\alpha m K}(1 + |x| + t)^{-K}$ order, respectively. We conclude that we must have $u^0 \in S(\mathbb{R}^3)$ and $f \in S(\mathbb{R}^3 \times [0,\infty))$. To each possible $u(x,t)$ so that (3) is true, the external force $f(x,t)$ and the pressure $p(x,t)$ can be fittingly constructed, in C^∞

class, verifying (8), and in a way to satisfy all the necessary conditions, finding, this way, a possible solution to (1), (2), (3), (4), (5) and (6), and only (7) wouldn't be satisfied, for $t > 0$, according to (10). We then show one example of breakdown solutions to case (C) of this millennium problem. This example, however, won't take to case (A) from [1], of existing and smoothness of solutions, because it violates (7) (case (A) also impose a null external force, $f = 0$).

An overview of the problem's conditions is listed below ($: \mathbb{R}^3$ and $: \mathbb{R}^3 \times [0, \infty)$ representing the respective functions domains).

$\nu > 0, n = 3$		
$\exists u^0(x): \mathbb{R}^3$		smooth (C^∞), divergence-free ($\nabla \cdot u^0 = 0$)
$\exists f(x,t): \mathbb{R}^3 \times [0, \infty)$		smooth (C^∞)
(4)	$\lvert \partial_x^\alpha u^0(x) \rvert \le C_{\alpha K}(1 + \lvert x \rvert)^{-K}: \mathbb{R}^3,\ \forall \alpha, K$	
(5)	$\lvert \partial_x^\alpha \partial_t^m f(x,t) \rvert \le C_{\alpha m K}(1 + \lvert x \rvert + t)^{-K}: \mathbb{R}^3 \times [0, \infty)\ , \forall \alpha, m, K$	
$\nexists (p, u): \mathbb{R}^3 \times [0, \infty)$ /		
(1)	$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i - \frac{\partial p}{\partial x_i} + f_i(x,t), 1 \le i \le 3\ \ (x \in \mathbb{R}^3, t \ge 0)$	
(2)	$\nabla \cdot u = 0$	
(3)	$u(x, 0) = u^0(x) \qquad (x \in \mathbb{R}^3)$	
(6)	$p, u \in C^\infty(\mathbb{R}^3 \times [0, \infty))$	
(7)	$\int_{\mathbb{R}^3} \lvert u(x,t) \rvert^2 dx < C, \forall t \ge 0$	(bounded energy)

It's important that we also analyse the solution's uniqueness question. As $u^0(x)$ and $f(x,t)$ are given of C^∞ class, chosen by us, and satisfying (4) and (5), i.e., belonging to the Schwartz space, with $\nabla \cdot u^0 = 0$, claim that there is no solution (p, u) to the system (1), (2), (3), (6) and (7) might assume that we explored, or proved to, the infinite possible combinations of p and u, i.e., of (p, u). So we need that exists uniqueness of solution for the speed that we build, eliminating other possible speeds for the same data used, $u^0(x)$ and $f(x,t)$, and involving in finite total kinetic energy.

The uniqueness of the solution (except due the pressure $p(x,t)$ with constant additional term or time-dependent) comes from classical results already known, for example described in the mentioned article of Fefferman [1]: the system of Navier-Stokes equations (1), (2), (3) it has (unique [10]) solution for all $t \ge 0$ or only for a finite time interval $[0, T)$ depending on the initial data, where T

is called "*blowup time*". When there is a solution with finite T then the velocity u becomes unbounded near the "*blowup time*".

We see that the existence of our solution in the given example is guaranteed by construction and direct substitution. Our velocity has no irregular behavior, any regularity loss, at no time t, in none position, that becomes unlimited, infinite, even for $t \to \infty$ or $|x| \to \infty$, therefore, there can be no *"blowup time"* in the example we gave, therefore the solution found in the previous case is unique at all times (unless pressure). But even if there were a finite T (in [11], [12] we see that $T > 0$), the uniqueness would exist in at least a small interval of time, which is enough to show that in this time range occurs the breakdown of Navier-Stokes solutions because it was disobeyed limited kinetic energy condition (7), making the case (C) true.

Although only exposed one case possible for infinite energy occurring at $t > 0$, when the velocity u takes the form $u(x,t) = u^0(x)e^{-t} + v(t)$, cases more generally of velocities $v(x,t)$ dependent explicitly the spatial coordinates x_1, x_2, x_3 probably occur also, with $v(x,t)$ a vector, $v(x,0) = 0$, as well as the velocities of the form $u(x,t) = u^0(x)v(x,t)$, with $v(x,t)$ a scalar function, $v(x,0) = 1$, or also other possible velocities $u(x,t)$. A large and important research in Analysis, Mathematical Physics and Applied Mathematics (e.g. [1], [10]-[16]).

Grateful to Professor Ricardo Rosa of the UFRJ University, mathematical expert on the Navier-Stokes equations, who explained to me about the case $\alpha = 0$ and its nature of multi-index.

$\square$

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08-Breakdown of Navier-Stokes Solutions – Unbounded Energy for t > 0

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Abstract – A solution to the 6th millenium problem, respect to breakdown of Navier-Stokes solutions and the bounded energy. We have proved that there are initial velocities $u^0(x)$ and forces $f(x,t)$ such that there is no physically reasonable solution to the Navier-Stokes equations for $t > 0$, which corresponds to the case (C) of the problem relating to Navier-Stokes equations available on the website of the Clay Institute.

Keywords – Navier-Stokes equations, continuity equation, breakdown, existence, smoothness, physically reasonable solutions, gradient field, conservative field, velocity, pressure, external force, bounded energy, millenium problem.

§ 1

A maneira mais simples que vejo para se provar a quebra de soluções (*breakdown solutions*) das equações de Navier-Stokes, seguindo o descrito em [1], refere-se à condição de energia limitada (*bounded energy*), a finitude da integral do quadrado da velocidade do fluido em todo o espaço.

Podemos certamente construir soluções de

(1) $$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i - \frac{\partial p}{\partial x_i} + f_i,\ 1 \le i \le 3,$$

que obedeçam à condição de divergente nulo para a velocidade (equação da continuidade para densidade de massa constante),

(2) $$div\, u \equiv \nabla \cdot u = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0,$$ (fluidos incompressíveis)

e à condição inicial

(3) $$u(x,0) = u^0(x),$$

onde u_i, p, f_i são funções da posição $x \in \mathbb{R}^3$ e do tempo $t \ge 0, t \in \mathbb{R}$. A constante $\nu \ge 0$ é o coeficiente de viscosidade, p representa a pressão e $u = (u_1, u_2, u_3)$ é a velocidade do fluido, medidas na posição x e tempo t, com $\nabla^2 = \nabla \cdot \nabla = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$. A função $f = (f_1, f_2, f_3)$ tem dimensão de aceleração ou força por unidade de massa, mas seguiremos denominando este vetor e suas componentes pelo nome genérico de força, tal como adotado em [1]. É a força externa aplicada ao fluido, por exemplo, gravidade.

As funções $u^0(x)$ e $f(x,t)$ devem obedecer, respectivamente,

(4) $\quad |\partial_x^\alpha u^0(x)| \leq C_{\alpha K}(1+|x|)^{-K}$ sobre $\mathbb{R}^3$, para quaisquer $\alpha \in \mathbb{N}_0^3$ e $K \in \mathbb{R}$,

e

(5) $\quad |\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1+|x|+t)^{-K}$ sobre $\mathbb{R}^3 \times [0,\infty)$, para quaisquer $\alpha \in \mathbb{N}_0^3$, $m \in \mathbb{N}_0$ e $K \in \mathbb{R}$,

com $\mathbb{N}_0 = \{0,1,2,3,\dots\}$ (derivadas de ordem zero não alteram o valor da função), e uma solução (p,u) de (1) para que seja considerada fisicamente razoável deve ser contínua e ter todas as derivadas, de infinitas ordens, também contínuas (*smooth*), i.e.,

(6) $\quad p, u \in C^\infty(\mathbb{R}^3 \times [0,\infty))$.

Dada uma velocidade inicial u^0 de classe C^∞ com divergente nulo (*divergence-free*, $\nabla \cdot u^0 = 0$) sobre $\mathbb{R}^3$ e um campo de forças externo f também de classe C^∞ sobre $\mathbb{R}^3 \times [0,\infty)$, quer-se, para que uma solução seja fisicamente razoável, além da validade de (6), que $u(x,t)$ não divirja para $|x| \to \infty$ e seja satisfeita a condição de energia limitada (*bounded energy*), i.e.,

(7) $\quad \int_{\mathbb{R}^3} |u(x,t)|^2 dx < C$, para todo $t \geq 0$.

Vemos que todas as condições acima, de (1) a (7), precisam ser obedecidas para se obter uma solução (p,u) considerada fisicamente razoável, contudo, para se obter uma quebra de soluções, (1), (2), (3), (6) ou (7) poderiam não ser satisfeitas para algum $t \geq 0$, em alguma posição $x \in \mathbb{R}^3$, mantendo-se ainda a validade de (4) e (5).

Uma maneira de fazer com que esta situação (*breakdown*) ocorra é quando (1) não tem solução possível para a pressão $p(x,t)$, quando o campo vetorial $\phi: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$ em

(8) $$\nabla p = \nu\, \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla)u + f = \phi$$

é não gradiente, não conservativo, em ao menos um $(x,t) \in \mathbb{R}^3 \times [0,\infty)$. Nesse caso, para $\phi = (\phi_1, \phi_2, \phi_3)$ ser não gradiente deve valer

(9) $$\frac{\partial \phi_i}{\partial x_j} \neq \frac{\partial \phi_j}{\partial x_i},\ i \neq j,$$

para algum par $(i,j), 1 \leq i,j \leq 3$, $x \in \mathbb{R}^3$ e tempo t não negativo (para mais detalhes veja, por exemplo, Apostol[2], cap. 10).

Se admitirmos, entretanto, que (1) tem solução (p,u) possível e esta também obedece (2), (3) e (6), a condição inicial $u^0(x)$ verifica (2) e (4), a força externa $f(x,t)$ verifica (5) e $u^0(x)$ e $f(x,t)$ são de classe C^∞, podemos tentar obter a condição de quebra de soluções em $t \geq 0$ violando-se a condição (7) de energia limitada (*bounded energy*), i.e., escolhendo-se $u^0(x)$ ou $u(x,t)$ que também obedeçam a

(10) $\int_{\mathbb{R}^3} |u(x,t)|^2 dx \to \infty$, para algum $t \geq 0$.

A descrição oficial do problema para este caso (C) de quebra de soluções é dada a seguir:

(C) Quebra das soluções da Equação de Navier-Stokes sobre $\mathbb{R}^3$. Para $\nu > 0$ e dimensão espacial $n = 3$ existem um campo vetorial suave e com divergência nula $u^0(x)$ sobre $\mathbb{R}^3$ e uma força externa suave $f(x,t)$ sobre $\mathbb{R}^3 \times [0,\infty)$ satisfazendo

(4) $|\partial_x^\alpha u^0(x)| \leq C_{\alpha K}(1+|x|)^{-K}$ sobre $\mathbb{R}^3$, $\forall \alpha, K$,

e

(5) $|\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1+|x|+t)^{-K}$ sobre $\mathbb{R}^3 \times [0,\infty)$, $\forall \alpha, m, K$,

tais que não existe solução (p,u) sobre $\mathbb{R}^3 \times [0,\infty)$ satisfazendo (1), (2), (3), (6) e (7).

Vê-se claramente que podemos resolver este problema buscando velocidades válidas cuja integral do seu quadrado em todo o espaço $\mathbb{R}^3$ é infinito, ou também, conforme indicamos em (8), buscando funções ϕ não gradientes, onde a pressão p não poderá ser considerada uma função potencial, para algum instante $t \geq 0$. Entendemos que os α, m indicados em (4) e (5) só fazem sentido para $|\alpha|, m \in \{0,1,2,3,4,\dots\}$ e os K negativos podem ser desprezados, pois não limitam o valor das funções u^0, f e suas derivadas quando $|x| \to \infty$ ou $t \to \infty$, com $C_{\alpha K}, C_{\alpha m K} > 0$.

§ 2

A inequação (4) traz implicitamente que $u^0(x)$ deve pertencer ao espaço vetorial das funções de rápido decrescimento, que tendem a zero em $|x| \to \infty$, conhecido como espaço de Schwartz, $S(\mathbb{R}^3)$, em homenagem ao matemático francês Laurent Schwartz (1915-2002) que o estudou [3]. Estas funções e suas infinitas derivadas são contínuas (C^∞) e decaem mais rápido que o inverso de qualquer polinômio, tais que

(11) $\lim_{|x|\to\infty} |x|^k D^\alpha \varphi(x) = 0$

para todo $\alpha = (\alpha_1, \dots, \alpha_n)$, α_i inteiro não negativo, e todo inteiro $k \geq 0$. α é um multi-índice, com a convenção

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}}, |\alpha| = \alpha_1 + \dots + \alpha_n, \alpha_i \in \{0, 1, 2, \dots\}. \tag{12}$$

D^0 é o operador identidade, D^α um operador diferencial. Um exemplo de função deste espaço é $u(x) = P(x)e^{-|x|^2}$, onde $P(x)$ é uma função polinomial.

Valem as seguintes propriedades [4]:

1) $S(\mathbb{R}^n)$ é um espaço vetorial; ele é fechado sobre combinações lineares.

2) $S(\mathbb{R}^n)$ é uma álgebra; o produto de funções em $S(\mathbb{R}^n)$ também pertence a $S(\mathbb{R}^n)$.

3) $S(\mathbb{R}^n)$ é fechado sobre multiplicação por polinômios.

4) $S(\mathbb{R}^n)$ é fechado sobre diferenciação.

5) $S(\mathbb{R}^n)$ é fechado sobre translações e multiplicação por exponenciais complexos $(e^{ix\cdot\xi})$.

6) funções de $S(\mathbb{R}^n)$ são integráveis: $\int_{\mathbb{R}^n} |f(x)|dx < \infty$ para $f \in S(\mathbb{R}^n)$. Isto segue do fato de que $|f(x)| \leq M(1 + |x|)^{-(n+1)}$ e, usando coordenadas polares, $\int_{\mathbb{R}^n} (1 + |x|)^{-(n+1)} dx = C \int_0^\infty (1 + r)^{-n-1}\, r^{n-1} dr < \infty$, i.e., o integrando decresce como r^{-2} (e $(1 + r)^{-2}$) no infinito e produz uma integral finita.

Da definição de $S(\mathbb{R}^3)$ e propriedades anteriores vemos que, como $u^0(x) \in S(\mathbb{R}^3)$, então $\int_{\mathbb{R}^3} |u^0(x)|dx \leq \int_{\mathbb{R}^3} M(1 + |x|)^{-4} dx \leq C \int_0^\infty (1 + r)^{-2}\, dr < \infty$ e quadrando $|u^0(x)|$ e $M(1 + |x|)^{-4}$ chegamos à desigualdade $\int_{\mathbb{R}^3} |u^0(x)|^2 dx < \infty$, que contradiz (10).

Outra forma de verificar isso é que o conjunto $S(\mathbb{R}^n)$ está contido em $L^p(\mathbb{R}^n)$ para todo p, $1 \leq p < \infty$ ([5]-[9]), e em particular para $p = 2$ e $n = 3$ segue a finitude de $\int_{\mathbb{R}^3} |u^0(x)|^2 dx$.

Portanto, se a condição (7) for desobedecida, conforme propomos neste artigo, será para $t > 0$, por exemplo, encontrando alguma função $u(x, t)$ da forma $u^0(x)v(x, t)$, $v(x, 0) = 1$, ou $u^0(x) + v(x, t)$, $v(x, 0) = 0$, com $\int_{\mathbb{R}^3} |v(x, t)|^2 dx \to \infty$ e $\int_{\mathbb{R}^3} |u(x, t)|^2 dx \to \infty$.

§ 3

De fato, escolhendo $u^0(x) \in S(\mathbb{R}^3)$ e $f(x, t) \in S(\mathbb{R}^3 \times [0, \infty))$, obedecendo-se assim (4) e (5), lembrando-se que não precisamos ter $u, p \in S(\mathbb{R}^3 \times [0, \infty))$ como solução, apenas $u, p \in C^\infty(\mathbb{R}^3 \times [0, \infty))$, então é possível construir uma solução para a velocidade da forma $u(x,t) = u^0(x)e^{-t} + v(t)$, com $v(0) = 0$, tal que $\int_{\mathbb{R}^3} |u(x,t)|^2 dx \to \infty$, pois, quando $\int_{\mathbb{R}^3}[|u^0(x)|^2 e^{-t} + 2u^0(x) \cdot v(t)]dx \geq 0$, por exemplo, quando cada componente de $u^0(x)$ tem o mesmo sinal da respectiva componente de $v(t)$ ou o produto entre elas é zero ou $\int_{\mathbb{R}^3} u^0(x) \cdot v(t) dx \geq 0$, teremos $\int_{\mathbb{R}^3} |u(x,t)|^2 dx \geq \int_{\mathbb{R}^3} |v(t)|^2 dx = |v(t)|^2 \int_{\mathbb{R}^3} dx \to \infty$, com $v(t) \neq 0, t > 0$. Também devemos escolher u, u^0 tais que $\nabla \cdot u = \nabla \cdot u^0 = 0$.

Em especial, escolhamos, para $1 \leq i \leq 3$,

(13.1) $$u^0(x) = e^{-(x_1^2+x_2^2+x_3^2)}(x_2 x_3, x_1 x_3, -2x_1 x_2),$$
(13.2) $$v_i(t) = w(t) = e^{-t}(1 - e^{-t}),$$
(13.3) $$u_i(x,t) = u_i^0(x)e^{-t} + v_i(t),$$
(13.4) $$f_i(x,t) = \left(-u_i^0 + e^{-t} \sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j} + \sum_{j=1}^{3} v_j \frac{\partial u_i^0}{\partial x_j} - \nu \nabla^2 u_i^0\right) e^{-t},$$

o que resulta para $p(x,t)$, como a única incógnita ainda a determinar,

(14) $$\nabla p + \frac{\partial v}{\partial t} = 0,$$

e então

(15) $$p(x,t) = -\frac{dw}{dt}(x_1 + x_2 + x_3) + \theta(t).$$

A pressão obtida tem uma dependência temporal genérica $\theta(t)$, que deve ser de classe $C^\infty([0,\infty))$ e podemos supor limitada, e diverge no infinito ($|x| \to \infty$), mas tenderá a zero em todo o espaço com o aumento do tempo (a menos eventualmente de $\theta(t)$), devido ao fator e^{-t} que aparece na derivada de $w(t)$,

(16) $$\frac{dw}{dt} = e^{-t}(2e^{-t} - 1).$$

Neste exemplo $\int_{\mathbb{R}^3} u^0(x) \cdot v(t) dx = 0$, e assim $\int_{\mathbb{R}^3} |u(x,t)|^2 dx \to \infty$ para $t > 0$, como queríamos. Mais simples ainda seria escolher $u^0(x) = 0$.

Interessante observarmos que não ocorre nenhuma descontinuidade na velocidade, nem singularidade (divergência: $|u| \to \infty$), entretanto a energia cinética total em todo o espaço diverge, $\int_{\mathbb{R}^3} |u|^2\, dx \to \infty$. Tivemos como dados de entrada $u^0 \in L^2(\mathbb{R}^3)$, $f \in L^2(\mathbb{R}^3 \times [0,\infty))$, mas por solução $u \notin L^2(\mathbb{R}^3 \times [0,\infty))$, assim como $p \notin L^2(\mathbb{R}^3 \times [0,\infty))$, para $t > 0$.

§ 4

Nosso exemplo obedece às condições de divergência nula (*divergence-free*, $\nabla \cdot u^0 = 0$), suavidade (*smoothness*, C^∞) e derivadas parciais de u^0 e f da ordem de $C_{\alpha K}(1+|x|)^{-K}$ e $C_{\alpha m K}(1+|x|+t)^{-K}$, respectivamente. Concluímos que deve ser $u^0 \in S(\mathbb{R}^3)$ e $f \in S(\mathbb{R}^3 \times [0,\infty))$. Para cada $u(x,t)$ possível tal que (3) seja verdadeira, a força externa $f(x,t)$ e a pressão $p(x,t)$ podem ser convenientemente construídas, na classe C^∞, verificando (8), e de modo a satisfazerem todas as condições necessárias, encontrando-se assim uma solução possível para (1), (2), (3), (4), (5) e (6), e apenas (7) não seria satisfeita, para $t > 0$, conforme (10). Mostramos então um exemplo de quebra de soluções para o caso (C) deste problema do milênio. Este exemplo, entretanto, não leva ao caso (A) de [1], de existência e suavidade das soluções, justamente por violar (7) (O caso (A) também impõe que seja nula a força externa, $f = 0$).

Um resumo das condições do problema está listado abaixo.

$\nu > 0, n = 3$		
$\exists u^0(x): \mathbb{R}^3$		smooth (C^∞), divergence-free ($\nabla \cdot u^0 = 0$)
$\exists f(x,t): \mathbb{R}^3 \times [0,\infty)$		smooth (C^∞)
(4)	$\lvert\partial_x^\alpha u^0(x)\rvert \le C_{\alpha K}(1+\lvert x\rvert)^{-K}: \mathbb{R}^3,\ \forall \alpha, K$	
(5)	$\lvert\partial_x^\alpha \partial_t^m f(x,t)\rvert \le C_{\alpha m K}(1+\lvert x\rvert+t)^{-K}: \mathbb{R}^3 \times [0,\infty)\ , \forall \alpha, m, K$	
$\nexists (p,u): \mathbb{R}^3 \times [0,\infty)\ /$		
(1)	$\frac{\partial u_i}{\partial t} + \sum_{j=1}^3 u_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i - \frac{\partial p}{\partial x_i} + f_i(x,t), 1 \le i \le 3 \ \ (x \in \mathbb{R}^3, t \ge 0)$	
(2)	$\nabla \cdot u = 0$	
(3)	$u(x,0) = u^0(x) \qquad (x \in \mathbb{R}^3)$	
(6)	$p, u \in C^\infty(\mathbb{R}^3 \times [0,\infty))$	
(7)	$\int_{\mathbb{R}^3} \lvert u(x,t)\rvert^2 dx < C, \forall t \ge 0$	(bounded energy)

É importante analisarmos também a questão da unicidade das soluções. Como $u^0(x)$ e $f(x,t)$ são dados, escolhidos por nós, de classe C^∞ e satisfazendo (4) e (5), i.e., pertencentes ao espaço de Schwartz, com $\nabla \cdot u^0 = 0$, afirmar que não existe solução (p,u) para o sistema (1), (2), (3), (6) e (7) pode pressupor que exploramos, ou provamos para, as infinitas combinações possíveis de p e de u, i.e., de (p,u). Sendo assim, precisamos que haja unicidade de solução para cada velocidade que construímos, o que elimina outras velocidades possíveis para os mesmos dados utilizados, $u^0(x)$ e $f(x,t)$, e que implicassem em energia cinética total finita.

A unicidade da solução (a menos da pressão $p(x,t)$ com o termo adicional constante ou dependente do tempo) vem dos resultados clássicos já conhecidos, descritos por exemplo no mencionado artigo de Fefferman [1]: o sistema das equações de Navier-Stokes (1), (2), (3) tem solução (única [10]) para todo $t \geq 0$ ou apenas para um intervalo de tempo $[0,T)$ finito dependente dos dados iniciais, onde T é chamado de "*blowup time*". Quando há uma solução com T finito então a velocidade u torna-se ilimitada próxima do "*blowup time*".

Vemos que a existência de nossa solução, no exemplo dado, está garantida por construção e substituição direta. Nossa velocidade não apresenta nenhum comportamento irregular, em instante t algum, em posição alguma, que a torne ilimitada, infinita, nem mesmo para $t \to \infty$ ou $|x| \to \infty$, sendo assim, não pode haver o "*blowup time*" no exemplo que demos, portanto a solução encontrada no caso anterior é única em todo tempo (a menos da pressão). Mas ainda que houvesse um T finito (em [11], [12] vemos que $T > 0$), a unicidade existiria em pelo menos um pequeno intervalo de tempo, o que já é suficiente para mostrar que neste intervalo ocorre a quebra das soluções de Navier-Stokes por ser desobedecida a condição de energia cinética limitada (7), tornando o caso (C) verdadeiro.

Embora só tenhamos exposto um caso possível para ocorrência de energia infinita em $t > 0$, quando a velocidade u toma a forma $u(x,t) = u^0(x)e^{-t} + v(t)$, casos mais gerais de velocidades $v(x,t)$ dependentes explicitamente das coordenadas espaciais x_1, x_2, x_3 provavelmente ocorram também, $v(x,t)$ um vetor com $v(x,0) = 0$, assim como para as velocidades da forma $u(x,t) = u^0(x)v(x,t)$, sendo $v(x,t)$ uma função escalar com $v(x,0) = 1$, ou ainda outras possíveis velocidades $u(x,t)$. Uma vasta e importante pesquisa em Análise, Física-Matemática e Matemática Aplicada (por exemplo, [1], [10]-[16]).

Grato ao professor Ricardo Rosa da UFRJ, matemático especialista nas equações de Navier-Stokes, que me explicou sobre o caso $\alpha = 0$ e sua natureza de multi-índice.

$\square$

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09 – Three Examples of Unbounded Energy for $t > 0$

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We have different gifts, according to the grace given to each of us.
If your gift is prophesying, then prophesy in accordance with your faith;
if it is serving, then serve;
if it is teaching, then teach;
if it is to encourage, then give encouragement;
if it is giving, then give generously;
if it is to lead, do it diligently;
if it is to show mercy, do it cheerfully.
Love must be sincere. Hate what is evil; cling to what is good.
(Romans 12, 6-9)

Abstract – A solution to the 6th millenium problem, respect to breakdown of Navier-Stokes solutions and the bounded energy. We have proved that there are initial velocities $u^0(x)$ and forces $f(x,t)$ such that there is no physically reasonable solution to the Navier-Stokes equations for $t > 0$, which corresponds to the case (C) of the problem relating to Navier-Stokes equations available on the website of the Clay Institute. Three examples are given.

Keywords – Navier-Stokes equations, continuity equation, breakdown, existence, smoothness, physically reasonable solutions, gradient field, conservative field, velocity, pressure, external force, unbounded energy, millennium problem, uniqueness, non uniqueness, 15th Problem of Smale, blowup time.

A great effort and expectation were used in this article, relying on the uniqueness of solutions for the velocity in at least a small time interval, which I later found not to be true. Yet it still contains several good results, despite the multiplicity of solutions.

The uniqueness mentioned by Leray, Ladyzhenskaya, Temam, etc. refers to the inclusion of boundary conditions, for example, velocity equal to zero at infinity. February-03-2017.

Um grande esforço e expectativa foram usados neste artigo, baseando-se na unicidade de soluções para a velocidade ao menos em um pequeno intervalo de tempo, que posteriormente verifiquei não ser verdade. Mesmo assim ele ainda contém vários bons resultados, a despeito da multiplicidade de soluções.

A unicidade mencionada por Leray, Ladyzhenskaya, Temam, etc. refere-se à inclusão de condições de contorno, por exemplo, velocidade igual a zero no infinito. 03/02/2017.

§ 1 – Introduction

The second way I see to prove the breakdown solutions of Navier-Stokes equations, following the described in [1], refers to the condition of bounded energy, the finiteness of the integral of the squared velocity of the fluid in the whole space.

We can certainly construct solutions for

(1) $$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i - \frac{\partial p}{\partial x_i} + f_i,\ 1 \leq i \leq 3,$$

that obey the condition of divergence-free to the velocity (continuity equation to the constant mass density),

(2) $$div\, u \equiv \nabla \cdot u = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0,$$ (incompressible fluids)

and the initial condition

(3) $$u(x,0) = u^0(x),$$

where u_i, p, f_i are functions of the position $x \in \mathbb{R}^3$ and the time $t \geq 0, t \in \mathbb{R}$. The constant $\nu \geq 0$ is the viscosity coefficient, p represents the pressure and $u = (u_1, u_2, u_3)$ is the fluid velocity, measured in the position x and time t, with $\nabla^2 = \nabla \cdot \nabla = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$. The function $f = (f_1, f_2, f_3)$ has the dimension as acceleration or force per mass unit, but we will keep on naming this vector and its components by its generic name of force, such as used in [1]. It's the externally applied force to the fluid, for example, gravity.

The functions $u^0(x)$ and $f(x,t)$ must obey, respectively,

(4) $$|\partial_x^\alpha u^0(x)| \leq C_{\alpha K}(1 + |x|)^{-K}$$ on $\mathbb{R}^3$, for any $\alpha \in \mathbb{N}_0^3$ and $K \in \mathbb{R}$,

and

(5) $$|\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1 + |x| + t)^{-K}$$ on $\mathbb{R}^3 \times [0,\infty)$, for any $\alpha \in \mathbb{N}_0^3$, $m \in \mathbb{N}_0$ and $K \in \mathbb{R}$,

with $\mathbb{N}_0 = \{0,1,2,3,\dots\}$ (derivatives of order zero does not change the value of function), and a solution (p, u) from (1) to be considered physically reasonable must be continuous and have all the derivatives, of infinite orders, also continuous (smooth), i.e.,

(6) $$p, u \in C^\infty(\mathbb{R}^3 \times [0,\infty)).$$

Given an initial velocity u^0 of C^∞ class, divergence-free ($\nabla \cdot u^0 = 0$) on $\mathbb{R}^3$ and an external forces field f also C^∞ class on $\mathbb{R}^3 \times [0,\infty)$, we want, for that a solution to be physically reasonable, beyond the validity of (6), that $u(x,t)$ does not diverge to $|x| \to \infty$ and satisfy the bounded energy condition, i.e.,

(7) $$\int_{\mathbb{R}^3} |u(x,t)|^2 dx < C, \text{ for all } t \geq 0.$$

We see that every condition above, from (1) to (7), need to be obeyed to get a solution (p,u) considered physically reasonable, however, to get the breakdown solutions, (1), (2), (3), (6) or (7) could not be satisfied to some $t \geq 0$, in some position $x \in \mathbb{R}^3$, still maintaining (4) and (5) validity.

A way to make this situation (breakdown) happens is when (1) have no possible solution to the pressure $p(x,t)$, when the vector field $\phi: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$ in

(8) $$\nabla p = \nu \, \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla)u + f = \phi$$

is not gradient, not conservative, in at least one $(x,t) \in \mathbb{R}^3 \times [0,\infty)$. In this case, to $\phi = (\phi_1, \phi_2, \phi_3)$ not to be gradient, it must be

(9) $$\frac{\partial \phi_i}{\partial x_j} \neq \frac{\partial \phi_j}{\partial x_i}, \; i \neq j,$$

to some pair $(i,j), 1 \leq i,j \leq 3$, $x \in \mathbb{R}^3$ and time t not negative (for details check, for example, Apostol[2], chapter 10).

If we admit, however, that (1) has a possible solution (p,u) and this also obey (2), (3) and (6), the initial condition $u^0(x)$ verifies (2) and (4), the external force $f(x,t)$ verifies (5) and both $u^0(x)$ and $f(x,t)$ are C^∞ class, we can try get a breakdown solutions in $t \geq 0$ violating the condition (7) (bounded energy), i.e., choosing $u^0(x)$ or $u(x,t)$ that also obey to

(10) $$\int_{\mathbb{R}^3} |u(x,t)|^2 dx \to \infty, \text{ for some } t \geq 0.$$

The official description of the problem to this (C) case of breakdown solutions is given below:

(C) Breakdown solutions of Navier-Stokes on $\mathbb{R}^3$. Take $\nu > 0$ and $n = 3$. Then there exist a smooth and divergence-free vector field $u^0(x)$ on $\mathbb{R}^3$ and a smooth external force $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ satisfying

(4) $$|\partial_x^\alpha u^0(x)| \leq C_{\alpha K}(1 + |x|)^{-K} \text{ on } \mathbb{R}^3, \forall \alpha, K,$$

and

(5) $$|\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1 + |x| + t)^{-K} \text{ on } \mathbb{R}^3 \times [0,\infty), \forall \alpha, m, K,$$

for which there exist no solutions (p, u) of (1), (2), (3), (6), (7) on $\mathbb{R}^3 \times [0, \infty)$.

It's clear to see that we can solve this problem searching valid velocities which the integral of its square in all space $\mathbb{R}^3$ is infinite, or also, as shown in (8), searching functions ϕ non gradients, where the pressure p won't be considered a potential function to some instant $t \geq 0$. We understand that the α, m shown in (4) and (5) just make sense to $|\alpha|, m \in \{0, 1, 2, 3, 4, \dots\}$ and the negatives K can be neglected because it does not limit the value of functions u^0, f and its derivatives when $|x| \to \infty$ or $t \to \infty$, with $C_{\alpha K}, C_{\alpha m K} > 0$.

§ 2 – The Schwartz Space S

The inequation (4) brings implicitly that $u^0(x)$ must belong to the vectorial space of rapidly decreasing functions, which tend to zero for $|x| \to \infty$, known as Schwartz space, $S(\mathbb{R}^3)$, named after the French mathematician Laurent Schwartz (1915-2002) which studied it [3]. These functions and its derivatives of all orders are continuous (C^∞) and decrease faster than the inverse of any polynomial, such that

(11) $$\lim_{|x| \to \infty} |x|^k D^\alpha \varphi(x) = 0$$

for all $\alpha = (\alpha_1, \dots, \alpha_n)$, α_i non negative integer, and all integer $k \geq 0$. α is a multi-index, with the convention

(12) $$D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}}, |\alpha| = \alpha_1 + \dots + \alpha_n, \alpha_i \in \{0, 1, 2, \dots\}.$$

D^0 is the operator identity, D^α is a differential operator. An example of function of this space is $u(x) = P(x)e^{-|x|^2}$, where $P(x)$ is a polynomial function.

The following properties are valid [4]:

1) $S(\mathbb{R}^n)$ is a vector space; it is closed under linear combinations.

2) $S(\mathbb{R}^n)$ is an algebra; the product of functions in $S(\mathbb{R}^n)$ also belongs to $S(\mathbb{R}^n)$ (this follows from Leibniz' formula for derivatives of products).

3) $S(\mathbb{R}^n)$ is closed under multiplication by polinomials, althrough polynomials are not in S.

4) $S(\mathbb{R}^n)$ is closed under differentiation.

5) $S(\mathbb{R}^n)$ is closed under translations and multiplication by complex exponentials ($e^{ix \cdot \xi}$).

6) $S(\mathbb{R}^n)$ functions are integrable: $\int_{\mathbb{R}^n}|f(x)|dx < \infty$ for $f \in S(\mathbb{R}^n)$. This follows from the fact that $|f(x)| \leq M(1+|x|)^{-(n+1)}$ and, using polar coordinates, $\int_{\mathbb{R}^n}(1+|x|)^{-(n+1)}dx = C\int_0^\infty(1+r)^{-n-1}\, r^{n-1}dr < \infty$, i.e., the function $|f|$ decreases like r^{-2} (and $(1+r)^{-2}$) at infinity and a finite integral is produced.

By $S(\mathbb{R}^3)$ definition and previous properties we see that, as $u^0(x) \in S(\mathbb{R}^3)$, then $\int_{\mathbb{R}^3}|u^0(x)|dx \leq \int_{\mathbb{R}^3} M(1+|x|)^{-4}dx \leq C\int_0^\infty(1+r)^{-2}\,dr < \infty$ and squared $|u^0(x)|$ and $M(1+|x|)^{-4}$ we come to the inequality $\int_{\mathbb{R}^3}|u^0(x)|^2dx < \infty$, that contradicts (10).

Another way to check this is that the set $S(\mathbb{R}^n)$ it is contained in $L^p(\mathbb{R}^n)$ for all p, $1 \leq p < \infty$ ([5]-[9]), and in particular for $p=2$ and $n=3$ follows the finiteness of $\int_{\mathbb{R}^3}|u^0(x)|^2dx$.

Therefore, if the condition (7) is disobeyed, as we propose in this article, will be for $t>0$, for example, finding some function $u(x,t)$ like $u^0(x)v(x,t)$, $v(x,0)=1$, or $u^0(x)+v(x,t)$, $v(x,0)=0$, with $\int_{\mathbb{R}^3}|v(x,t)|^2dx \to \infty$ and $\int_{\mathbb{R}^3}|u(x,t)|^2dx \to \infty$.

§ 3 – Example 1

Really, choosing $u^0(x) \in S(\mathbb{R}^3)$ and $f(x,t) \in S(\mathbb{R}^3 \times [0,\infty))$, obeying this way (4) and (5), remembering that we do not need have $u,p \in S(\mathbb{R}^3 \times [0,\infty))$ as a solution, but only $u,p \in C^\infty(\mathbb{R}^3 \times [0,\infty))$, then it is possible to build a solution to the velocity like $u(x,t) = u^0(x)e^{-t} + v(t)$, with $v(0)=0$, such that $\int_{\mathbb{R}^3}|u(x,t)|^2dx \to \infty$, because when $\int_{\mathbb{R}^3}[|u^0(x)|^2e^{-t} + 2u^0(x)\cdot v(t)]dx \geq 0$, for example, when each component of $u^0(x)$ has the same sign of the respective component of $v(t)$ or the product between them is zero or $\int_{\mathbb{R}^3}u^0(x)\cdot v(t)dx \geq 0$, we will have $\int_{\mathbb{R}^3}|u(x,t)|^2dx \geq \int_{\mathbb{R}^3}|v(t)|^2dx = |v(t)|^2\int_{\mathbb{R}^3}dx \to \infty$, with $v(t) \neq 0$, $t>0$. We must also choose u, u^0 such that $\nabla\cdot u = \nabla\cdot u^0 = 0$.

In particular, we choose, for $1 \leq i \leq 3$,

(13.1) $$u^0(x) = e^{-(x_1^2+x_2^2+x_3^2)}(x_2x_3, x_1x_3, -2x_1x_2),$$

(13.2) $$v_i(t) = w(t) = e^{-t}(1-e^{-t}),$$

(13.3) $$u_i(x,t) = u_i^0(x)e^{-t} + v_i(t),$$

(13.4) $$f_i(x,t) = \left(-u_i^0 + e^{-t}\sum_{j=1}^3 u_j^0\frac{\partial u_i^0}{\partial x_j} + \sum_{j=1}^3 v_j\frac{\partial u_i^0}{\partial x_j} - \nu\nabla^2 u_i^0\right)e^{-t},$$

which results to $p(x,t)$, as the only unknown dependent variable yet to be determined,

(14) $$\nabla p + \frac{\partial v}{\partial t} = 0,$$

and then

(15) $$p(x,t) = -\frac{dw}{dt}(x_1 + x_2 + x_3) + \theta(t).$$

The resulting pressure has a general time dependence $\theta(t)$, should be class $C^\infty([0,\infty))$ and we can assume limited, and diverges at infinity ($|x| \to \infty$), but tends to zero at all space with the increased time (unless possibly $\theta(t)$), due to the factor e^{-t} that appears in the derivative of $w(t)$,

(16) $$\frac{dw}{dt} = e^{-t}(2e^{-t} - 1).$$

In this example $\int_{\mathbb{R}^3} u^0(x) \cdot v(t) dx = 0$, and so $\int_{\mathbb{R}^3} |u(x,t)|^2 dx \to \infty$ for $t > 0$, as we wanted. Simpler it would be to choose $u^0(x) = 0$.

Interesting to note that there is no discontinuity in velocity, no singularity (divergence: $|u| \to \infty$), however diverges the total kinetic energy in the whole space, $\int_{\mathbb{R}^3} |u|^2\, dx \to \infty,\ t > 0$. We had as input data $u^0 \in L^2(\mathbb{R}^3)$, $f \in L^2\big(\mathbb{R}^3 \times [0,\infty)\big)$, but the solution $u \notin L^2\big(\mathbb{R}^3 \times [0,\infty)\big)$, as $p \notin L^2\big(\mathbb{R}^3 \times [0,\infty)\big)$.

§ 4 – Example 2 – General Idea

Another interesting example, using the same previous initial velocity, but making v explicitly depend on the position coordinates x_1, x_2 in the direction e_1, e_2, besides time t, and be equal to zero in the direction e_3, with $v(x,0) = 0$, $\nabla \cdot v = 0$, $v \not\equiv 0$ (v not identically zero), and also obeys all the conditions (1) to (6), is, for $1 \le i \le 3$,

(17.1) $$u^0(x) = e^{-(x_1^2+x_2^2+x_3^2)}(x_2x_3, x_1x_3, -2x_1x_2),$$

(17.2) $$v(x,t) = e^{-t} w(x,t),$$

(17.3) $$w(x,t) = (w_1(x_1,x_2,t), w_2(x_1,x_2,t), 0),$$

$$w(x,0) = 0,\ \nabla \cdot w = 0,\ w_3 = v_3 = 0,\ w \not\equiv 0,$$

(17.4) $$u_i(x,t) = u_i^0(x)e^{-t} + v_i(x,t) = [u_i^0(x) + w_i(x,t)]e^{-t},$$

(17.5) $$f_i(x,t) = \left(-u_i^0 + e^{-t} \textstyle\sum_{j=1}^3 [u_j^0 \frac{\partial u_i^0}{\partial x_j} + u_j^0 \frac{\partial w_i}{\partial x_j} + w_j \frac{\partial u_i^0}{\partial x_j}] - \nu \nabla^2 u_i^0\right) e^{-t}$$

$$= \left(-u_i^0 + \textstyle\sum_{j=1}^3 [e^{-t} u_j^0 \frac{\partial u_i^0}{\partial x_j} + u_j^0 \frac{\partial v_i}{\partial x_j} + v_j \frac{\partial u_i^0}{\partial x_j}] - \nu \nabla^2 u_i^0\right) e^{-t},$$

which results for $p(x,t)$ and $v(x,t)$, as unknowns still to be determined,

(18) $$\frac{\partial p}{\partial x_i} + \frac{\partial v_i}{\partial t} + \sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial x_j} = \nu \nabla^2 v_i,$$

the Navier-Stokes equations without external force.

We know that for $n = 2$ the equation (18) has a solution whose existence and uniqueness is already demonstrated ([10]-[13]), therefore, we will transform our three-dimensional system (18) in a two-dimensional system in v, which offers as solution a pressure p and a velocity v, *a priori*, with spatially two-dimensional domain, i.e., in the variables (x_1, x_2, t). Resolved, by hypothesis, the equation (18) above, with $v(x,0) = 0$, $\nabla \cdot v = 0$, but v not identically zero, we add the third spatial coordinate $v_3 \equiv 0$ in the definitive solution for $u(x,t)$, spatially three-dimensional, in (17.4), and calculate the external force in (17.5). Choosing $v \in S(\mathbb{R}^2 \times [0,\infty))$ or v polinomial, sine, cosine or their sums to be used in (18), we guarantee that $f \in S(\mathbb{R}^3 \times [0,\infty))$, obeying up (5), with $u^0 \in S(\mathbb{R}^3)$, according (4). Making v limited in module (norm in Euclidean space) we make that u not diverge at $|x| \to \infty$, which is a physically reasonable and desirable condition in [1]. Then build a velocity v not identically zero, with $v(x,0) = 0$, $\nabla \cdot v = 0$, such that it is relatively simple to solve (18), which is limited in module, can (preferably) go to zero at infinity in at least some situations and can be integrated in $\mathbb{R}^2$, it is C^∞ class and satisfies (5).

Equation (18) also admit a general time dependence to the pressure that is of the form

(19) $$p(x,t) = p_1(x_1, x_2, t) + \theta(t),\ x \in \mathbb{R}^3,$$

i.e., besides the conventional solution p_1 to the pressure of the two-dimensional problem of the Navier-Stokes equations (18) in the independent variables (x_1, x_2, t), add to p a generic parcel $\theta(t)$ only dependent on the time and/or a constant as the definitive pressure solution in the original three-dimensional problem, as we have seen in (15).

The infinitude of the total kinetic energy in this second example, occurs due to the integration of a two-dimensional function ($|v|^2$ or $|w|^2$) not identically zero in the infinite three dimensional space ($\mathbb{R}^3$).

The total kinetic energy of the problem is, for $v = e^{-t} w$,

(20) $$\int_{\mathbb{R}^3} |u|^2 dx = \int_{\mathbb{R}^3} (e^{-2t}|u^0|^2 + 2e^{-t} u^0 \cdot v + |v|^2) dx$$
$$= e^{-2t} \int_{\mathbb{R}^3} (|u^0|^2 + 2u^0 \cdot w + |w|^2) dx.$$

Although $\int_{\mathbb{R}^3}(|u^0|^2 + 2u^0 \cdot w)dx$ is finite, by the properties of functions belonging to the Schwartz space and integrable (the case $u^0 = 0$ is elementary), the third parcel in (20) will be infinite in $\mathbb{R}^3$ for $v, w \not\equiv 0$, though the function can converge and be finite in $\mathbb{R}^2$, that is, if $|v|$ is not identically zero and $t > 0$,

(21) $$\int_{\mathbb{R}^3} |v|^2 dx = \int_{-\infty}^{+\infty} \left(\int_{\mathbb{R}^2} |v|^2 dx\right) dx_3 = C_2 \int_{-\infty}^{+\infty} dx_3 \to \infty,$$

hence, for strictly positive and finite t,

(22) $$\int_{\mathbb{R}^3} |u|^2 dx \to \infty, t > 0, v \not\equiv 0,$$

the violation of condition (7).

§ 5 – Example 2 – Exact Solution

Let us now solve (18) explicitly, first in the domain $\mathbb{R}^2 \times [0, \infty)$. In the example 3 your domain will be $\mathbb{R}^3 \times [0, \infty)$. We show that a solution of the type

(23) $$v(x_1, x_2, t) = (X(x_1 - x_2)T(t),\ X(x_1 - x_2)T(t)),$$

with a given pressure such that

(24) $$\frac{\partial p}{\partial x_1} = -\frac{\partial p}{\partial x_2} = aQ(x_1 - x_2)R(t) + b,$$

a, b constants, $a \neq 0, Q$ a function of the difference of the spatial coordinates, R a function of time, Q, R not identically zero functions, solve (18) and eliminate the non-linear term, in which case if $T(0) = 0$ resolves (17) and the original system (1), (2), (3). X and T not identically zero, of course.

If $v_i = v_j = V$ in (18), we have to the nonlinear terms

(25) $$\sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial x_j} = \sum_{j=1}^{3} V \frac{\partial V}{\partial x_j} = V \sum_{j=1}^{3} \frac{\partial V}{\partial x_j}.$$

Doing $\sum_{j=1}^{3} \frac{\partial V}{\partial x_j} = 0$ in (25) eliminates the nonlinear term, equality which is true when the necessary condition incompressible fluid imposed by us, $\nabla \cdot v = 0$, is satisfied, i.e.,

(26) $$\sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} = \sum_{j=1}^{3} \frac{\partial V}{\partial x_j} = 0.$$

Defining $V(x, t) = X(\xi(x))T(t)$, with $x \in \mathbb{R}^n$, then

(27)

$$\sum_{j=1}^{n}\frac{\partial V}{\partial x_j}=T(t)\sum_{j=1}^{n}\frac{\partial X(\xi(x))}{\partial x_j}=T(t)\sum_{j=1}^{n}X'(\xi)\frac{\partial \xi(x)}{\partial x_j}=T(t)X'(\xi)\sum_{j=1}^{n}\frac{\partial \xi(x)}{\partial x_j}.$$

Functions $\xi(x)$ such that $\sum_{j=1}^{n}\frac{\partial \xi(x)}{\partial x_j}=0$ then result in $\sum_{j=1}^{n}\frac{\partial V}{\partial x_j}=0$, according (27), following the example of $\xi=x_1-x_2$ in spatial dimension $n=2$, as it was used in (23).

Substituting (24) in (18), already no nonlinear terms $\sum_{j=1}^{3}v_j\frac{\partial v_i}{\partial x_j}$, and for simplicity making $a=1, b=0$, comes

(28) $$Q(x_1-x_2)R(t)+\frac{\partial V}{\partial t}=\nu\nabla^2 V,$$

with $V=X(x_1-x_2)T(t)$. We thus transform a system of n partial differential equations nonlinear in a single linear partial differential equation.

Defining $\xi=x_1-x_2$, equation (28) becomes

(29) $$Q(\xi)R(t)+X(\xi)\frac{dT}{dt}=\nu T\nabla^2 X(\xi).$$

We want to get a function $T(t)$ such that $T(0)=0$, in order that in $t=0$ we have $v(x,0)=0$, according (23). Let us choose, for example, among other possibilities endless,

(30) $$T(t)=(1-e^{-t})e^{-t},$$

limited function in range $0\le T(t)\le 1,\ t\ge 0$, going to zero for $t\to\infty$.

Thus, by (29), with

(31) $$\frac{dT}{dt}=e^{-t}(2e^{-t}-1),$$

comes

(32) $$Q(\xi)R(t)+X(\xi)e^{-t}(2e^{-t}-1)=\nu(1-e^{-t})e^{-t}\nabla^2 X(\xi).$$

Defining $Q(\xi)=X(\xi)$ in (32), to separate our equation with the traditional method of separation of variables used in D.P.E. theory,

(33) $$[R(t)+e^{-t}(2e^{-t}-1)]X(\xi)=\nu(1-e^{-t})e^{-t}\nabla^2 X(\xi).$$

The linear partial differential equation (33) may be resolved by some alternative combinations:

(34) $$\begin{cases} R(t)+e^{-t}(2e^{-t}-1)=\pm\nu(1-e^{-t})e^{-t} \\ X(\xi)=\pm\nabla^2 X(\xi) \end{cases}$$

or

(35) $$\begin{cases} R(t) + e^{-t}(2e^{-t} - 1) = \pm(1 - e^{-t})e^{-t} \\ X(\xi) = \pm\nu\nabla^2 X(\xi) \end{cases}$$

or more generally, with $\nu_1 \cdot \nu_2 = \nu > 0,\ \nu_1, \nu_2 > 0$,

(36) $$\begin{cases} R(t) + e^{-t}(2e^{-t} - 1) = \pm\nu_1(1 - e^{-t})e^{-t} \\ X(\xi) = \pm\nu_2\nabla^2 X(\xi) \end{cases}$$

The differential equation of second order in X, depending on which of the signals we use to $\pm$, leads us to the Helmholtz equation (negative sign) or a moving steady state governed by Schrödinger equation independent of time (positive signal or negative).

Not intending to use any specific boundary condition for $X(\xi)$ and we do make use of series and Fourier integrals, we choose here the negative sign in $\pm$ (the option should be the same in both equations system), and let us make X be a trigonometric function, sum of sine and cosine in ξ, i.e.,

(37) $$X(\xi) = A\,cos(B\xi) + C\,sin(D\xi).$$

With $\xi = x_1 - x_2$ we have

(38) $$\begin{aligned} \nabla^2 X &= \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)[A\,cos(B\xi) + C\,sin(D\xi)] \\ &= \frac{\partial^2}{\partial x_1^2}[A\,cos(B\xi) + C\,sin(D\xi)] + \frac{\partial^2}{\partial x_2^2}[A\,cos(B\xi) + C\,sin(D\xi)] \\ &= -2[AB^2\,cos(B\xi) + CD^2\,sin(D\xi)]. \end{aligned}$$

From $X(\xi) = -\nu_2\nabla^2 X(\xi)$ in (36) comes

(39) $$\nu_2 = \frac{1}{2B^2} = \frac{1}{2D^2},\ \nu_1 = 2B^2\nu = 2D^2\nu,\ |B| = |D|,$$

whatever the values of A and C (if $A = C = 0$ or $B = D = 0$ we have the trivial and unwanted solution $v(x,t) \equiv 0$).

The solution for $R(t)$ obtained is then, using $\nu_1 = 2B^2\nu$ given in (39) and the negative sign in (36),

(40) $$R(t) = -e^{-t}[2B^2\nu(1 - e^{-t}) + 2e^{-t} - 1],$$

being $R(0) = -1$.

From (23), (30) and (37) comes up as a possible solution, for $x \in \mathbb{R}^3$ and implicitly introducing the third coordinate space $v_3 \equiv 0$ into v, to

(41) $$\begin{aligned} v(x,t) &= X(x_1 - x_2)T(t)(1,1,0) \\ &= [A\,cos(B\xi) + C\,sin(\pm B\xi)](1 - e^{-t})e^{-t}(1,1,0), \end{aligned}$$

which as we can see is not actually a single solution for speed, because of the endless possibilities that we had to set the time dependence $T(t)$, as well as the temporal dependence $X(\xi)$, $\xi = x_1 - x_2$, beyond the arbitrary constants A, B, C in (41). Even without uniqueness of solution, it meets the requirements we expected: it is limited, continuous of class C^∞, equal to zero at the initial time, tends to zero with increasing time, and has divergent null ($\nabla \cdot v = 0$). Furthermore, when used in the expression (17.5) obtained for the external force, it does not remove to the force f the condition that belong to Schwartz space in relation to space $\mathbb{R}^3$ and to the time, i.e., $f \in S(\mathbb{R}^3 \times [0, \infty))$, as can be shown of the S properties that we saw in section § 2 above.

The pressure is obtained by integrating (24) with respect to the difference $\xi = x_1 - x_2$, with $a = 1, b = 0, Q(\xi) = X(\xi)$ and $R(t)$ given in (40),

$$
\begin{aligned}
(42) \qquad p(x,t) - p_0(t) &= R(t) \int_{\xi_0}^{\xi} Q(\xi) d\xi \\
&= -e^{-t}[2B^2\nu(1 - e^{-t}) + 2e^{-t} - 1]\, S(\xi), \\
S(\xi) &= \frac{A}{B}[sin(B\xi) - sin(B\xi_0)] \pm \frac{A}{B}[cos(\pm B\xi) - cos(\pm B\xi_0)],
\end{aligned}
$$

where ξ_0 is the surface $\xi = \xi_0$ and where the pressure is p_0 at time t. Again we see that this solution is not unique, not only due solely to the function $p_0(t)$ and respective ξ_0, but also because of the arbitrary constants A and B, the signal $\pm$, beyond from the way $R(t)$ and $Q(\xi)$ were obtained, with a certain freedom of possibilities. $p_0(t)$ substitute the function $\theta(t)$ used in (15) and (19), our generic function of time, or a constant, which must be class $C^\infty([0, \infty))$ and we can assume limited.

Completing the main solution (p, u) that we seek to equation (1), we finally have

$$
(43) \qquad u(x,t) = u^0(x)e^{-t} + v(x,t),
$$

with $u^0(x)$ given in (17.1), $v(x,t)$ in (41) and $f(x,t)$ in (17.5).

The velocity (secondary) v we choose makes the velocity (main) u a function with some properties similar to it: u it is limited oscillating, contains a sum of sine and cosine of a difference in spatial coordinates and decays exponentially over time, or does, not belong to a Schwartz space over the position, nor is square integrable (violating so the inequality (7) in $t > 0$), but is continuous of class C^∞ and does not diverge when $|x| \to \infty$. Their behavior in relation to $x_1 - x_2$ and the divergence of the total kinetic energy obviously not withdraw of $f(x,t)$ the condition to be pertaining to $S\big(\mathbb{R}^3 \times [0, \infty)\big)$, equivalent to inequality (5), since it only depends on $u^0(x)$ and $v(x,t)$. We also have $v(x,0) = 0$, $\nabla \cdot v = 0$, $v \in C^\infty\big(\mathbb{R}^3 \times [0, \infty)\big)$, the validity of (1), (2), (3), (4) and (6), $u(x,0) = u^0(x)$, $u^0 \in S(\mathbb{R}^3)$, with $\nabla \cdot u = 0$ and $u \in C^\infty\big(\mathbb{R}^3 \times [0, \infty)\big)$, as we wanted.

§ 6 – The non uniqueness in $n = 2$ spatial dimension

What's with the proofs of uniqueness of solutions of the Navier-Stokes equations in spatial dimension $n = 2$?

Not possible to examine all the available proofs, you can at least understand that such proofs should not take into account the absence of the nonlinear term in the Navier-Stokes equations, $\sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} \equiv ((u \cdot \nabla)u)_i, 1 \leq i \leq n$, and it was this lack that we use in our second example.

Similarly to this cause, also realize that different equations of the type Navier-Stokes equations with the absence of one or more terms of their complete equation, and which nevertheless have the same initial condition $u(x, 0) = u^0(x)$, will probably, in the general case, different solutions $u(x, t)$ among them, and so there can be no uniqueness of solution from the full Navier-Stokes equation, with all terms. If all always presented the same and only solution would suffice for us to solve the simplest of them only, for example, $\nabla p = -\frac{\partial u}{\partial t}$ ou $\nabla p = \nu \nabla^2 u$ (Poisson Equation if $\nabla p \not\equiv 0$ or of Laplace if $\nabla p \equiv 0$) or $\frac{\partial u}{\partial t} = \nu \nabla^2 u$ (Heat Equation with $\nabla p = 0$), all with $u(x, 0) = u^0(x)$, and check if the sum of the other missing terms is zero to apply the solution u obtained in the reduced equation. If so, the solution of the reduced equation is also solution of the complete equation. Important example of this absence are the Euler equations, which differ from the Navier-Stokes equations by the absence of differential operator Laplacian applied to u, $\nabla^2 u \equiv \Delta u$, due to the viscosity coefficient be zero, $\nu = 0$.

It is easy to prove that the above three equations, as well as the equation $\nabla p + \frac{\partial u}{\partial t} = \nu \nabla^2 u$, not may actually have a unique solution, given only the initial condition for velocity $u(x, 0) = u^0(x)$. On the contrary, the complete form of the Navier-Stokes equations, where we assume that $\sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} \equiv ((u \cdot \nabla)u)_i \not\equiv 0,\ 1 \leq i \leq n$, it has uniqueness of solution for $n = 2$ and at least a short period of time not null $[0, T]$ for $n = 3$, where T is known as *blowup time*. Let us add all of these equations the condition of incompressibility, $\nabla \cdot u = 0$.

This is so an interesting problem of Combinatorial Analysis applied to Mathematical Analysis and Mathematical Physics.

§ 7 – Uniqueness in $n = 2$ spatial dimension

We found in section § 5 that the system

(44) $$\begin{cases} \nabla p + \frac{\partial v}{\partial t} = \nu \nabla^2 v \\ (v \cdot \nabla) v = 0 \\ \nabla \cdot v = 0 \\ v(x, 0) = 0 \end{cases}$$

has infinite solutions to the velocity of the form

(45) $$v(x,t) = X(\xi)T(t)(1,1), \xi = x_1 - x_2,$$

with $T(0) = 0$, however there are known proofs of the uniqueness of

(46) $$\begin{cases} \nabla p + \frac{\partial v}{\partial t} + (v \cdot \nabla) v = \nu \nabla^2 v \\ \nabla \cdot v = 0 \\ v(x, 0) = 0 \end{cases}$$

contradicting what we got.

No need to linger in the known proofs, exposing all its details, repeating his steps, you can be seen in Leray [10], Ladyzhenskaya [11], Kreiss and Lorenz [14], among others, that the proofs of existence and uniqueness are based on the complete form of the Navier-Stokes equations, for example (46), and not in a dismembered form of the Navier-Stokes equations, as (44).

The Navier-Stokes equations without external force with $n = 2$ are (using $x \equiv x_1$ and $y \equiv x_2$)

(47) $$\begin{cases} \frac{\partial p}{\partial x} + \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} = \nu \nabla^2 u_1 \\ \frac{\partial p}{\partial y} + \frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} = \nu \nabla^2 u_2 \end{cases}$$

We can dispose the system up in a similar form to a system of linear equations,

(48) $$\begin{cases} u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} = \nu \nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t} \\ u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} = \nu \nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t} \end{cases}$$

and then in the form of a matrix equation,

(49) $$\begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \nu \nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t} \\ \nu \nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t} \end{pmatrix}.$$

Calling

(50) $$A = \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} \end{pmatrix},$$

(51) $$U = \begin{pmatrix} u_1 \\ \\ u_2 \end{pmatrix},$$

(52) $$B = \begin{pmatrix} \nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t} \\ \\ \nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t} \end{pmatrix},$$

the solution for U of the equation (49), $AU = B$, is

(53) $$U = A^{-1}B,$$

that for its existence and unique solution must be

(54) $$\det A = \frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x} \neq 0,$$

that is,

(55) $$\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} \neq \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x},$$

rule should also be obeyed for $t = 0$ (again can lead us to cases (C) and (D) of [1] applying the method in matrix 3×3, i.e., $n = 3$, however, with appropriate choice of p or $\partial u/\partial t$ the system will be possible).

If we use the condition of incompressibility $\nabla \cdot u = 0$,

(56) $$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0,$$

i.e.,

(57) $$\frac{\partial u_1}{\partial x} = -\frac{\partial u_2}{\partial y},$$

becomes the condition (55) in

(58) $$-\left(\frac{\partial u_2}{\partial y}\right)^2 \neq \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x},$$

or equivalently,

(59) $$-\left(\frac{\partial u_1}{\partial x}\right)^2 \neq \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}.$$

Since this condition must be valid for all t, in $t = 0$ must obey to

(60) $$-\left(\frac{\partial u_1^0}{\partial x}\right)^2 \neq \frac{\partial u_1^0}{\partial y}\frac{\partial u_2^0}{\partial x}$$

and

(61) $$-\left(\frac{\partial u_2^0}{\partial y}\right)^2 \neq \frac{\partial u_1^0}{\partial y}\frac{\partial u_2^0}{\partial x},$$

using $u(x,y,0) = u^0(x,y) = (u_1^0(x,y), u_2^0(x,y))$.

If the initial velocity u^0 is such that either disobeyed (60) or (61) then either there is no solution to the system (47) (impossible system) or there will be a non-unique solution (indeterminate system), as in theory linear systems.

Defining

(62) $$U_1 = \begin{pmatrix} \nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t} & \frac{\partial u_1}{\partial y} \\ \nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t} & \frac{\partial u_2}{\partial y} \end{pmatrix}$$

and

(63) $$U_2 = \begin{pmatrix} \frac{\partial u_1}{\partial x} & \nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial x} & \nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t} \end{pmatrix},$$

the solution for u_1, u_2 will be

(64) $$u_1 = \frac{\det U_1}{\det A}$$

and

(65) $$u_2 = \frac{\det U_2}{\det A}.$$

Being

(66) $$\det U_1 = \left(\nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t}\right)\frac{\partial u_2}{\partial y} - \left(\nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t}\right)\frac{\partial u_1}{\partial y}$$

and

(67) $$\det U_2 = \left(\nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t}\right)\frac{\partial u_1}{\partial x} - \left(\nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t}\right)\frac{\partial u_2}{\partial x},$$

with $\det A$ given in (54), then we have

(68) $$u_1 = \frac{\det U_1}{\det A} = \frac{\left(\nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t}\right)\frac{\partial u_2}{\partial y} - \left(\nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t}\right)\frac{\partial u_1}{\partial y}}{\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}}$$

and

(69) $$u_2 = \frac{\det U_2}{\det A} = \frac{\left(\nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t}\right)\frac{\partial u_1}{\partial x} - \left(\nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t}\right)\frac{\partial u_2}{\partial x}}{\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}}.$$

Using the incompressibility equation in the determinant of A,

(70) $$u_1 = -\frac{\left(\nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t}\right)\frac{\partial u_2}{\partial y} - \left(\nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t}\right)\frac{\partial u_1}{\partial y}}{\left(\frac{\partial u_2}{\partial y}\right)^2 + \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}}$$

and

(71) $$u_2 = -\frac{\left(\nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t}\right)\frac{\partial u_1}{\partial x} - \left(\nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t}\right)\frac{\partial u_2}{\partial x}}{\left(\frac{\partial u_1}{\partial x}\right)^2 + \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}}.$$

It is true that the solutions (equations) above are as or more complicated as the original equations (47), and seems there no use whatsoever in resolving them.

But this complicated form can be reached with more certainty to the following conclusion: the Navier-Stokes (and Euler) equations have a symmetry between the variables, both dependent as independent. The same can also be realized directly in (47).

The symmetry in this case of $n = 2$ is

(72.1) $u_1 \leftrightarrow u_2$

(72.2) $x \leftrightarrow y$

p and t being unchanged:

(73.1) $p \leftrightarrow p$

(73.2) $t \leftrightarrow t$.

This suggests, if not completely solves, the question of the solution of these equations. If the equations themselves are symmetrical with respect to certain transformations, so we hope that their solutions are also under these transformations. The same method can be applied also for $n \geq 3$, with the rule (e.g.)

(74.1) $u_i \mapsto u_{i+1}$, $u_{n+1} \equiv u_1$,

(74.2) $x_i \mapsto x_{i+1}$, $x_{n+1} \equiv x_1$,

(74.3) $p \leftrightarrow p$,

(74.4) $t \leftrightarrow t$.

In this case it is necessary that the initial condition $u(x,0)=u^0(x)$ also obey these symmetries, but remains unchanged the condition of incompressibility: $\sum_{i=1}^{n}\frac{\partial u_i}{\partial x_i}=\sum_{i=1}^{n}\frac{\partial u_i^0}{\partial x_i}=0$.

If we provide $u_2(x,y,t)$ as input data in our system then we can conclude that the solution for u_1, supposedly symmetrical to u_2 by the rule (72) previous, is

(75) $$u_1(x,y,t)=u_2(y,x,t),$$

i.e., we exchange x by y, and vice versa, in the solution previously given for u_2 and we equate to u_1 the result of this transformation. Lack get the pressure p or else if it has also been given, check that the variables $u_1,\ u_2,\ p$ really satisfy the original system.

The general form of the solution to the pressure p, which must satisfy

(76) $$\nabla p+\frac{\partial u}{\partial t}+(u\cdot\nabla)u=\nu\nabla^2 u,$$

is

(77) $$p-p_0(t)=\int_{(x_0,y_0)}^{(x,y)}\left[\nu\nabla^2 u-\frac{\partial u}{\partial t}-(u\cdot\nabla)u\right]\cdot dl,$$

where we assume that in the position $(x,y)=(x_0,y_0)$ and at the instant t the pressure is equal to $p_0(t)$. The integration occurs in any way between (x_0,y_0) and (x,y), because the pressure should be a potential function of the integral of (77) in order that (47) has solution.

It is also expected that p be symmetric with respect to variables x and y, in other words,

(78) $$p(x,y,t)=p(y,x,t),$$

as well as in 3 dimensions, using $x\equiv x_1, y\equiv x_2, z\equiv x_3$,

(79) $$p(x,y,z,t)=p(y,z,x,t)=p(z,x,y,t).$$

Of course (74), (75), (78) and (79) implicitly admits that we have rectangular symmetry in the initial and contour conditions of the system. Since this symmetry does not occur, for example, have another type of symmetry, spherical, cylindrical, or even none symmetry (general case), the equalities (74), (75), (78) and (79) do not need to be met. Thus, the solution for the case that there is none symmetry is still a problem to be solved, assuming that there is at least one solution (when the system is possible; as we said, it can be proved that the system is always possible, for example, with appropriate choice of p or $\partial u/\partial t$).

Finally then developed the foregoing, our example 3, which seeks a unique solution to the Navier-Stokes system with $n=3$, all terms of the equation, nonzero external force, and provides infinite total kinetic energy to the system (1) to (6) in $t > 0$ will be based on the example 2, but again need to resort to the absence of non-linear term in the equation auxiliary with $n = 3$. Since (18), the Navier-Stokes equation without external force, has as initial condition the zero initial velocity, the only possible velocity for your solution with all the terms is also the zero velocity due to the uniqueness of the solutions in the form complete this equation (abstracting constant generic pressures and/or time functions), solution that does not interest us. So, we need again that (18) does not have the non-linear term. The uniqueness of the main equation solution in three dimensions, however, at least in short time, is guaranteed because it contains all terms (again, except for not unique pressure), including the applied external force (which itself depends of not unique solution of the auxiliary equation with $n = 3$).

§ 8 – Example 3

The third example is a generalization of the example 2, with the velocity components v_2 and v_3 proportional to the component v_1,

(80.1) $$v_1 = X(\xi)T(t),\ \xi = x_1 + \frac{1}{\alpha}x_2 - 2\frac{1}{\beta}x_3,\ \alpha \neq 0,\ \beta \neq 0,$$

(80.2) $$v_2 = \alpha v_1,$$

(80.3) $$v_3 = \beta v_1,$$

α and β non-zero constant. We could also use other coefficients combinations in the variables x_i in ξ, whenever $\nabla \cdot (\xi I) = 0$, with $I = (1,1,1)$. In the example 2 we use $\alpha = 1,\ \beta = 0$.

We will choose the components of the initial velocity u^0 with some property of symmetry. It is not easy to think of not constant velocities with symmetrical components u_i^0 and simultaneously whose divergent $\nabla \cdot u^0$ is null. The velocities with symmetry which the i-th component does not contain the i-th coordinate space, for all i (natural) in $1 \leq i \leq n$, fulfill this requirement: $\frac{\partial u_i^0}{\partial x_i} = 0$. Alternatively we can use the known vector equality $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$, ie, choosing a vector u^0 that has a potential vector $\boldsymbol{A}$, i.e., $u^0 = \nabla \times \boldsymbol{A}$. So we choose primarily a vector $\boldsymbol{A}$ that has the symmetry properties we expect.

Be $\boldsymbol{A} = (A_1, A_2, A_3)$ the potential vector we want. Doing $A_1 = A_2 = A_3 = e^{-r^2}$, with $r^2 = x_1^2 + x_2^2 + x_3^2$, the value assigned to the initial velocity $u^0(x)$ will be

(81) $$u^0(x) = rot\ \boldsymbol{A} = 2e^{-r^2}(-x_2 + x_3, -x_3 + x_1, -x_1 + x_2).$$

Following the equations 17 of Example 2, let us now for $x \in \mathbb{R}^3$, (82.1) $v(x,t) = e^{-t}w(x,t),$

(82.2) $w(x,t) = \left(w_1(x_1,x_2,x_3,t), w_2(x_1,x_2,x_3,t), w_3(x_1,x_2,x_3,t)\right),$

$w(x,0) = 0,\ \nabla\cdot w = 0,\ w \not\equiv 0,$

(82.3) $u_i(x,t) = u_i^0(x)e^{-t} + v_i(x,t) = [u_i^0(x) + w_i(x,t)]e^{-t},$

(82.4) $$f_i(x,t) = \left(-u_i^0 + e^{-t}\sum_{j=1}^3[u_j^0\frac{\partial u_i^0}{\partial x_j} + u_j^0\frac{\partial w_i}{\partial x_j} + w_j\frac{\partial u_i^0}{\partial x_j}] - \nu\nabla^2 u_i^0\right)e^{-t}$$
$$= \left(-u_i^0 + \sum_{j=1}^3[e^{-t}u_j^0\frac{\partial u_i^0}{\partial x_j} + u_j^0\frac{\partial v_i}{\partial x_j} + v_j\frac{\partial u_i^0}{\partial x_j}] - \nu\nabla^2 u_i^0\right)e^{-t},$$

which results for $p(x,t)$ and $v(x,t)$, as unknowns variables still to be determined,

(83) $$\frac{\partial p}{\partial x_i} + \frac{\partial v_i}{\partial t} + \sum_{j=1}^3 v_j\frac{\partial v_i}{\partial x_j} = \nu\nabla^2 v_i,$$

the Navier-Stokes equations without external force.

Equations (80) applied to (83) result in

(84) $$\begin{cases} \frac{\partial p}{\partial x_1} + \frac{\partial v_1}{\partial t} + v_1\left(\frac{\partial v_1}{\partial x_1} + \alpha\frac{\partial v_1}{\partial x_2} + \beta\frac{\partial v_1}{\partial x_3}\right) = \nu\nabla^2 v_1 \\ \frac{\partial p}{\partial x_2} + \alpha\frac{\partial v_1}{\partial t} + \alpha v_1\left(\frac{\partial v_1}{\partial x_1} + \alpha\frac{\partial v_1}{\partial x_2} + \beta\frac{\partial v_1}{\partial x_3}\right) = \nu\alpha\nabla^2 v_1 \\ \frac{\partial p}{\partial x_3} + \beta\frac{\partial v_1}{\partial t} + \beta v_1\left(\frac{\partial v_1}{\partial x_1} + \alpha\frac{\partial v_1}{\partial x_2} + \beta\frac{\partial v_1}{\partial x_3}\right) = \nu\beta\nabla^2 v_1 \end{cases}$$

As

(85) $$\frac{\partial v_1}{\partial x_1} + \alpha\frac{\partial v_1}{\partial x_2} + \beta\frac{\partial v_1}{\partial x_3} = T(t)\frac{dX}{d\xi}\left(\frac{\partial \xi}{\partial x_1} + \alpha\frac{\partial \xi}{\partial x_2} + \beta\frac{\partial \xi}{\partial x_3}\right)$$
$$= T(t)\frac{dX}{d\xi}(1+1-2) = 0,$$

by the definition of ξ we use in (80.1), then (84) becomes

(86) $$\begin{cases} \frac{\partial p}{\partial x_1} + \frac{\partial v_1}{\partial t} = \nu\nabla^2 v_1 \\ \frac{\partial p}{\partial x_2} + \alpha\frac{\partial v_1}{\partial t} = \alpha\nu\nabla^2 v_1 \\ \frac{\partial p}{\partial x_3} + \beta\frac{\partial v_1}{\partial t} = \beta\nu\nabla^2 v_1 \end{cases}$$

or equivalently,

(87) $$\begin{cases} \frac{\partial p}{\partial x_1} = \nu\nabla^2 v_1 - \frac{\partial v_1}{\partial t} \\ \frac{\partial p}{\partial x_2} = \alpha\left[\nu\nabla^2 v_1 - \frac{\partial v_1}{\partial t}\right] = \alpha\frac{\partial p}{\partial x_1} \\ \frac{\partial p}{\partial x_3} = \beta\left[\nu\nabla^2 v_1 - \frac{\partial v_1}{\partial t}\right] = \beta\frac{\partial p}{\partial x_1} \end{cases}$$

Similar to what we saw in section § 5, equation (24) for $a = 1$ and $b = 0$, we will make the pressure to be defined as

(88) $\frac{\partial p}{\partial \xi} = Q(\xi)R(t),$

and the velocity

(89) $v_i = c_i X\big(\xi(x)\big)\, T(t),\ c_1 = 1, c_2 = \alpha, c_3 = \beta,$

with ξ defined in (80.1),

(90) $\xi = x_1 + \frac{1}{\alpha}x_2 - 2\frac{1}{\beta}x_3,\ \alpha \neq 0,\ \beta \neq 0.$

Then it will be sufficient, besides the equation (88) to the pressure, solve one linear partial differential equation involving v_1, instead of a system of three nonlinear partial differential equations involving v_1, v_2, v_3.

The development of the solution here follows the same steps already seen in section § 5, equations (29) to (43), being the main change the expression for ξ given in (90), with increased dimensions and the proportionality between v_2, v_3 and v_1. We come to

(91)
$$v(x,t) = X\left(x_1 + \frac{1}{\alpha}x_2 - 2\frac{1}{\beta}x_3\right) T(t)(1,\ \alpha,\ \beta)$$
$$= [A\, cos(B\xi) + C\, sin(\pm B\xi)](1 - e^{-t})e^{-t}(1,\ \alpha,\ \beta),\ \alpha, \beta \neq 0,$$

keeping valid the solutions (42) and (43) for the pressure p and velocity u, respectively. Initial velocity equal to (81). We also have the validity of $\nabla \cdot v = 0$ and the corresponding integral $\int_{\mathbb{R}^3} |v|^2 dx$ infinite, portion of kinetic energy total system (1) to (6).

§ 9 – Conclusion

All three examples obey the necessary conditions of divergence-free ($\nabla \cdot u^0 = 0$), smoothness (C^∞) and partial derivatives of u^0 and f of $C_{\alpha K}$ $(1 + |x|)^{-K}$ and $C_{\alpha m K}$ $(1 + |x| + t)^{-K}$ order, respectively. We conclude that we must have $u^0 \in S(\mathbb{R}^3)$ and $f \in S(\mathbb{R}^3 \times [0, \infty))$. To each possible $u(x,t)$ so that (3) is true, the external force $f(x,t)$ and the pressure $p(x,t)$ can be fittingly constructed, in C^∞ class, verifying (8), and in a way to satisfy all the necessary conditions, finding, this way, a possible solution to (1), (2), (3), (4), (5) and (6), and only (7) wouldn't be satisfied, for $t > 0$, according to (10). We then show examples of breakdown solutions to case (C) of this millennium problem. These examples, however, won't take to case (A) from [1], of existence and smoothness of solutions, because they violate (7) (case (A) also impose a null external force, $f = 0$).

An overview of the problem's conditions is listed below ($: \mathbb{R}^3$ and $: \mathbb{R}^3 \times [0, \infty)$ representing the respective functions domains).

$\nu > 0, n = 3$

$\exists u^0(x)\colon \mathbb{R}^3$ smooth (C^∞), divergence-free ($\nabla \cdot u^0 = 0$)

$\exists f(x,t)\colon \mathbb{R}^3 \times [0,\infty)$ smooth (C^∞)

(4) $|\partial_x^\alpha u^0(x)| \leq C_{\alpha K}(1+|x|)^{-K}\colon \mathbb{R}^3,\ \forall \alpha, K$

(5) $|\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1+|x|+t)^{-K}\colon \mathbb{R}^3 \times [0,\infty)$, $\forall \alpha, m, K$

$\nexists (p,u)\colon \mathbb{R}^3 \times [0,\infty)$ /

(1) $\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i - \frac{\partial p}{\partial x_i} + f_i(x,t), 1 \leq i \leq 3$ $(x \in \mathbb{R}^3, t \geq 0)$

(2) $\nabla \cdot u = 0$

(3) $u(x,0) = u^0(x)$ $(x \in \mathbb{R}^3)$

(6) $p, u \in C^\infty(\mathbb{R}^3 \times [0,\infty))$

(7) $\int_{\mathbb{R}^3} |u(x,t)|^2 dx < C, \forall t \geq 0$ (bounded energy)

In all three examples the head velocity u we used was of the form

(92) $$u(x,t) = [u^0(x) + w(x)(1-e^{-t})]e^{-t};$$

in example 1, $w(x) = 1$, in example 2, $w(x) = X(\xi)(1,1,0)$, $\xi = x_1 - x_2$, and in example 3, $w(x) = X(\xi)(1,\alpha,\beta)$, $\xi = x_1 + \frac{1}{\alpha}x_2 - 2\frac{1}{\beta}x_3$, $\alpha, \beta\ cst. \neq 0$, examples 2 and 3 with $X(\xi) = [A\cos(B\xi) + C\sin(\pm B\xi)]$.

It's important that we also analyze the solution's uniqueness question. As $u^0(x)$ and $f(x,t)$ are given of C^∞ class, chosen by us, and satisfying (4) and (5), i.e., belonging to the Schwartz space, with $\nabla \cdot u^0 = 0$, claim that there is no solution (p,u) to the system (1), (2), (3), (6) and (7) might assume that we explored, or proved to, the infinite possible combinations of p and u, i.e., of (p,u). So we need that exists uniqueness of solution for the velocity that we build, eliminating other possible velocities for the same data used, $u^0(x)$ and $f(x,t)$, and involving in finite total kinetic energy.

The uniqueness of the solution (except due the pressure $p(x,t)$ with constant additional term or time-dependent $\theta(t)$, and other cases of non-uniqueness of pressure on x and $T(t)$) comes from classical results already known, for example described in the mentioned article of Fefferman [1]: the system of Navier-Stokes equations (1), (2), (3) it has (unique [15]) solution for all $t \geq 0$ or only for a finite time interval $[0,T)$ depending on the initial data, where T is called "*blowup time*". When there is a solution with finite T then the velocity u becomes unbounded near the "*blowup time*".

We see that the existence of each our solution in the given examples are guaranteed by construction and direct substitution. Our velocities has no irregular behavior, any regularity loss, at no time t, in none position, that becomes one unlimited, infinite, even for $t \to \infty$ or $|x| \to \infty$, therefore, there can be no *"blowup time"* in the examples we gave, therefore the solutions found in the previous cases are unique at all times (unless pressure). But even if there were a finite T (in [14], [16] we see that $T > 0$), the uniqueness would exist in at least a small interval of time, which is enough to show that in this time range occurs the breakdown of Navier-Stokes solutions because it was disobeyed the limited kinetic energy condition (7), making the case (C) true.

We must understand that uniqueness is in the main velocity u (equation 1), it is not necessary that is also in secondary velocity v (equations 14, 18 and 83), which as we have seen in the examples 2 and 3 it can have infinite solutions, due to the absence of n nonlinear terms $\sum_{j=1}^{n} v_j \frac{\partial v_i}{\partial x_j}$. Chosen a velocity v, however, applying in it the external force f (equations 13.4, 17.5, 82.4), results finally in the uniqueness of u (according 13.3, 17.4, 43, 82.3), solution of an equation with all the terms, of its kinetic energy and the corresponding divergence of the total kinetic energy $\int_{\mathbb{R}^3} |u|^2\, dx$ in $t > 0$ due to the term $\int_{\mathbb{R}^3} |v|^2\, dx \to \infty$. The pressure p, we already know, it is not unique, but this does not change, qualitatively, the fact that the total kinetic energy of the system is infinite or not. This is better understood with examples 1 and 2: v being any constant or time dependent exclusively, or with $x \in \mathbb{R}$ or with $x \in \mathbb{R}^2$, since not identically zero, and whatever the pressure p, null or not, the condition (7) is violated due to integration of $|v|^2$ in the whole space $\mathbb{R}^3$.

§ 10 – Final Comments

It is not difficult to extend the results obtained earlier in the § 5 with the two-dimensional speed to a speed v with three non-zero spatial components, as we saw in section § 8.

In examples 2 and 3 we had to solve an ordinary differential equation to get $X(\xi)$. We will now, however, find a solution non unique for the velocity in the Navier-Stokes equations, but without solving any differential equation aid. You just have to make an integration necessary to obtain pressure. Out of curiosity, the initial speed may be different from zero, as well as the external force, and we are not concerned to seek just endless kinetic energies or velocities belonging to the Schwartz space. We are not looking now a *breakdown solution*, on the contrary, we seek endless (many) *solutions*.

We will solve the system (1), (2), (3) for the special case in which

$$(93) \qquad u(x_1, x_2, x_3, t) = X(x_1 + x_2 + x_3)T(t)\ (1, 1, -2) = X(\xi)T(t)J,$$
$$\xi(x) = x_1 + x_2 + x_3,\ J = (1, 1, -2),$$

being worth $\nabla \cdot (\xi J) = 0$. This gives us $\nabla \cdot u = \nabla \cdot u^0 = 0$ and the elimination of non-linear terms $(u \cdot \nabla)u \equiv \left(\sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j},\right)_{1 \le i \le 3} = 0$ of the Navier-Stokes equations, with or without external force. So the solution of (1) will be reduced to the solution of one linear partial differential equation, the Heat Equation three-dimensional inhomogeneous,

$$(94) \qquad \frac{\partial p}{\partial x_i} = \nu \nabla^2 u_i - \frac{\partial u_i}{\partial t} + f_i = \phi_i, 1 \le i \le 3,$$

need to be true

$$(95) \qquad \frac{\partial \phi_i}{\partial x_j} = \frac{\partial \phi_j}{\partial x_i}, i \ne j.$$

As $\frac{\partial p}{\partial x_i} = \frac{\partial p}{\partial \xi}\frac{\partial \xi}{\partial x_i} = \frac{\partial p}{\partial \xi}$, $\forall i$, as well the differential operators $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x_i} = \frac{\partial}{\partial \xi}$ and $\left(\frac{\partial}{\partial x_i}\right)^2 = \left(\frac{\partial}{\partial \xi}\right)^2$, $\forall i$, i.e., we have a pressure that may be expressed as a function of ξ, as well as the velocity components u, and the x_i are shown symmetrically and linearly relative to $\xi = x_1 + x_2 + x_3$, with the transformation of infinitesimal element of integration $d\xi = \frac{\partial \xi}{\partial x_1}dx_1 + \frac{\partial \xi}{\partial x_2}dx_2 + \frac{\partial \xi}{\partial x_3}dx_3 = dx_1 + dx_2 + dx_3$, equality (95) is true, it is valid $\frac{\partial^2 p}{\partial x_j \partial x_i} = \frac{\partial^2 p}{\partial x_i \partial x_j}$, and we have the following solution to the pressure:

$$(96) \qquad p(x, t) - p_0(t) = \int_{\xi_0}^{\xi(x)} \left(\nu \mathrm{T} \nabla^2 X - X\frac{dT}{dt} + f\right) d\xi,$$

with

$$(97) \qquad \frac{\partial p}{\partial x_1} = \frac{\partial p}{\partial x_2} = \frac{\partial p}{\partial x_3},$$

assuming that the force $f(x, t)$ is of the form $Y(\xi)Z(t)(1, 1, -2)$, such as $u(x, t) = X(\xi)T(t)(1, 1, -2)$. Let us consider $p_0(t)$ as the pressure at the instant t and the surface $\xi = \xi_0$. This solves the system we wanted, since the integration in (96) is possible, and so we do not solve any intermediate ordinary differential equation to find $X(\xi)$, because we can prefix which the expression for $X(\xi)$ we want to use, among infinite possibilities, and such that have $u(x, 0) = u^0(x)$.

Other combinations of the components of the vector J may be used, as well as other combinations of the coefficients of x_1, x_2, x_3 in ξ, provided that they eliminate non-linear terms and check it (2) and (95). Thus, more complex forms to ξ are also possible, in addition to linear, which provides a robust way to achieve solutions for u. For example, defining

(98) $$u_i = \alpha_i(x,t)u_1,\ 1 \le i \le n,\ \alpha_1 = 1,$$

the condition to be obeyed by X and ξ in order to eliminate the nonlinear terms is

(99) $$\alpha_i \frac{dX(\xi)}{d\xi} \sum_{j=1}^{n} \alpha_j \frac{\partial \xi}{\partial x_j} + X(\xi) \sum_{j=1}^{n} \alpha_j \frac{\partial \alpha_i}{\partial x_j} = 0,$$

for all i (natural) in $1 \le i \le n$. For each determined i liminates the nonlinear term of the line (or coordinate) i if (99) is satisfied.

One way to do (99) true is when

(100) $$\sum_{j=1}^{n} \alpha_j \frac{\partial \xi}{\partial x_j} = \sum_{j=1}^{n} \alpha_j \frac{\partial \alpha_i}{\partial x_j} = 0.$$

When the α_i are constant or time dependent only the condition to be obeyed for ξ is

(101) $$\sum_{j=1}^{n} \alpha_j \frac{\partial \xi}{\partial x_j} = 0,$$

which is in accordance with examples 2 and 3 above.

Including also the incompressibility condition for u, must be valid also the relation

(102) $$\sum_{j=1}^{n} \frac{\partial(\alpha_j u_1)}{\partial x_j} = u_1 \sum_{j=1}^{n} \frac{\partial \alpha_j}{\partial x_j} + \sum_{j=1}^{n} \alpha_j \frac{\partial u_1}{\partial x_j}$$
$$= T(t)[X(\xi) \sum_{j=1}^{n} \frac{\partial \alpha_j}{\partial x_j} + \frac{dX(\xi)}{d\xi} \sum_{j=1}^{n} \alpha_j \frac{\partial \xi}{\partial x_j}] = 0.$$

As (102) must be valid for all t, then we need to be valid

(103) $$X(\xi) \sum_{j=1}^{n} \frac{\partial \alpha_j}{\partial x_j} + \frac{dX(\xi)}{d\xi} \sum_{j=1}^{n} \alpha_j \frac{\partial \xi}{\partial x_j} = 0.$$

When the α_j are constant or time dependent only, the condition to be obeyed for ξ is equal to the condition (101) previous,

(104) $$\sum_{j=1}^{n} \alpha_j \frac{\partial \xi}{\partial x_j} = 0.$$

Notice that the function $T(t)$ in (93) must not have singularities in case it is desired that the velocity u is regular, limited in module, notwithstanding, $T(t)$ singular, infinite for one or more values of time t, the function can be considered as a "highlighter" of *blowups*, and so we can build solutions with instants of *blowup* τ_* well determined, to our will, such that $T(\tau_*) \to \infty$.

In the absence of singularities of $T(t)$ and $X(\xi(x))$, however, only wishing regular velocities, it follows that it is possible for a three-dimensional Navier-Stokes equation (generally, n-dimensional) *"well behaved"* have more than one solution for the same initial velocity. To the special form given to the solution $u(x,t)$ in (93), with $T(0)=0$ or not, for a same initial velocity $u(x,0) = X\big(\xi(x)\big)T(0)J = u^0(x)$, with $J=(1,1,-2)$, can be generated, in principle, infinite different velocities $u(x,t) = X(\xi(x))T(t)J$, for different functions of the position $X(\xi(x))$ and time $T(t)$, solutions that solve the Navier-Stokes equation (1). If the external force is zero, this brings us to the negative answer to 15th problem of Smale [12], as we have seen before thinking only in the non uniqueness of pressure due to the additional term $\theta(t)+q$, where $q \neq 0$ is a constant and $\theta(t)$ an explicit function of time (in the original Smale problem pressure does not vary in time).

In next article the corresponding section § 7 in three dimensions.

Thankful, friend God. For peace between religions and between people.

Dedicated to John Nash. In memoriam.

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09 – Three Examples of Unbounded Energy for $t > 0$

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Temos dons diferentes, de acordo com a graça dada a cada um de nós:
se é a profecia, exerçamo-la em harmonia com a fé;
se é o serviço, pratiquemos o serviço;
se é o dom de ensinar, consagremo-nos ao ensino;
se é o dom de exortar, exortemos.
Quem distribui donativos, faça-o com simplicidade;
quem preside, presida com solicitude;
quem se dedica a obras de misericórdia, faça-o com alegria.
O amor seja sincero. Detestai o mal, apegai-vos ao bem.
(Romanos 12, 6-9)

Abstract – A solution to the 6th millenium problem, respect to breakdown of Navier-Stokes solutions and the bounded energy. We have proved that there are initial velocities $u^0(x)$ and forces $f(x,t)$ such that there is no physically reasonable solution to the Navier-Stokes equations for $t > 0$, which corresponds to the case (C) of the problem relating to Navier-Stokes equations available on the website of the Clay Institute. Three examples are given.

Keywords – Navier-Stokes equations, continuity equation, breakdown, existence, smoothness, physically reasonable solutions, gradient field, conservative field, velocity, pressure, external force, unbounded energy, millenium problem, uniqueness, non uniqueness, 15th Problem of Smale, blowup time.

§ 1 - Introdução

A segunda maneira que vejo para se provar a quebra de soluções (*breakdown solutions*) das equações de Navier-Stokes, seguindo o descrito em [1], refere-se à condição de energia limitada (*bounded energy*), a finitude da integral do quadrado da velocidade do fluido em todo o espaço.

Podemos certamente construir soluções de

$$(1) \qquad \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i - \frac{\partial p}{\partial x_i} + f_i,\ 1 \le i \le 3,$$

que obedeçam à condição de divergente nulo para a velocidade (equação da continuidade para densidade de massa constante),

$$(2) \qquad div\, u \equiv \nabla \cdot u = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0, \qquad \text{(fluidos incompressíveis)}$$

e à condição inicial

(3) $$u(x,0)=u^0(x),$$

onde u_i, p, f_i são funções da posição $x\in\mathbb{R}^3$ e do tempo $t\geq 0, t\in\mathbb{R}$. A constante $\nu\geq 0$ é o coeficiente de viscosidade, p representa a pressão e $u=(u_1,u_2,u_3)$ é a velocidade do fluido, medidas na posição x e tempo t, com $\nabla^2=\nabla\cdot\nabla=\sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}$. A função $f=(f_1,f_2,f_3)$ tem dimensão de aceleração ou força por unidade de massa, mas seguiremos denominando este vetor e suas componentes pelo nome genérico de força, tal como adotado em [1]. É a força externa aplicada ao fluido, por exemplo, gravidade.

As funções $u^0(x)$ e $f(x,t)$ devem obedecer, respectivamente,

(4) $|\partial_x^\alpha u^0(x)|\leq C_{\alpha K}(1+|x|)^{-K}$ sobre $\mathbb{R}^3$, para quaisquer $\alpha\in\mathbb{N}_0^3$ e $K\in\mathbb{R}$,

e

(5) $|\partial_x^\alpha\partial_t^m f(x,t)|\leq C_{\alpha mK}(1+|x|+t)^{-K}$ sobre $\mathbb{R}^3\times[0,\infty)$, para quaisquer $\alpha\in\mathbb{N}_0^3$, $m\in\mathbb{N}_0$ e $K\in\mathbb{R}$,

com $\mathbb{N}_0=\{0,1,2,3,\dots\}$ (derivadas de ordem zero não alteram o valor da função), e uma solução (p,u) de (1) para que seja considerada fisicamente razoável deve ser contínua e ter todas as derivadas, de infinitas ordens, também contínuas (*smooth*), i.e.,

(6) $$p,u\in C^\infty\qquad(\mathbb{R}^3\times[0,\infty)).$$

Dada uma velocidade inicial u^0 de classe C^∞ com divergente nulo (*divergence-free*, $\nabla\cdot u^0=0$) sobre $\mathbb{R}^3$ e um campo de forças externo f também de classe C^∞ sobre $\mathbb{R}^3\times[0,\infty)$, quer-se, para que uma solução seja fisicamente razoável, além da validade de (6), que $u(x,t)$ não divirja para $|x|\to\infty$ e seja satisfeita a condição de energia limitada (*bounded energy*), i.e.,

(7) $\int_{\mathbb{R}^3}|u(x,t)|^2dx<C$, para todo $t\geq 0$.

Vemos que todas as condições acima, de (1) a (7), precisam ser obedecidas para se obter uma solução (p,u) considerada fisicamente razoável, contudo, para se obter uma quebra de soluções, (1), (2), (3), (6) ou (7) poderiam não ser satisfeitas para algum $t\geq 0$, em alguma posição $x\in\mathbb{R}^3$, mantendo-se ainda a validade de (4) e (5).

Uma maneira de fazer com que esta situação (*breakdown*) ocorra é quando (1) não tem solução possível para a pressão $p(x,t)$, quando o campo vetorial $\phi:\mathbb{R}^3\times[0,\infty)\to\mathbb{R}^3$ em

(8) $$\nabla p = \nu\, \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla)u + f = \phi$$

é não gradiente, não conservativo, em ao menos um $(x, t) \in \mathbb{R}^3 \times [0, \infty)$. Nesse caso, para $\phi = (\phi_1, \phi_2, \phi_3)$ ser não gradiente deve valer

(9) $$\frac{\partial \phi_i}{\partial x_j} \neq \frac{\partial \phi_j}{\partial x_i}, i \neq j,$$

para algum par $(i, j), 1 \leq i, j \leq 3$, $x \in \mathbb{R}^3$ e tempo t não negativo (para mais detalhes veja, por exemplo, Apostol[2], cap. 10).

Se admitirmos, entretanto, que (1) tem solução (p, u) possível e esta também obedece (2), (3) e (6), a condição inicial $u^0(x)$ verifica (2) e (4), a força externa $f(x, t)$ verifica (5) e $u^0(x)$ e $f(x, t)$ são de classe C^∞, podemos tentar obter a condição de quebra de soluções em $t \geq 0$ violando-se a condição (7) de energia limitada (*bounded energy*), i.e., escolhendo-se $u^0(x)$ ou $u(x, t)$ que também obedeçam a

(10) $$\int_{\mathbb{R}^3} |u(x, t)|^2 dx \to \infty, \text{ para algum } t \geq 0.$$

A descrição oficial do problema para este caso (C) de quebra de soluções é dada a seguir:

(C) Quebra das soluções da Equação de Navier-Stokes sobre $\mathbb{R}^3$. Para $\nu > 0$ e dimensão espacial $n = 3$ existem um campo vetorial suave e com divergência nula $u^0(x)$ sobre $\mathbb{R}^3$ e uma força externa suave $f(x, t)$ sobre $\mathbb{R}^3 \times [0, \infty)$ satisfazendo

(4) $$|\partial_x^\alpha u^0(x)| \leq C_{\alpha K}(1 + |x|)^{-K} \text{ sobre } \mathbb{R}^3, \forall \alpha, K,$$

e

(5) $$|\partial_x^\alpha \partial_t^m f(x, t)| \leq C_{\alpha m K}(1 + |x| + t)^{-K} \text{ sobre } \mathbb{R}^3 \times [0, \infty), \forall \alpha, m, K,$$

tais que não existe solução (p, u) sobre $\mathbb{R}^3 \times [0, \infty)$ satisfazendo (1), (2), (3), (6) e (7).

Vê-se claramente que podemos resolver este problema buscando velocidades válidas cuja integral do seu quadrado em todo o espaço $\mathbb{R}^3$ é infinito, ou também, conforme indicamos em (8), buscando funções ϕ não gradientes, onde a pressão p não poderá ser considerada uma função potencial, para algum instante $t \geq 0$. Entendemos que os α, m indicados em (4) e (5) só fazem sentido para $|\alpha|, m \in \{0, 1, 2, 3, 4, \dots\}$ e os K negativos podem ser desprezados, pois não limitam o valor das funções u^0, f e suas derivadas quando $|x| \to \infty$ ou $t \to \infty$, com $C_{\alpha K}, C_{\alpha m K} > 0$.

§ 2 – O espaço de Schwartz S

A inequação (4) traz implicitamente que $u^0(x)$ deve pertencer ao espaço vetorial das funções de rápido decrescimento, que tendem a zero em $|x| \to \infty$, conhecido como espaço de Schwartz, $S(\mathbb{R}^3)$, em homenagem ao matemático francês Laurent Schwartz (1915-2002) que o estudou [3]. Estas funções e suas infinitas derivadas são contínuas (C^∞) e decaem mais rápido que o inverso de qualquer polinômio, tais que

(11) $$\lim_{|x|\to\infty} |x|^k D^\alpha \varphi(x) = 0$$

para todo $\alpha = (\alpha_1, \dots, \alpha_n)$, α_i inteiro não negativo, e todo inteiro $k \geq 0$. α é um multi-índice, com a convenção

(12) $$D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}}, |\alpha| = \alpha_1 + \cdots + \alpha_n, \alpha_i \in \{0, 1, 2, \dots\}.$$

D^0 é o operador identidade, D^α um operador diferencial. Um exemplo de função deste espaço é $u(x) = P(x)e^{-|x|^2}$, onde $P(x)$ é uma função polinomial.

Valem as seguintes propriedades [4]:

1) $S(\mathbb{R}^n)$ é um espaço vetorial; ele é fechado sobre combinações lineares.

2) $S(\mathbb{R}^n)$ é uma álgebra; o produto de funções em $S(\mathbb{R}^n)$ também pertence a $S(\mathbb{R}^n)$.

3) $S(\mathbb{R}^n)$ é fechado sobre multiplicação por polinômios.

4) $S(\mathbb{R}^n)$ é fechado sobre diferenciação.

5) $S(\mathbb{R}^n)$ é fechado sobre translações e multiplicação por exponenciais complexos ($e^{ix\cdot\xi}$).

6) funções de $S(\mathbb{R}^n)$ são integráveis: $\int_{\mathbb{R}^n} |f(x)| dx < \infty$ para $f \in S(\mathbb{R}^n)$. Isto segue do fato de que $|f(x)| \leq M(1 + |x|)^{-(n+1)}$ e, usando coordenadas polares, $\int_{\mathbb{R}^n} (1 + |x|)^{-(n+1)} dx = C \int_0^\infty (1 + r)^{-n-1}\, r^{n-1} dr < \infty$, i.e., o integrando decresce como r^{-2} (e $(1 + r)^{-2}$) no infinito e produz uma integral finita.

Da definição de $S(\mathbb{R}^3)$ e propriedades anteriores vemos que, como $u^0(x) \in S(\mathbb{R}^3)$, então $\int_{\mathbb{R}^3} |u^0(x)| dx \leq \int_{\mathbb{R}^3} M(1 + |x|)^{-4} dx \leq C \int_0^\infty (1 + r)^{-2}\, dr < \infty$ e quadrando $|u^0(x)|$ e $M(1 + |x|)^{-4}$ chegamos à desigualdade $\int_{\mathbb{R}^3} |u^0(x)|^2 dx < \infty$, que contradiz (10).

Outra forma de verificar isso é que o conjunto $S(\mathbb{R}^n)$ está contido em $L^p(\mathbb{R}^n)$ para todo p, $1 \leq p < \infty$ ([5]-[9]), e em particular para $p = 2$ e $n = 3$ segue a finitude de $\int_{\mathbb{R}^3} |u^0(x)|^2 dx$.

Portanto, se a condição (7) for desobedecida, conforme propomos neste artigo, será para $t > 0$, por exemplo, encontrando alguma função $u(x,t)$ da forma $u^0(x)v(x,t)$, $v(x,0) = 1$, ou $u^0(x) + v(x,t)$, $v(x,0) = 0$, com $\int_{\mathbb{R}^3} |v(x,t)|^2 dx \to \infty$ e $\int_{\mathbb{R}^3} |u(x,t)|^2 dx \to \infty$.

§ 3 – Exemplo 1

De fato, escolhendo $u^0(x) \in S(\mathbb{R}^3)$ e $f(x,t) \in S(\mathbb{R}^3 \times [0,\infty))$, obedecendo-se assim (4) e (5), lembrando-se que não precisamos ter $u, p \in S(\mathbb{R}^3 \times [0,\infty))$ como solução, apenas $u, p \in C^\infty(\mathbb{R}^3 \times [0,\infty))$, então é possível construir uma solução para a velocidade da forma $u(x,t) = u^0(x)e^{-t} + v(t)$, com $v(0) = 0$, tal que $\int_{\mathbb{R}^3} |u(x,t)|^2 dx \to \infty$, pois, quando $\int_{\mathbb{R}^3} [|u^0(x)|^2 e^{-t} + 2u^0(x) \cdot v(t)] dx \geq 0$, por exemplo, quando cada componente de $u^0(x)$ tem o mesmo sinal da respectiva componente de $v(t)$ ou o produto entre elas é zero ou $\int_{\mathbb{R}^3} u^0(x) \cdot v(t) dx \geq 0$, teremos $\int_{\mathbb{R}^3} |u(x,t)|^2 dx \geq \int_{\mathbb{R}^3} |v(t)|^2 dx = |v(t)|^2 \int_{\mathbb{R}^3} dx \to \infty$, com $v(t) \neq 0, t > 0$. Também devemos escolher u, u^0 tais que $\nabla \cdot u = \nabla \cdot u^0 = 0$.

Em especial, escolhamos, para $1 \leq i \leq 3$,

(13.1) $$u^0(x) = e^{-(x_1^2+x_2^2+x_3^2)}(x_2x_3, x_1x_3, -2x_1x_2),$$

(13.2) $$v_i(t) = w(t) = e^{-t}(1 - e^{-t}),$$

(13.3) $$u_i(x,t) = u_i^0(x)e^{-t} + v_i(t),$$

(13.4) $$f_i(x,t) = \left(-u_i^0 + e^{-t} \sum_{j=1}^3 u_j^0 \frac{\partial u_i^0}{\partial x_j} + \sum_{j=1}^3 v_j \frac{\partial u_i^0}{\partial x_j} - \nu \nabla^2 u_i^0\right) e^{-t},$$

o que resulta para $p(x,t)$, como a única incógnita ainda a determinar,

(14) $$\nabla p + \frac{\partial v}{\partial t} = 0,$$

e então

(15) $$p(x,t) = -\frac{dw}{dt}(x_1 + x_2 + x_3) + \theta(t).$$

A pressão obtida tem uma dependência temporal genérica $\theta(t)$, que deve ser de classe $C^\infty([0,\infty))$ e podemos supor limitada, e diverge no infinito ($|x| \to \infty$), mas tenderá a zero em todo o espaço com o aumento do tempo (a menos eventualmente de $\theta(t)$), devido ao fator e^{-t} que aparece na derivada de $w(t)$,

(16) $$\frac{dw}{dt} = e^{-t}(2e^{-t} - 1).$$

Neste exemplo $\int_{\mathbb{R}^3} u^0(x) \cdot v(t) dx = 0$, e assim $\int_{\mathbb{R}^3} |u(x,t)|^2 dx \to \infty$ para $t > 0$, como queríamos. Mais simples ainda seria escolher $u^0(x) = 0$.

Interessante observarmos que não ocorre nenhuma descontinuidade na velocidade, nem singularidade (divergência: $|u| \to \infty$), entretanto a energia cinética total em todo o espaço diverge, $\int_{\mathbb{R}^3} |u|^2\, dx \to \infty$. Tivemos como dados de entrada $u^0 \in L^2(\mathbb{R}^3), f \in L^2\big(\mathbb{R}^3 \times [0,\infty)\big)$, mas por solução $u \notin L^2\big(\mathbb{R}^3 \times [0,\infty)\big)$, assim como $p \notin L^2\big(\mathbb{R}^3 \times [0,\infty)\big)$.

§ 4 – Exemplo 2 – Ideia Geral

Outro exemplo interessante, utilizando a mesma velocidade inicial anterior, mas fazendo v depender explicitamente das coordenadas de posição x_1, x_2 nas direções e_1, e_2, além do tempo t, e ser igual a zero na direção e_3, com $v(x,0) = 0$, $\nabla \cdot v = 0$, $v \not\equiv 0$ (v não identicamente nulo), e que também obedece a todas as condições de (1) a (6), é, para $1 \le i \le 3$,

(17.1) $$u^0(x) = e^{-(x_1^2+x_2^2+x_3^2)}(x_2x_3, x_1x_3, -2x_1x_2),$$

(17.2) $$v(x,t) = e^{-t}w(x,t),$$

(17.3) $$w(x,t) = (w_1(x_1,x_2,t), w_2(x_1,x_2,t), 0),$$
$$w(x,0) = 0,\ \nabla \cdot w = 0,\ w_3 = v_3 = 0,\ w \not\equiv 0,$$

(17.4) $$u_i(x,t) = u_i^0(x)e^{-t} + v_i(x,t) = [u_i^0(x) + w_i(x,t)]e^{-t},$$

(17.5) $$f_i(x,t) = \left(-u_i^0 + e^{-t}\sum_{j=1}^3 [u_j^0 \frac{\partial u_i^0}{\partial x_j} + u_j^0 \frac{\partial w_i}{\partial x_j} + w_j \frac{\partial u_i^0}{\partial x_j}] - \nu\nabla^2 u_i^0\right) e^{-t}$$
$$= \left(-u_i^0 + \sum_{j=1}^3 [e^{-t}u_j^0 \frac{\partial u_i^0}{\partial x_j} + u_j^0 \frac{\partial v_i}{\partial x_j} + v_j \frac{\partial u_i^0}{\partial x_j}] - \nu\nabla^2 u_i^0\right) e^{-t},$$

o que resulta para $p(x,t)$, como a única incógnita ainda a determinar,

(18) $$\frac{\partial p}{\partial x_i} + \frac{\partial v_i}{\partial t} + \sum_{j=1}^3 v_j \frac{\partial v_i}{\partial x_j} = \nu\nabla^2 v_i,$$

as equações de Navier-Stokes sem força externa.

Nós sabemos que para $n = 2$ a equação (18) tem solução cuja existência e unicidade já está provada ([10]-[13]), sendo assim, transformemos nosso sistema tridimensional (18) em um sistema bidimensional em v, o que fornecerá como solução uma pressão p e uma velocidade v, *a priori*, com domínio espacialmente bidimensional, i.e., nas variáveis (x_1, x_2, t). Resolvida, por hipótese, a equação (18) acima, com $v(x,0) = 0$, $\nabla \cdot v = 0$, mas v não identicamente nula, acrescentemos a terceira coordenada espacial $v_3 \equiv 0$ na solução definitiva para $u(x,t)$, espacialmente tridimensional, em (17.4), e calculemos a força externa em (17.5). Escolhendo $v \in S(\mathbb{R}^2 \times [0,\infty))$ ou v polinomial, seno, cosseno ou suas somas para ser usada em (18), garantiremos que $f \in S(\mathbb{R}^3 \times [0,\infty))$, obedecendo-se (5), com $u^0 \in S(\mathbb{R}^3)$, conforme (4). Fazendo que v seja limitada em módulo (norma no espaço euclidiano) faremos com que u não divirja em $|x| \to \infty$, que é uma condição fisicamente razoável e desejável em [1]. Construamos então uma velocidade v não identicamente nula, com $v(x,0) = 0$, $\nabla \cdot v = 0$, tal que seja relativamente simples resolver (18), que seja limitada em módulo, possa (de preferência) tender a zero no infinito em ao menos determinadas situações e se possível ser integrável em $\mathbb{R}^2$, seja de classe C^∞ e satisfaça (5).

A equação (18) admitirá ainda uma dependência temporal genérica para a pressão da forma

(19) $$p(x,t) = p_1(x_1, x_2, t) + \theta(t),\ x \in \mathbb{R}^3,$$

i.e., além da solução convencional p_1 para a pressão do problema bidimensional das equações de Navier-Stokes (18) nas variáveis independentes (x_1, x_2, t), acrescente-se a p uma parcela genérica $\theta(t)$ dependente apenas do tempo e/ou uma constante como a solução definitiva da pressão no problema tridimensional original, conforme já vimos em (15).

A infinitude da energia cinética total, neste segundo exemplo, ocorre devido à integração de uma função bidimensional ($|v|^2$ ou $|w|^2$) não identicamente nula no espaço tridimensional infinito ($\mathbb{R}^3$).

A energia cinética total do problema é, para $v = e^{-t}w$,

(20) $$\begin{aligned}\int_{\mathbb{R}^3} |u|^2 dx &= \int_{\mathbb{R}^3} (e^{-2t}|u^0|^2 + 2e^{-t}u^0 \cdot v + |v|^2) dx \\ &= e^{-2t} \int_{\mathbb{R}^3} (|u^0|^2 + 2u^0 \cdot w + |w|^2) dx.\end{aligned}$$

Embora $\int_{\mathbb{R}^3} (|u^0|^2 + 2u^0 \cdot w) dx$ seja finito, das propriedades das funções pertencentes ao espaço de Schwartz e integráveis (o caso $u^0 = 0$ é elementar), a terceira parcela em (20) divergirá em $\mathbb{R}^3$ para $v, w \not\equiv 0$, ainda que possa convergir e ser finita em $\mathbb{R}^2$, ou seja, se $|v|$ não for identicamente nulo e $t > 0$,

(21) $$\int_{\mathbb{R}^3} |v|^2 dx = \int_{-\infty}^{+\infty} \left(\int_{\mathbb{R}^2} |v|^2 dx \right) dx_3 = C_2 \int_{-\infty}^{+\infty} dx_3 \to \infty,$$

donde, para t estritamente positivo e finito,

(22) $$\int_{\mathbb{R}^3} |u|^2 dx \to \infty, t > 0, v \not\equiv 0,$$

a violação da condição (7).

§ 5 – Exemplo 2 – Solução Exata

Vamos agora resolver (18) de maneira explícita, primeiramente no domínio $\mathbb{R}^2 \times [0, \infty)$. No exemplo 3 seu domínio será $\mathbb{R}^3 \times [0, \infty)$. Mostraremos que uma solução do tipo

(23) $$v(x_1, x_2, t) = (X(x_1 - x_2)T(t),\ X(x_1 - x_2)T(t)),$$

com uma pressão dada tal que

(24) $$\frac{\partial p}{\partial x_1} = -\frac{\partial p}{\partial x_2} = aQ(x_1 - x_2)R(t) + b,$$

a, b constantes, $a \neq 0, Q$ função da diferença das coordenadas espaciais, R função do tempo, Q, R funções não identicamente nulas, resolve (18) e elimina seu termo não linear, e nesse caso se $T(0) = 0$ resolve-se (17) e o sistema (1), (2), (3) original. X e T não identicamente nulas, evidentemente.

Se $v_i = v_j = V$ em (18), teremos para os seus termos não lineares

(25) $$\sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial x_j} = \sum_{j=1}^{3} V \frac{\partial V}{\partial x_j} = V \sum_{j=1}^{3} \frac{\partial V}{\partial x_j}.$$

Fazendo $\sum_{j=1}^{3} \frac{\partial V}{\partial x_j} = 0$ em (25) elimina-se então o termo não linear, igualdade que é verdadeira quando a condição necessária de fluídos incompressíveis imposta por nós, $\nabla \cdot v = 0$, é satisfeita, i.e.,

(26) $$\sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} = \sum_{j=1}^{3} \frac{\partial V}{\partial x_j} = 0.$$

Definindo $V(x, t) = X\big(\xi(x)\big)T(t)$, com $x \in \mathbb{R}^n$, então

(27)
$$\sum_{j=1}^{n} \frac{\partial V}{\partial x_j} = T(t) \sum_{j=1}^{n} \frac{\partial X(\xi(x))}{\partial x_j} = T(t) \sum_{j=1}^{n} X'(\xi) \frac{\partial \xi(x)}{\partial x_j} = T(t)X'(\xi) \sum_{j=1}^{n} \frac{\partial \xi(x)}{\partial x_j}.$$

Funções $\xi(x)$ tais que $\sum_{j=1}^{n} \frac{\partial \xi(x)}{\partial x_j} = 0$ resultarão então em $\sum_{j=1}^{n} \frac{\partial V}{\partial x_j} = 0$, conforme (27), a exemplo de $\xi = x_1 - x_2$ em dimensão espacial $n = 2$, tal qual utilizado em (23).

Substituindo (24) em (18), já sem os termos não lineares $\sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial x_j}$, e por simplicidade fazendo $a = 1, b = 0$, vem

(28) $$Q(x_1 - x_2)R(t) + \frac{\partial V}{\partial t} = \nu\nabla^2 V,$$

com $V = X(x_1 - x_2)T(t)$. Transformamos assim um sistema de n equações diferenciais parciais não lineares em uma única equação diferencial parcial linear.

Definindo $\xi = x_1 - x_2$, a equação (28) fica

(29) $$Q(\xi)R(t) + X(\xi)\frac{dT}{dt} = \nu T\nabla^2 X(\xi).$$

Queremos obter uma função $T(t)$ tal que $T(0) = 0$, para que em $t = 0$ tenhamos $v(x,0) = 0$, conforme (23). Escolhamos, por exemplo, dentre infinitas outras possibilidades,

(30) $$T(t) = (1 - e^{-t})e^{-t},$$

função limitada no intervalo $0 \le T(t) \le 1,\ t \ge 0$, que vai a zero para $t \to \infty$.

Assim, de (29), com

(31) $$\frac{dT}{dt} = e^{-t}(2e^{-t} - 1),$$

vem

(32) $$Q(\xi)R(t) + X(\xi)e^{-t}(2e^{-t} - 1) = \nu(1 - e^{-t})e^{-t}\nabla^2 X(\xi).$$

Definindo $Q(\xi) = X(\xi)$ em (32), a fim de separar nossa equação com o tradicional método de separação de variáveis usado na teoria de E.D.P.,

(33) $$[R(t) + e^{-t}(2e^{-t} - 1)]X(\xi) = \nu(1 - e^{-t})e^{-t}\nabla^2 X(\xi).$$

A equação diferencial parcial linear (33) pode ser resolvida por algumas alternativas de combinações:

(34) $$\begin{cases} R(t) + e^{-t}(2e^{-t} - 1) = \pm\nu(1 - e^{-t})e^{-t} \\ X(\xi) = \pm\nabla^2 X(\xi) \end{cases}$$

ou

(35) $$\begin{cases} R(t) + e^{-t}(2e^{-t} - 1) = \pm(1 - e^{-t})e^{-t} \\ X(\xi) = \pm\nu\nabla^2 X(\xi) \end{cases}$$

ou de forma mais geral, com $\nu_1 \cdot \nu_2 = \nu > 0,\ \nu_1, \nu_2 > 0$,

(36) $$\begin{cases} R(t) + e^{-t}(2e^{-t} - 1) = \pm\nu_1(1 - e^{-t})e^{-t} \\ X(\xi) = \pm\nu_2\nabla^2 X(\xi) \end{cases}$$

A equação diferencial de segunda ordem em X nos sistemas acima, dependendo de qual dos sinais usemos em $\pm$, remete-nos à Equação de Helmholtz (sinal negativo) ou a algum movimento em estado estacionário regido pela Equação de Schrödinger independente do tempo (sinal positivo ou negativo).

Não pretendendo usar nenhuma condição de contorno específica para $X(\xi)$ e que nos faça recorrer às séries e integrais de Fourier, escolhemos aqui o sinal negativo em $\pm$ (a opção deve ser a mesma nas duas equações do sistema), e façamos X ser uma função trigonométrica, soma de seno e cosseno em ξ, i.e.,

(37) $$X(\xi) = A\, cos(B\xi) + C\, sen(D\xi).$$

Com $\xi = x_1 - x_2$ temos

(38) $$\begin{aligned} \nabla^2 X &= \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)[A\, cos(B\xi) + C\, sen(D\xi)] \\ &= \frac{\partial^2}{\partial x_1^2}[A\, cos(B\xi) + C\, sen(D\xi)] + \frac{\partial^2}{\partial x_2^2}[A\, cos(B\xi) + C\, sen(D\xi)] \\ &= -2[AB^2\, cos(B\xi) + CD^2\, sen(D\xi)]. \end{aligned}$$

De $X(\xi) = -\nu_2 \nabla^2 X(\xi)$ em (36) vem

(39) $$\nu_2 = \frac{1}{2B^2} = \frac{1}{2D^2},\ \nu_1 = 2B^2\nu = 2D^2\nu,\ |B| = |D|,$$

quaisquer que sejam os valores de A e C (se $A = C = 0$ ou $B = D = 0$ teremos a solução trivial e indesejada $v(x,t) \equiv 0$).

A solução para $R(t)$ que se obtém é então, usando $\nu_1 = 2B^2\nu$ dado em (39) e o sinal negativo em (36),

(40) $$R(t) = -e^{-t}[2B^2\nu(1 - e^{-t}) + 2e^{-t} - 1],$$

valendo $R(0) = -1$.

De (23), (30) e (37) chega-se, como um caso possível de solução, para $x \in \mathbb{R}^3$ e introduzindo implicitamente a terceira coordenada espacial $v_3 \equiv 0$ em v, a

(41) $$\begin{aligned} v(x,t) &= X(x_1 - x_2)T(t)(1,1,0) \\ &= [A\, cos(B\xi) + C\, sen(\pm B\xi)](1 - e^{-t})e^{-t}(1,1,0), \end{aligned}$$

que como podemos perceber não é de fato uma solução única para a velocidade, devido às infinitas possibilidades que tivemos para definir a dependência temporal $T(t)$, bem como a dependência espacial $X(\xi)$, $\xi = x_1 - x_2$, além das constantes arbitrárias A, B, C em (41). Mesmo sem unicidade de solução, ela satisfaz aos requisitos que esperávamos: é limitada, contínua de classe C^∞, igual a zero no instante inicial, tende a zero com o aumento do tempo, e tem divergente nulo ($\nabla \cdot v = 0$). Além disso, quando utilizada na expressão (17.5) obtida para a força externa, não retira da força f a condição de pertencer ao espaço de Schwartz em relação ao espaço $\mathbb{R}^3$ e ao tempo, i.e., $f \in S(\mathbb{R}^3 \times [0, \infty))$, conforme é possível provar das propriedades de S que vimos na seção § 2 anterior.

A pressão é obtida integrando-se (24) em relação à diferença $\xi = x_1 - x_2$, com $a = 1, b = 0, Q(\xi) = X(\xi)$ e $R(t)$ dado em (40),

(42) $$\begin{aligned} p(x,t) - p_0(t) &= R(t) \int_{\xi_0}^{\xi} Q(\xi) d\xi \\ &= -e^{-t}[2B^2\nu(1 - e^{-t}) + 2e^{-t} - 1]\, S(\xi), \\ S(\xi) &= \tfrac{A}{B}[sen(B\xi) - sen(B\xi_0)] \pm \tfrac{A}{B}[cos(\pm B\xi) - cos(\pm B\xi_0)], \end{aligned}$$

onde ξ_0 é a superfície $\xi = \xi_0$ e onde a pressão é p_0 no instante t. Novamente vemos que esta solução não é única, não apenas devido exclusivamente à função $p_0(t)$ e respectivo ξ_0, mas também devido às constantes arbitrárias A e B, o sinal $\pm$, além da maneira como $R(t)$ e $Q(\xi)$ foram obtidas, com certa liberdade de possibilidades. $p_0(t)$ substitui a função $\theta(t)$ usada em (15) e (19), nossa função genérica do tempo, ou uma constante, que deve ser de classe $C^\infty([0, \infty))$ e podemos supor limitada.

Completando a solução principal (p, u) que buscamos para a equação (1), temos finalmente

(43) $$u(x,t) = u^0(x)e^{-t} + v(x,t),$$

com $u^0(x)$ dado em (17.1), $v(x,t)$ em (41) e $f(x,t)$ em (17.5).

A velocidade (secundária) v que escolhemos torna a velocidade (principal) u uma função com algumas propriedades semelhantes a ela: u é limitada oscilante, contém uma soma de seno e cosseno em relação à diferença das coordenadas espaciais, e decai exponencialmente em relação ao tempo, ou seja, não pertence a um espaço de Schwartz em relação à posição, nem é de quadrado integrável (violando assim a inequação (7) em $t > 0$), mas é contínua de classe C^∞ e não diverge quando $|x| \to \infty$. Seu comportamento em relação a $x_1 - x_2$ e a divergência da energia cinética total, obviamente, não retiram de $f(x,t)$ a condição de ser pertencente a $S(\mathbb{R}^3 \times [0, \infty))$, equivalente à inequação (5), já que esta só depende de $u^0(x)$ e $v(x,t)$. Também temos $v(x, 0) = 0$, $\nabla \cdot v = 0$, $v \in C^\infty(\mathbb{R}^3 \times [0, \infty))$, a validade de (1), (2), (3), (4) e (6), $u(x, 0) = u^0(x)$, $u^0 \in S(\mathbb{R}^3)$, com $\nabla \cdot u = 0$ e $u \in C^\infty(\mathbb{R}^3 \times [0, \infty))$, conforme queríamos.

§ 6 – A não unicidade em dimensão espacial $n = 2$

O que há com as provas de unicidade das soluções das equações de Navier-Stokes em dimensão espacial $n = 2$?

Não sendo possível analisar todas as provas existentes, é possível ao menos entender que tais provas não devem levar em consideração a ausência do termo não linear nas Equações de Navier-Stokes, $\sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} \equiv ((u \cdot \nabla)u)_i, 1 \leq i \leq n$, e foi a esta ausência que recorremos em nosso segundo exemplo.

Semelhantemente a esta causa, também se percebe que diferentes equações do tipo de Navier-Stokes, com ausência de um ou mais termos da respectiva equação completa, e que não obstante tenham a mesma condição inicial $u(x,0) = u^0(x)$, terão provavelmente, no caso geral, diferentes soluções $u(x,t)$ entre elas, e assim não poderá haver unicidade de solução em relação à equação de Navier-Stokes completa, com todos os termos. Se todas apresentassem sempre a mesma e única solução, bastaria para nós resolver somente a mais simples delas, por exemplo, $\nabla p = -\frac{\partial u}{\partial t}$ ou $\nabla p = \nu \nabla^2 u$ (Equação de Poisson se $\nabla p \not\equiv 0$ ou de Laplace se $\nabla p \equiv 0$) ou $\frac{\partial u}{\partial t} = \nu \nabla^2 u$ (Equação do Calor com $\nabla p = 0$), todas com $u(x,0) = u^0(x)$, e conferir se a soma dos demais termos faltantes é igual a zero ao aplicar a solução u obtida na equação reduzida. Se sim, a solução da equação reduzida é também solução da equação completa. Importante exemplo desta ausência são as Equações de Euler, que diferem das Equações de Navier-Stokes pela ausência do operador diferencial laplaciano aplicado a u, $\nabla^2 u \equiv \Delta u$, devido ao coeficiente de viscosidade ser nulo, $\nu = 0$.

É fácil provar que as três equações acima, assim como a equação $\nabla p + \frac{\partial u}{\partial t} = \nu \nabla^2 u$, não podem realmente ter uma única solução, dada apenas a condição inicial para a velocidade $u(x,0) = u^0(x)$. Pelo contrário, a forma completa das equações de Navier-Stokes, onde supomos que $\sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} \equiv ((u \cdot \nabla)u)_i \not\equiv 0,\ 1 \leq i \leq n$, tem unicidade de solução para $n = 2$ e em ao menos um pequeno intervalo de tempo não nulo $[0,T]$ para $n = 3$, onde T é conhecido como *blowup time*. Acrescentemos em todas estas equações a condição de incompressibilidade, $\nabla \cdot u = 0$.

Trata-se assim de um interessante problema de Análise Combinatória aplicada à Análise Matemática e Física-Matemática.

§ 7 – Unicidade em dimensão espacial $n = 2$

Verificamos na seção § 5 que o sistema

$$(44) \qquad \begin{cases} \nabla p + \frac{\partial v}{\partial t} = \nu \nabla^2 v \\ (v \cdot \nabla)v = 0 \\ \nabla \cdot v = 0 \\ v(x,0) = 0 \end{cases}$$

tem infinitas soluções para a velocidade da forma

(45) $$v(x,t) = X(\xi)T(t)(1,1), \xi = x_1 - x_2,$$

com $T(0) = 0$, não obstante existem as provas conhecidas da unicidade de

(46) $$\begin{cases} \nabla p + \frac{\partial v}{\partial t} + (v \cdot \nabla)v = \nu \nabla^2 v \\ \nabla \cdot v = 0 \\ v(x,0) = 0 \end{cases}$$

contradizendo o que obtivemos.

Sem ser necessário nos demorarmos nas provas conhecidas, expondo todos os seus detalhes, repetindo suas passagens, é possível constatar em Leray [10], Ladyzhenskaya [11], Kreiss and Lorenz [14], dentre outros, que as provas de existência e unicidade baseiam-se na forma completa das equações de Navier-Stokes, por exemplo (46), e não em uma forma desmembrada das equações de Navier-Stokes, como (44).

As equações de Navier-Stokes sem força externa com $n = 2$ são (usando $x \equiv x_1$ e $y \equiv x_2$)

(47) $$\begin{cases} \frac{\partial p}{\partial x} + \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} = \nu \nabla^2 u_1 \\ \frac{\partial p}{\partial y} + \frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} = \nu \nabla^2 u_2 \end{cases}$$

Podemos dispor o sistema acima de forma parecida com um sistema de equações lineares,

(48) $$\begin{cases} u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} = \nu \nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t} \\ u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} = \nu \nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t} \end{cases}$$

e a seguir em forma de uma equação matricial,

(49) $$\begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \nu \nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t} \\ \nu \nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t} \end{pmatrix}.$$

Chamando

(50) $$A = \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} \end{pmatrix},$$

(51) $$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

(52) $$B = \begin{pmatrix} \nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t} \\ \nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t} \end{pmatrix},$$

a solução para U da equação (49), $AU = B$, é

(53) $$U = A^{-1}B,$$

que para existir e ter solução única deve-se ter

(54) $$\det A = \frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x} \neq 0,$$

ou seja,

(55) $$\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} \neq \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x},$$

regra que também deve ser obedecida para $t = 0$ (de novo pode nos levar aos casos (C) e (D) de [1] aplicando-se o método em matriz 3×3, i.e., $n = 3$, entretanto, com conveniente escolha de p ou $\partial u/\partial t$ o sistema será possível).

Se usarmos a condição de incompressibilidade $\nabla \cdot u = 0$,

(56) $$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0,$$

i.e.,

(57) $$\frac{\partial u_1}{\partial x} = -\frac{\partial u_2}{\partial y},$$

transforma-se a condição (55) em

(58) $$-\left(\frac{\partial u_2}{\partial y}\right)^2 \neq \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x},$$

ou equivalentemente,

(59) $$-\left(\frac{\partial u_1}{\partial x}\right)^2 \neq \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}.$$

Como esta condição deve ser válida para todo t, em $t = 0$ deve-se obedecer a

(60) $$-\left(\frac{\partial u_1^0}{\partial x}\right)^2 \neq \frac{\partial u_1^0}{\partial y}\frac{\partial u_2^0}{\partial x}$$

e

(61) $$-\left(\frac{\partial u_2^0}{\partial y}\right)^2 \neq \frac{\partial u_1^0}{\partial y}\frac{\partial u_2^0}{\partial x},$$

usando $u(x,y,0) = u^0(x,y) = (u_1^0(x,y), u_2^0(x,y))$.

Se a velocidade inicial u^0 for tal que sejam desobedecidas (60) ou (61) então ou não haverá solução para o sistema (47) (sistema impossível) ou não haverá uma única solução (sistema indeterminado), tal como na teoria de sistemas lineares.

Definindo

(62) $$U_1 = \begin{pmatrix} \nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t} & \frac{\partial u_1}{\partial y} \\ \nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t} & \frac{\partial u_2}{\partial y} \end{pmatrix}$$

e

(63) $$U_2 = \begin{pmatrix} \frac{\partial u_1}{\partial x} & \nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial x} & \nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t} \end{pmatrix},$$

a solução para u_1, u_2 será

(64) $$u_1 = \frac{\det U_1}{\det A}$$

e

(65) $$u_2 = \frac{\det U_2}{\det A}.$$

Sendo

(66) $$\det U_1 = \left(\nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t}\right)\frac{\partial u_2}{\partial y} - \left(\nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t}\right)\frac{\partial u_1}{\partial y}$$

e

(67) $$\det U_2 = \left(\nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t}\right)\frac{\partial u_1}{\partial x} - \left(\nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t}\right)\frac{\partial u_2}{\partial x},$$

com $\det A$ dado em (54), temos então

(68) $$u_1 = \frac{\det U_1}{\det A} = \frac{\left(\nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t}\right)\frac{\partial u_2}{\partial y} - \left(\nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t}\right)\frac{\partial u_1}{\partial y}}{\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}}$$

e

$$(69)\qquad u_2 = \frac{\det U_2}{\det A} = \frac{\left(\nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t}\right)\frac{\partial u_1}{\partial x} - \left(\nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t}\right)\frac{\partial u_2}{\partial x}}{\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}}.$$

Usando a equação de incompressibilidade no determinante de A,

$$(70)\qquad u_1 = -\frac{\left(\nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t}\right)\frac{\partial u_2}{\partial y} - \left(\nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t}\right)\frac{\partial u_1}{\partial y}}{\left(\frac{\partial u_2}{\partial y}\right)^2 + \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}}$$

e

$$(71)\qquad u_2 = -\frac{\left(\nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t}\right)\frac{\partial u_1}{\partial x} - \left(\nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t}\right)\frac{\partial u_2}{\partial x}}{\left(\frac{\partial u_1}{\partial x}\right)^2 + \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}}.$$

É verdade que as soluções (equações) acima são tão ou mais complicadas quanto às equações originais (47), e parece não haver utilidade em resolvê-las.

Mas desta forma complicada se pode chegar com mais certeza à seguinte constatação: as equações de Navier-Stokes (e Euler) têm uma simetria entre as variáveis, tanto as dependentes quanto as independentes. A mesma também pode ser percebida diretamente em (47).

A simetria neste caso de $n = 2$ é

(72.1) $\quad u_1 \leftrightarrow u_2$

(72.2) $\quad x \leftrightarrow y$

ficando p e t inalterados:

(73.1) $\quad p \leftrightarrow p$

(73.2) $\quad t \leftrightarrow t.$

Isso sugere, se não resolve completamente, a questão da solução destas equações. Se as equações em si são simétricas em relação a determinadas transformações, então esperamos que suas soluções também o sejam sob estas transformações. O mesmo método pode ser aplicado também para $n \geq 3$, com a regra (por exemplo)

(74.1) $\quad u_i \mapsto u_{i+1},\ u_{n+1} \equiv u_1,$

(74.2) $\quad x_i \mapsto x_{i+1},\ x_{n+1} \equiv x_1,$

(74.3) $\quad p \leftrightarrow p,$

(74.4) $\quad t \leftrightarrow t.$

Nesse caso é preciso que a condição inicial $u(x,0) = u^0(x)$ obedeça também a estas simetrias, mas permanece inalterada a condição de incompressibilidade: $\sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial u_i^0}{\partial x_i} = 0$.

Se fornecemos $u_2(x,y,t)$ como dado de entrada no nosso sistema então podemos concluir que a solução para u_1, supostamente simétrica a u_2 pela regra (72) anterior, seja

(75) $$u_1(x,y,t) = u_2(y,x,t),$$

i.e., trocamos x por y, e vice-versa, na solução dada previamente para u_2 e igualamos a u_1 o resultado desta transformação. Restará obter a pressão p ou então, caso ela também tenha sido dada, verificar se as variáveis u_1, u_2, p realmente satisfazem o sistema original.

A forma geral da solução para a pressão p, que deve satisfazer

(76) $$\nabla p + \frac{\partial u}{\partial t} + (u \cdot \nabla)u = \nu \nabla^2 u,$$

é

(77) $$p - p_0(t) = \int_{(x_0,y_0)}^{(x,y)} \left[\nu \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla)u\right] \cdot dl,$$

onde supomos que na posição $(x,y) = (x_0,y_0)$ e no instante t a pressão é igual a $p_0(t)$. A integração se dá em qualquer caminho entre (x_0,y_0) e (x,y), pois a pressão deve ser uma função potencial do integrando de (77) para que (47) tenha solução.

É de se esperar ainda que p seja simétrica em relação às variáveis x e y, ou seja,

(78) $$p(x,y,t) = p(y,x,t),$$

assim como em 3 dimensões, usando $x \equiv x_1, y \equiv x_2, z \equiv x_3$,

(79) $$p(x,y,z,t) = p(y,z,x,t) = p(z,x,y,t).$$

Claro que (74), (75), (78) e (79) admitem implicitamente que temos simetria retangular nas condições iniciais e de contorno do sistema. Uma vez que esta simetria não ocorra, por exemplo, tenhamos outro tipo de simetria, esférica, cilíndrica, ou mesmo simetria nenhuma (caso geral), as igualdades (74), (75), (78) e (79) não têm necessidade de serem satisfeitas. Sendo assim, a solução para o caso em que não há simetria alguma ainda é um problema a resolver, admitindo-se que há ao menos uma solução (quando o sistema é possível; conforme dissemos, pode-se provar que o sistema sempre é possível, por exemplo, com escolha apropriada de p ou $\partial u/\partial t$).

Finalmente então, desenvolvidas as considerações anteriores, nosso exemplo 3, que busca uma solução única para o sistema de Navier-Stokes em $n = 3$, com todos os termos da equação, força externa não nula, e que fornece energia cinética total infinita para o sistema (1) a (6) em $t > 0$, será baseado no exemplo 2, mas precisaremos novamente recorrer à ausência do termo não linear na equação auxiliar com $n = 3$. Uma vez que (18), a Equação de Navier-Stokes sem força externa, tem como condição inicial a velocidade inicial nula, a única velocidade possível para sua solução com todos os termos é também a velocidade nula, devido à unicidade das soluções na forma completa desta equação (abstraindo-se de pressões genéricas constantes e/ou funções do tempo), solução que não nos interessa. Sendo assim, precisaremos novamente que (18) não tenha o termo não linear. A unicidade da solução da equação principal em três dimensões, entretanto, ao menos em pequeno intervalo de tempo, é garantida por esta conter todos os termos (novamente, exceção feita à pressão não única), inclusive a força externa aplicada (que por si depende da solução não única da equação auxiliar com $n = 3$).

§ 8 – Exemplo 3

O terceiro exemplo é uma generalização do exemplo 2, com as componentes de velocidade v_2 e v_3 proporcionais à componente v_1,

(80.1) $$v_1 = X(\xi)T(t),\ \xi = x_1 + \frac{1}{\alpha}x_2 - 2\frac{1}{\beta}x_3,\ \alpha \neq 0,\ \beta \neq 0,$$

(80.2) $$v_2 = \alpha v_1,$$

(80.3) $$v_3 = \beta v_1,$$

α e β constantes não nulas. Também poderíamos usar outras combinações de coeficientes nas variáveis x_i em ξ, desde que $\nabla \cdot (\xi I) = 0$, com $I = (1, 1,1)$. No exemplo 2 usamos $\alpha = 1,\ \beta = 0$.

Vamos escolher as componentes da velocidade inicial u^0 com alguma propriedade de simetria. Não é fácil pensar em velocidades não constantes com componentes simétricas u_i^0 e ao mesmo tempo cujo divergente $\nabla \cdot u^0$ seja nulo. As velocidades com simetria cuja i-ésima componente não contém a i-ésima coordenada espacial, para todo i (natural) em $1 \leq i \leq n$, cumprem este requisito: $\frac{\partial u_i^0}{\partial x_i} = 0$. Alternativamente podemos utilizar a conhecida igualdade vetorial $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$, ou seja, escolher um vetor u^0 que tenha um potencial vetor $\boldsymbol{A}$, i.e., $u^0 = \nabla \times \boldsymbol{A}$. Então escolhamos primeiramente um vetor $\boldsymbol{A}$ que tenha as propriedades de simetria que esperamos.

Seja $\boldsymbol{A} = (A_1, A_2, A_3)$ o potencial vetor que queremos. Fazendo $A_1 = A_2 = A_3 = e^{-r^2}$, com $r^2 = x_1^2 + x_2^2 + x_3^2$, o valor que atribuímos para a velocidade inicial $u^0(x)$ será

(81) $$u^0(x) = rot\,\boldsymbol{A} = 2e^{-r^2}(-x_2 + x_3, -x_3 + x_1, -x_1 + x_2).$$

Seguindo as equações 17 do exemplo 2, façamos agora para $x \in \mathbb{R}^3$,

(82.1) $$v(x,t) = e^{-t} w(x,t),$$

(82.2) $$w(x,t) = \big(w_1(x_1,x_2,x_3,t), w_2(x_1,x_2,x_3,t), w_3(x_1,x_2,x_3,t)\big),$$
$$w(x,0) = 0,\ \nabla \cdot w = 0,\ w \not\equiv 0,$$

(82.3) $$u_i(x,t) = u_i^0(x)e^{-t} + v_i(x,t) = [u_i^0(x) + w_i(x,t)]e^{-t},$$

(82.4) $$f_i(x,t) = \left(-u_i^0 + e^{-t}\textstyle\sum_{j=1}^3 [u_j^0 \frac{\partial u_i^0}{\partial x_j} + u_j^0 \frac{\partial w_i}{\partial x_j} + w_j \frac{\partial u_i^0}{\partial x_j}] - \nu\nabla^2 u_i^0\right) e^{-t}$$
$$= \left(-u_i^0 + \textstyle\sum_{j=1}^3 [e^{-t} u_j^0 \frac{\partial u_i^0}{\partial x_j} + u_j^0 \frac{\partial v_i}{\partial x_j} + v_j \frac{\partial u_i^0}{\partial x_j}] - \nu\nabla^2 u_i^0\right) e^{-t},$$

o que resulta para $p(x,t)$ e $v(x,t)$, como incógnitas ainda a determinar,

(83) $$\frac{\partial p}{\partial x_i} + \frac{\partial v_i}{\partial t} + \textstyle\sum_{j=1}^3 v_j \frac{\partial v_i}{\partial x_j} = \nu\nabla^2 v_i,$$

as equações de Navier-Stokes sem força externa.

As equações (80) aplicadas em (83) resultam em

(84) $$\begin{cases} \dfrac{\partial p}{\partial x_1} + \dfrac{\partial v_1}{\partial t} + v_1\left(\dfrac{\partial v_1}{\partial x_1} + \alpha\dfrac{\partial v_1}{\partial x_2} + \beta\dfrac{\partial v_1}{\partial x_3}\right) = \nu\nabla^2 v_1 \\ \dfrac{\partial p}{\partial x_2} + \alpha\dfrac{\partial v_1}{\partial t} + \alpha v_1\left(\dfrac{\partial v_1}{\partial x_1} + \alpha\dfrac{\partial v_1}{\partial x_2} + \beta\dfrac{\partial v_1}{\partial x_3}\right) = \nu\alpha\nabla^2 v_1 \\ \dfrac{\partial p}{\partial x_3} + \beta\dfrac{\partial v_1}{\partial t} + \beta v_1\left(\dfrac{\partial v_1}{\partial x_1} + \alpha\dfrac{\partial v_1}{\partial x_2} + \beta\dfrac{\partial v_1}{\partial x_3}\right) = \nu\beta\nabla^2 v_1 \end{cases}$$

Como

(85) $$\frac{\partial v_1}{\partial x_1} + \alpha\frac{\partial v_1}{\partial x_2} + \beta\frac{\partial v_1}{\partial x_3} = T(t)\frac{dX}{d\xi}\left(\frac{\partial \xi}{\partial x_1} + \alpha\frac{\partial \xi}{\partial x_2} + \beta\frac{\partial \xi}{\partial x_3}\right)$$
$$= T(t)\frac{dX}{d\xi}(1 + 1 - 2) = 0,$$

pela definição de ξ que usamos em (80.1), então (84) fica

(86) $$\begin{cases} \dfrac{\partial p}{\partial x_1} + \dfrac{\partial v_1}{\partial t} = \nu\nabla^2 v_1 \\ \dfrac{\partial p}{\partial x_2} + \alpha\dfrac{\partial v_1}{\partial t} = \alpha\nu\nabla^2 v_1 \\ \dfrac{\partial p}{\partial x_3} + \beta\dfrac{\partial v_1}{\partial t} = \beta\nu\nabla^2 v_1 \end{cases}$$

ou equivalentemente,

(87) $$\begin{cases} \dfrac{\partial p}{\partial x_1} = \nu\nabla^2 v_1 - \dfrac{\partial v_1}{\partial t} \\ \dfrac{\partial p}{\partial x_2} = \alpha\left[\nu\nabla^2 v_1 - \dfrac{\partial v_1}{\partial t}\right] = \alpha\dfrac{\partial p}{\partial x_1} \\ \dfrac{\partial p}{\partial x_3} = \beta\left[\nu\nabla^2 v_1 - \dfrac{\partial v_1}{\partial t}\right] = \beta\dfrac{\partial p}{\partial x_1} \end{cases}$$

Semelhantemente ao que vimos na seção § 5, equação (24), para $a = 1$ e $b = 0$, vamos fazer a pressão ser definida como

(88) $$\frac{\partial p}{\partial \xi} = Q(\xi)R(t),$$

e a velocidade

(89) $$v_i = c_i\, X\big(\xi(x)\big)\, T(t),\ c_1 = 1, c_2 = \alpha, c_3 = \beta,$$

com ξ definido em (80.1),

(90) $$\xi = x_1 + \frac{1}{\alpha}x_2 - 2\frac{1}{\beta}x_3,\ \alpha \neq 0,\ \beta \neq 0.$$

Será suficiente então, além da equação (88) para a pressão, resolvermos uma única equação diferencial parcial linear, envolvendo v_1, ao invés de um sistema de três equações diferenciais parciais não lineares, envolvendo v_1, v_2, v_3.

O desenvolvimento da solução aqui segue os mesmos passos já vistos na seção § 5, equações (29) a (43), sendo a principal mudança a expressão para ξ dada em (90), com o aumento de dimensões e a proporcionalidade entre v_2, v_3 e v_1. Chegamos a

(91) $$\begin{aligned} v(x,t) &= X\left(x_1 + \frac{1}{\alpha}x_2 - 2\frac{1}{\beta}x_3\right)T(t)(1,\ \alpha,\ \beta) \\ &= [A\cos(B\xi) + C\, sen(\pm B\xi)](1 - e^{-t})e^{-t}(1,\ \alpha,\ \beta),\ \alpha, \beta \neq 0, \end{aligned}$$

mantendo-se válidas as soluções (42) e (43) para a pressão p e velocidade u, respectivamente. Velocidade inicial igual a (81). Também temos a validade de $\nabla \cdot v = 0$ e a correspondente integral $\int_{\mathbb{R}^3} |v|^2 dx$ infinita, parcela da energia cinética total do sistema (1) a (6).

§ 9 – Conclusão

Todos os três exemplos obedecem às condições de divergência nula (*divergence-free*, $\nabla \cdot u^0 = 0$), suavidade (*smoothness*, C^∞) e derivadas parciais de u^0 e f da ordem de $C_{\alpha K}\,(1 + |x|)^{-K}$ e $C_{\alpha m K}\,(1 + |x| + t)^{-K}$, respectivamente. Concluímos que deve ser $u^0 \in S(\mathbb{R}^3)$ e $f \in S(\mathbb{R}^3 \times [0, \infty))$. Para cada $u(x,t)$ possível tal que (3) seja verdadeira, a força externa $f(x,t)$ e a pressão $p(x,t)$ podem ser convenientemente construídas, na classe C^∞, verificando (8), e de modo a satisfazerem todas as condições necessárias, encontrando-se assim uma solução possível para (1), (2), (3), (4), (5) e (6), e apenas (7) não seria satisfeita, para $t > 0$, conforme (10). Mostramos então exemplos de quebra de soluções para o caso (C) deste problema do milênio. Estes exemplos, entretanto, não levam ao caso (A) de [1], de existência e suavidade das soluções, justamente por violarem (7) (O caso (A) também impõe que seja nula a força externa, $f = 0$).

Um resumo das condições do problema está listado abaixo ($:\mathbb{R}^3$ e $:\mathbb{R}^3 \times [0,\infty)$ representam o domínio das respectivas funções).

$\nu > 0, n = 3$

$\exists u^0(x): \mathbb{R}^3$ smooth (C^∞), divergence-free ($\nabla \cdot u^0 = 0$)

$\exists f(x,t): \mathbb{R}^3 \times [0,\infty)$ smooth (C^∞)

(4) $|\partial_x^\alpha u^0(x)| \leq C_{\alpha K}(1+|x|)^{-K}: \mathbb{R}^3, \ \forall \alpha, K$

(5) $|\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1+|x|+t)^{-K}: \mathbb{R}^3 \times [0,\infty)\ , \forall \alpha, m, K$

$\nexists (p,u): \mathbb{R}^3 \times [0,\infty)$ /

(1) $\frac{\partial u_i}{\partial t} + \sum_{j=1}^3 u_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i - \frac{\partial p}{\partial x_i} + f_i(x,t), 1 \leq i \leq 3 \ \ (x \in \mathbb{R}^3, t \geq 0)$

(2) $\nabla \cdot u = 0$

(3) $u(x,0) = u^0(x) \qquad (x \in \mathbb{R}^3)$

(6) $p, u \in C^\infty(\mathbb{R}^3 \times [0,\infty))$

(7) $\int_{\mathbb{R}^3} |u(x,t)|^2 dx < C, \forall t \geq 0$ (bounded energy)

Em todos os três exemplos a velocidade principal u que utilizamos foi da forma

(92) $$u(x,t) = [u^0(x) \ + w(x)(1-e^{-t})]e^{-t};$$

no exemplo 1, $w(x) = 1$, no exemplo 2, $w(x) = X(\xi)(1,1,0)$, $\xi = x_1 - x_2$, e no exemplo 3, $w(x) = X(\xi)(1,\alpha,\beta)$, $\xi = x_1 + \frac{1}{\alpha}x_2 - 2\frac{1}{\beta}x_3$, $\alpha, \beta\ cte. \neq 0$, exemplos 2 e 3 com $X(\xi) = [A\ cos(B\xi) + C\ sen(\pm B\xi)]$.

É importante analisarmos também a questão da unicidade das soluções. Como $u^0(x)$ e $f(x,t)$ são dados, escolhidos por nós, de classe C^∞ e satisfazendo (4) e (5), i.e., pertencentes ao espaço de Schwartz, com $\nabla \cdot u^0 = 0$, afirmar que não existe solução (p,u) para o sistema (1), (2), (3), (6) e (7) pode pressupor que exploramos, ou provamos para, as infinitas combinações possíveis de p e de u, i.e., de (p,u). Sendo assim, precisamos que haja unicidade de solução para cada velocidade que construímos, o que elimina outras velocidades possíveis para os mesmos dados utilizados, $u^0(x)$ e $f(x,t)$, e que implicassem em energia cinética total finita.

A unicidade da solução (a menos da pressão $p(x,t)$ com o termo adicional constante ou dependente do tempo $\theta(t)$, além de outros casos de não unicidade da pressão sobre x e $T(t)$) vem dos resultados clássicos já conhecidos, descritos por exemplo no mencionado artigo de Fefferman [1]: o sistema das equações de Navier-Stokes (1), (2), (3) tem solução (única [15]) para todo $t \geq 0$ ou apenas para um intervalo de tempo $[0,T)$ finito dependente dos dados iniciais, onde T é chamado de "*blowup time*". Quando há uma solução com T finito então a velocidade u torna-se ilimitada próxima do "*blowup time*".

Vemos que a existência de cada solução nossa, nos exemplos dados, está garantida por construção e substituição direta. Nossas velocidades não apresentam nenhum comportamento irregular, em instante t algum, em posição alguma, que as tornem ilimitadas, infinitas, nem mesmo para $t \to \infty$ ou $|x| \to \infty$, sendo assim, não pode haver o "*blowup time*" nos exemplos que demos, portanto cada solução encontrada nos casos anteriores é única em todo tempo (a menos da pressão). Mas ainda que houvesse um T finito (em [14], [16] vemos que $T > 0$), a unicidade existiria em pelo menos um pequeno intervalo de tempo, o que já é suficiente para mostrar que neste intervalo ocorre a quebra das soluções de Navier-Stokes por ser desobedecida a condição de energia cinética limitada (7), tornando o caso (C) verdadeiro.

Entendamos que a unicidade está na velocidade principal u (equação 1), não sendo preciso que esteja também na velocidade secundária v (equações 14, 18 e 83), que conforme vimos nos exemplos 2 e 3 pode ter infinitas soluções, devido à ausência dos n termos não lineares $\sum_{j=1}^{n} v_j \frac{\partial v_i}{\partial x_j}$. Escolhida uma velocidade v, entretanto, aplicando-a na força externa f (equações 13.4, 17.5, 82.4), resulta enfim na unicidade de u (conforme 13.3, 17.4, 43, 82.3), solução de uma equação com todos os termos, de sua energia cinética, e na correspondente divergência da energia cinética total $\int_{\mathbb{R}^3} |u|^2 \, dx$ em $t > 0$ devido ao termo $\int_{\mathbb{R}^3} |v|^2 \, dx \to \infty$. A pressão p, já sabemos, não é única, mas esta não altera, qualitativamente, o fato da energia cinética total do sistema ser infinita ou não. Isto é mais fácil de perceber com os exemplos 1 e 2: fosse v qualquer função constante, ou dependente exclusivamente do tempo, ou com $x \in \mathbb{R}$ ou com $x \in \mathbb{R}^2$, desde que não identicamente nula, e qualquer que fosse a pressão p, nula ou não, a condição (7) seria violada, devido à integração de $|v|^2$ em todo o espaço $\mathbb{R}^3$.

§ 10 – Comentários Finais

Não é difícil estender o resultado obtido anteriormente na seção § 5 com a velocidade bidimensional para uma velocidade v com três componentes espaciais não nulas, conforme vimos na seção § 8.

Nos exemplos 2 e 3 tivemos que resolver uma equação diferencial ordinária para obter $X(\xi)$. Vamos agora, entretanto, achar uma solução não única para a velocidade nas Equações de Navier-Stokes, mas sem precisar resolver nenhuma equação diferencial auxiliar. Só será preciso efetuar uma integração, necessária à obtenção da pressão. A título de curiosidade, a velocidade inicial poderá ser diferente de zero, assim como a força externa aplicada, e não estaremos preocupados em buscar apenas energias cinéticas infinitas ou velocidades pertencentes ao espaço de Schwartz. Não estamos buscando agora uma *breakdown solution*, pelo contrário, buscamos infinitas *solutions*.

Vamos resolver o sistema (1), (2), (3) para o caso especial em que

$$u(x_1, x_2, x_3, t) = X(x_1 + x_2 + x_3)T(t)\ (1, 1, -2) = X(\xi)T(t)J, \quad \xi(x) = x_1 + x_2 + x_3,\ J = (1, 1, -2), \tag{93}$$

valendo $\nabla \cdot (\xi J) = 0$. Isso nos dá $\nabla \cdot u = \nabla \cdot u^0 = 0$ e a eliminação dos termos não lineares $(u \cdot \nabla)u \equiv \left(\sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j},\right)_{1 \le i \le 3} = 0$ das Equações de Navier-Stokes, com ou sem força externa. Assim a solução de (1) será reduzida à solução de uma equação diferencial parcial linear, a Equação do Calor não homogênea tridimensional,

$$\frac{\partial p}{\partial x_i} = \nu \nabla^2 u_i - \frac{\partial u_i}{\partial t} + f_i = \phi_i,\ 1 \le i \le 3, \tag{94}$$

devendo valer

$$\frac{\partial \phi_i}{\partial x_j} = \frac{\partial \phi_j}{\partial x_i},\ i \neq j. \tag{95}$$

Como $\frac{\partial p}{\partial x_i} = \frac{\partial p}{\partial \xi}\frac{\partial \xi}{\partial x_i} = \frac{\partial p}{\partial \xi}$, $\forall i$, assim como os operadores diferenciais $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x_i} = \frac{\partial}{\partial \xi}$ e $\left(\frac{\partial}{\partial x_i}\right)^2 = \left(\frac{\partial}{\partial \xi}\right)^2$, $\forall i$, i.e., temos uma pressão que pode ser expressa como função de ξ, assim como as componentes da velocidade u, e os x_i apresentam-se de forma simétrica e linear em relação a $\xi = x_1 + x_2 + x_3$, com a transformação do elemento infinitesimal de integração $d\xi = \frac{\partial \xi}{\partial x_1}dx_1 + \frac{\partial \xi}{\partial x_2}dx_2 + \frac{\partial \xi}{\partial x_3}dx_3 = dx_1 + dx_2 + dx_3$, a igualdade (95) é verdadeira, é válido $\frac{\partial^2 p}{\partial x_j \partial x_i} = \frac{\partial^2 p}{\partial x_i \partial x_j}$, e teremos a seguinte solução para a pressão:

(96) $$p(x,t) - p_0(t) = \int_{\xi_0}^{\xi(x)} \left(\nu \mathrm{T} \nabla^2 X - X \frac{dT}{dt} + f\right) d\xi,$$

com

(97) $$\frac{\partial p}{\partial x_1} = \frac{\partial p}{\partial x_2} = \frac{\partial p}{\partial x_3},$$

supondo que a força $f(x,t)$ seja da forma $Y(\xi)Z(t)(1,1,-2)$, tal qual $u(x,t) = X(\xi)T(t)(1,1,-2)$. Consideremos $p_0(t)$ como a pressão no instante t e na superfície $\xi = \xi_0$. Isto resolve o sistema que pretendíamos, desde que a integração em (96) seja possível, e assim não precisamos resolver nenhuma equação diferencial ordinária intermediária para encontrarmos $X(\xi)$, pois podemos prefixar qual a expressão para $X(\xi)$ que desejamos utilizar, dentre infinitas possibilidades, e tal que tenhamos $u(x,0) = u^0(x)$.

Outras combinações das componentes do vetor J podem ser usadas, assim como outras combinações dos coeficientes dos x_1, x_2, x_3 em ξ, desde que eliminem-se os termos não lineares e verifique-se (2) e (95). Assim sendo, formas mais complicadas para ξ também são possíveis, além das lineares, o que traz uma robusta maneira de se obter as soluções para u. Por exemplo, definindo-se

(98) $$u_i = \alpha_i(x,t) u_1,\ 1 \leq i \leq n,\ \alpha_1 = 1,$$

a condição a ser obedecida por X e ξ a fim de se eliminar os termos não lineares é

(99) $$\alpha_i \frac{dX(\xi)}{d\xi} \sum_{j=1}^{n} \alpha_j \frac{\partial \xi}{\partial x_j} + X(\xi) \sum_{j=1}^{n} \alpha_j \frac{\partial \alpha_i}{\partial x_j} = 0,$$

para todo i (natural) em $1 \leq i \leq n$. Para cada determinado i elimina-se o termo não linear da respectiva linha (ou coordenada) i se (99) for satisfeita.

Uma maneira de fazer (99) ser verdadeira é quando

(100) $$\sum_{j=1}^{n} \alpha_j \frac{\partial \xi}{\partial x_j} = \sum_{j=1}^{n} \alpha_j \frac{\partial \alpha_i}{\partial x_j} = 0.$$

Quando os α_i são constantes ou dependentes apenas do tempo a condição a ser obedecida para ξ é

(101) $$\sum_{j=1}^{n} \alpha_j \frac{\partial \xi}{\partial x_j} = 0,$$

o que está de acordo com os exemplos 2 e 3 anteriores.

Incluindo-se ainda a condição de incompressibilidade para u, deve ser válida também a relação

(102) $$\sum_{j=1}^{n} \frac{\partial(\alpha_j u_1)}{\partial x_j} = u_1 \sum_{j=1}^{n} \frac{\partial \alpha_j}{\partial x_j} + \sum_{j=1}^{n} \alpha_j \frac{\partial u_1}{\partial x_j}$$
$$= T(t)[X(\xi) \sum_{j=1}^{n} \frac{\partial \alpha_j}{\partial x_j} + \frac{dX(\xi)}{d\xi} \sum_{j=1}^{n} \alpha_j \frac{\partial \xi}{\partial x_j}] = 0.$$

Como (102) deve ser válida para todo t, então precisamos que seja

(103) $$X(\xi) \sum_{j=1}^{n} \frac{\partial \alpha_j}{\partial x_j} + \frac{dX(\xi)}{d\xi} \sum_{j=1}^{n} \alpha_j \frac{\partial \xi}{\partial x_j} = 0.$$

Quando os α_j são constantes ou dependentes apenas do tempo a condição a ser obedecida para ξ é igual à condição (101) anterior,

(104) $$\sum_{j=1}^{n} \alpha_j \frac{\partial \xi}{\partial x_j} = 0.$$

Observemos que a função $T(t)$ em (93) não deve ter singularidades no caso de se desejar que a velocidade u seja regular, limitada em módulo, não obstante, $T(t)$ singular, infinita para um ou mais valores do tempo t, pode ser considerada como um "marcador" de *blowups*, e assim podemos construir soluções com instantes de *blowup* τ_* bem determinados, à nossa vontade, tais que $T(\tau_*) \to \infty$.

Na ausência de singularidades de $T(t)$ e $X(\xi(x))$, entretanto, desejando apenas velocidades regulares, conclui-se que é possível a uma equação de Navier-Stokes tridimensional (em geral, n-dimensional) *"bem comportada"* ter mais de uma solução para a mesma velocidade inicial. Da especial forma dada à solução $u(x,t)$ em (93), com $T(0) = 0$ ou não, para uma mesma velocidade inicial $u(x,0) = X(\xi(x))T(0)J = u^0(x)$, com $J = (1,1,-2)$, é possível gerar, em princípio, infinitas velocidades diferentes $u(x,t) = X(\xi(x))T(t)J$, para diferentes funções da posição $X(\xi(x))$ e do tempo $T(t)$, que resolvem a equação de Navier-Stokes (1). Se a força externa é zero, isso nos remete à resposta negativa ao 15º problema de Smale [12], como já havíamos visto anteriormente pensando apenas na não unicidade da pressão devido ao termo adicional $\theta(t) + q$, onde $q \neq 0$ é uma constante e $\theta(t)$ uma função explícita do tempo (no problema original de Smale a pressão não varia no tempo).

Em próximo artigo o correspondente à seção § 7 em três dimensões.

Grato, amigo Deus. Pela paz entre as religiões, e entre as pessoas.

Dedicado à memória de John Nash.

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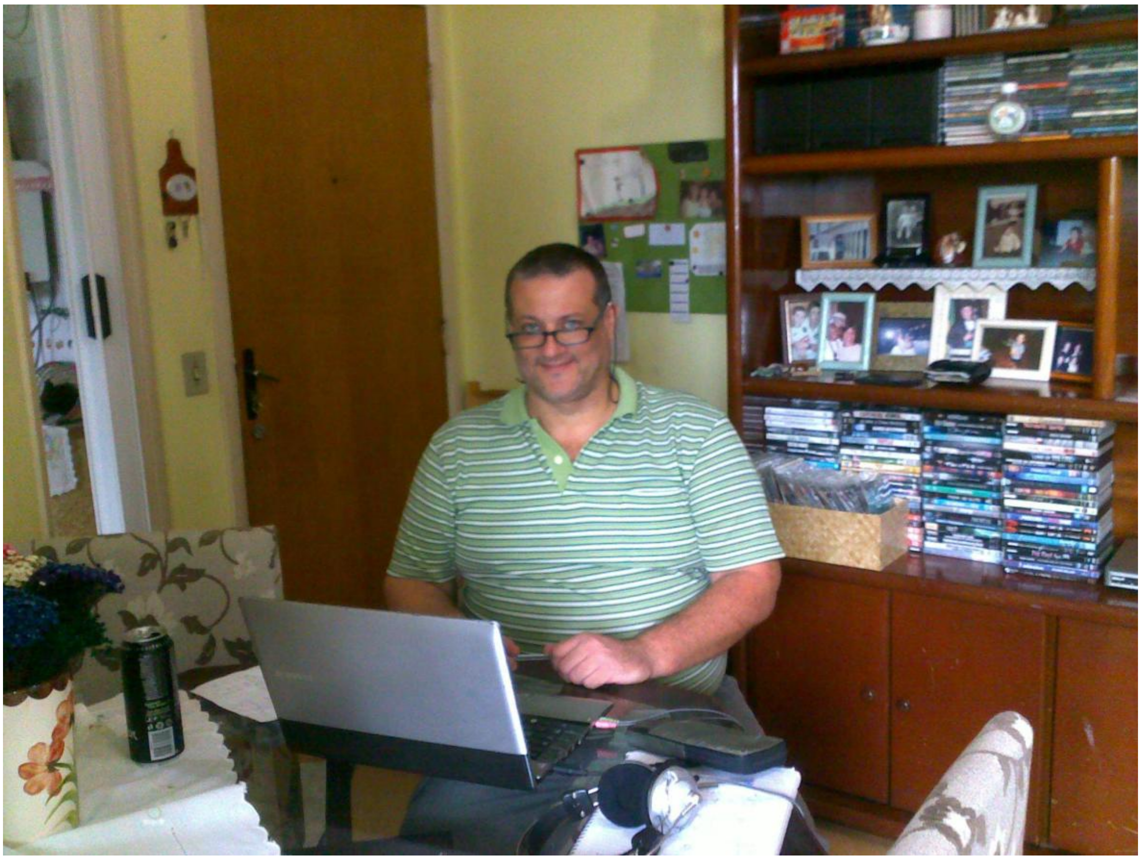

10 – Reviewing the 15th Problem of Smale: Navier-Stokes Equations

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Abstract – A most deep solution to the fifteenth problem of Smale, the Navier-Stokes equations in three spatial dimensions. The answer is negative.

Keywords – Navier-Stokes equations, existence, inexistence, smoothness, gradient field, conservative field, velocity, pressure, uniqueness, non uniqueness, 15th Problem of Smale.

§ 1 – Introdução

Steve Smale escreveu um estimulante artigo em 1998 onde propõe 18 problemas ainda não resolvidos na época[1], em atenção à solicitação de V.I.Arnold. Ambos, Arnold e Smale, por sua vez se inspiraram na famosa lista de 23 problemas de David Hilbert[2] (da lista de Hilbert, 10 estão completamente resolvidos, 8 parcialmente resolvidos e 5 a resolver).

A proposta deste artigo é resolver o 15º problema da lista de Smale, sobre a unicidade das soluções suaves (*smooth*) das equações de Navier-Stokes, de maneira não trivial, em um interessante nível de profundidade.

"*Do the Navier-Stokes equations on a 3-dimensional domain Ω in $\mathbb{R}^3$ have a unique smooth solution for all time?*"

Resposta: Não.

A resposta que aqui é dada já foi vista em [3], usando um argumento simples, a não unicidade da pressão devido ao acréscimo de uma constante não nula. Smale define a pressão no domínio $\Omega \subseteq \mathbb{R}^3$, sem variação no tempo, ao contrário do que normalmente se faz, por exemplo, em Fefferman[4]. Para a generalização de nossa exposição, entretanto, suponhamos inicialmente a pressão variando no tempo, além da variação espacial.

Se (u, p) é uma solução suave (lisa, regular, *smooth*, de classe C^∞) da equação de Navier-Stokes,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \nabla^2 u + \nabla p = 0, \tag{1}$$

com

$$\nabla \cdot u = 0, \qquad (2)$$

velocidade prescrita em $t = 0$,

$$u(x, 0) = u^0(x), \qquad (3)$$

e no contorno (ou borda, fronteira) $\partial\Omega$,

$$u|_{x \in \partial\Omega} = u^\partial(x, t), \qquad (4)$$

então $(u, p + \theta(t))$ também é uma solução, pois $\nabla p(x,t) = \nabla(p(x,t) + \theta(t))$, supondo que $\theta(t)$ não apresente singularidades, seja contínua e possa ser derivável espacialmente (obviamente, $\nabla\theta(t) = 0$), i.e., seja tão bem comportada quanto o que se espera para $p(x,t)$ neste problema.

Está muito claro que $p(x,t)$ e $p(x,t) + \theta(t)$ não são necessariamente a mesma solução p em (u, p) para o sistema (1) a (4), exceto se $\theta(t) = 0$, portanto a resposta deste problema não pode ser Sim.

Raciocínio análogo pode ser feito com as funções $q(x,t)$ tais que $\nabla q = 0$ (vetor nulo), cuja solução é uma constante em x ou variável apenas com o tempo. Temos, neste caso,

$$\nabla p(x,t) = \nabla(p(x,t) + \theta(t) + q), \qquad (5)$$

$q \in \mathbb{R}$. Então, se p faz parte da solução (u, p) de (1) e (2), também são soluções de (1) a (4) infinitos outros pares (u, r) tais que

$$r(x,t) = p(x,t) + \theta(t) + q, \qquad (6)$$

com $q \in \mathbb{R}$, $x \in \mathbb{R}^3$, $t \in [0, \infty)$, e as funções $p, r \colon \Omega \times [0, \infty) \to \mathbb{R}$, $\theta \colon [0, \infty) \to \mathbb{R}$, $p, r, \theta \in C^\infty$ em Ω para todo $t \geq 0$, $\Omega \subseteq \mathbb{R}^3$, i.e., todas estas funções e soluções são regulares (suaves, lisas, *smooth*).

Também há o caso de ser a função φ em

$$\nabla p = \nu \nabla^2 u - \frac{\partial u}{\partial t} - (u \cdot \nabla)u = \varphi \qquad (7)$$

não gradiente, o que impossibilitará de ser encontrado um valor para p, desde $t = 0$ ou a partir de algum $t = T_N$ ou mais genericamente em algum conjunto de valores de t tais que φ não seja uma função gradiente[5] nestes instantes de tempo t. Isto pode acontecer já em $t = 0$, com a imposição de uma adequada condição inicial adicional $\frac{\partial u}{\partial t}|_{t=0}$ ou então, por exemplo, para $\frac{Du}{Dt}|_{t=0} = \left(\frac{\partial u}{\partial t} + (u \cdot \nabla)u\right)|_{t=0}$.

Então pode não haver solução (u, p) para o sistema de equações (1) a (4) em algum $t \geq 0$, mas quando há solução ela não é única, pelo menos devido à infinidade de outras soluções $(u, p + \theta(t) + q)$ possíveis para o sistema, com $q \neq 0$ e $\theta(t) \neq 0$.

Vejam que não haver solução para p não é a mesma coisa que admitir que a pressão é nula, $p = 0$, ou mais genericamente impor como condição de contorno uma determinada pressão $p(x,t)$. Nessas situações ∇p existirá em geral, mas o problema original é outro, pois p deve ser uma variável dependente incógnita, não uma função pré-fixada.

Lembremos também que tanto neste problema descrito por Smale[1] quanto no correspondente (e mais detalhado) descrito por Fefferman[4] não é dada nenhuma condição inicial para a pressão $p(x,t)$, apenas para a velocidade inicial $u(x,0)$. Smale, ao contrário de Fefferman, inclui uma condição de contorno para $u(x,t)$ sobre $\partial\Omega$, a nossa equação (4).

Ainda mais uma observação é necessária. Conforme dissemos, ao contrário de Fefferman e da pressão real, no cotidiano, em máquinas e na natureza, poder variar com o tempo, Smale define o domínio da pressão como igual a $\Omega \subseteq \mathbb{R}^3$, i.e., sem variar no tempo. Seja por equívoco ou não, mesmo admitindo-se $p, r: \Omega \to \mathbb{R}$, com $p, r \in C^\infty$ em Ω para todo $t \geq 0$ e $q \in \mathbb{R}$ uma constante, sem utilizarmos a função $\theta(t)$, temos

$$\nabla p(x) = \nabla(p(x) + q), \tag{8}$$

resultado que também proporcionará infinitas outras soluções (u, r) admissíveis para (1) a (4), sendo (u, p) uma solução regular (suave, lisa, *smooth*) e

$$r(x) = p(x) + q, \tag{9}$$

$q \neq 0$, como já dissemos em [6] com outras palavras, usando $\theta(t)$, e mostramos inicialmente para o caso mais geral da pressão variável com o tempo e a posição, $p(x,t)$.

Naturalmente, estamos admitindo que $r(x)$ em (9) está definida em Ω, tal qual $p(x)$. Isto não oferece nenhuma dificuldade de entendimento no caso especial de ser $\Omega = \mathbb{R}^3$. Outros domínios, entretanto, também são facilmente estendidos para a função $r(x)$. Se $p(x)$ existe em Ω e tem imagem $\mathbb{R}$, então para todo $x \in \Omega$ a função $r(x) = p(x) + q$ também existe, está bem definida e tem imagem $\mathbb{R}$. Comentário similar se faz a respeito de $r(x,t)$ sobre $\Omega \times [0, \infty)$, dada em (6).

Assim, concluindo, a resposta ao problema é Não. Nem sempre, nem única.

Nas próximas seções analisaremos situações mais gerais para que ocorra a não unicidade destas soluções, além da possibilidade de não haver solução alguma.

§ 2 – Uma solução mais completa

Na seção anterior mostramos de maneira quase trivial a não unicidade da solução (u, p) das equações de Navier-Stokes devido à pressão p, com o correspondente acréscimo de um termo constante q ou dependente do tempo $\theta(t)$. Vamos agora achar uma solução não única para a velocidade nas equações de Navier-Stokes, sem precisar resolver nenhuma equação diferencial auxiliar. Só será preciso efetuar uma integração, necessária à obtenção da pressão. Em benefício da generalização, a velocidade inicial poderá ser diferente de zero, assim como a força externa aplicada, e não estaremos preocupados em buscar apenas energias cinéticas infinitas ou velocidades pertencentes ao espaço de Schwartz. Não estamos buscando uma *breakdown solution*, como fizemos em [7], pelo contrário, buscamos infinitas *smooth solutions*.

Vamos resolver o sistema (1), (2), (3) para o caso especial em que

$$u(x_1, x_2, x_3, t) = X(x_1 + x_2 + x_3)T(t)\,(1, 1, -2) = X(\xi)T(t)J, \quad (10)$$
$$\xi(x) = x_1 + x_2 + x_3,\ J = (1, 1, -2),$$

valendo $\nabla \cdot (\xi J) = 0$, onde estamos supondo $X \in C^\infty(\mathbb{R}^3)$ e $T \in C^\infty([0, \infty))$. Isso nos dá $\nabla \cdot u = \nabla \cdot u^0 = 0$ e a eliminação dos termos não lineares $(u \cdot \nabla)u \equiv \left(\sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j},\right)_{1 \le i \le 3} = 0$ das Equações de Navier-Stokes, com ou sem força externa. Assim a solução de (1) será reduzida à solução de uma equação diferencial parcial linear, a Equação do Calor não homogênea tridimensional,

$$\frac{\partial p}{\partial x_i} = \nu \nabla^2 u_i - \frac{\partial u_i}{\partial t} + f_i = \phi_i,\ 1 \le i \le 3, \quad (11)$$

devendo valer

$$\frac{\partial \phi_i}{\partial x_j} = \frac{\partial \phi_j}{\partial x_i},\ i \ne j. \quad (12)$$

Como $\frac{\partial p}{\partial x_i} = \frac{\partial p}{\partial \xi}\frac{\partial \xi}{\partial x_i} = \frac{\partial p}{\partial \xi}$, $\forall i$, assim como os operadores diferenciais $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x_i} = \frac{\partial}{\partial \xi}$ e $\left(\frac{\partial}{\partial x_i}\right)^2 = \left(\frac{\partial}{\partial \xi}\right)^2$, $\forall i$, i.e., temos uma pressão que pode ser expressa como função de ξ, assim como as componentes da velocidade u, e os x_i apresentam-se de forma simétrica e linear em relação a $\xi = x_1 + x_2 + x_3$, com a transformação do elemento infinitesimal de integração $d\xi = \frac{\partial \xi}{\partial x_1}dx_1 + \frac{\partial \xi}{\partial x_2}dx_2 + \frac{\partial \xi}{\partial x_3}dx_3 = dx_1 + dx_2 + dx_3$, a igualdade (12) é verdadeira, é válido $\frac{\partial^2 p}{\partial x_j \partial x_i} = \frac{\partial^2 p}{\partial x_i \partial x_j}$, e teremos a seguinte solução para a pressão:

$$p(x,t) - p_0(t) = \int_{\xi_0}^{\xi(x)} \left(\nu \mathrm{T} \nabla^2 X - X \frac{dT}{dt} + f \right) d\xi, \tag{13}$$

com

$$\frac{\partial p}{\partial x_1} = \frac{\partial p}{\partial x_2} = \frac{\partial p}{\partial x_3}, \tag{14}$$

supondo que a força $f(x,t) \in C^\infty(\mathbb{R}^3 \times [0,\infty))$ seja da forma $Y(\xi)Z(t)(1,1,-2)$, tal qual $u(x,t) = X(\xi)T(t)(1,1,-2)$. Consideremos $p_0(t) \in C^\infty([0,\infty))$ como a pressão no instante t e na superfície $\xi = \xi_0$. Isto resolve o sistema que pretendíamos, desde que a integração em (13) seja possível, e assim não precisamos resolver nenhuma equação diferencial ordinária intermediária para encontrarmos $X(\xi)$, pois podemos prefixar qual a expressão para $X(\xi)$ que desejamos utilizar, dentre infinitas possibilidades, e tal que tenhamos $u(x,0) = u^0(x)$.

Outras combinações das componentes do vetor J podem ser usadas, assim como outras combinações dos coeficientes dos x_1, x_2, x_3 em ξ, desde que eliminem-se os termos não lineares e verifique-se (2) e (12). Assim sendo, formas mais complicadas para ξ também são possíveis, além das lineares, o que traz uma robusta maneira de se obter as soluções para u. Por exemplo, definindo-se

$$u_i = \alpha_i(x,t)u_1,\ 1 \le i \le n,\ \alpha_1 = 1, \tag{15}$$

a condição a ser obedecida por X e ξ a fim de se eliminar os termos não lineares é

$$\alpha_i \frac{dX(\xi)}{d\xi} \sum_{j=1}^{n} \alpha_j \frac{\partial \xi}{\partial x_j} + X(\xi) \sum_{j=1}^{n} \alpha_j \frac{\partial \alpha_i}{\partial x_j} = 0, \tag{16}$$

para todo i (natural) em $1 \le i \le n$. Para cada determinado i elimina-se o termo não linear da respectiva linha (ou coordenada) i se (16) for satisfeita.

Uma maneira de fazer (16) ser verdadeira é quando

$$\sum_{j=1}^{n} \alpha_j \frac{\partial \xi}{\partial x_j} = \sum_{j=1}^{n} \alpha_j \frac{\partial \alpha_i}{\partial x_j} = 0. \tag{17}$$

Quando os α_i são constantes ou dependentes apenas do tempo a condição a ser obedecida para ξ é

$$\sum_{j=1}^{n} \alpha_j \frac{\partial \xi}{\partial x_j} = 0, \tag{18}$$

o que está de acordo com a expressão de ξ escolhida em (10).

Incluindo-se ainda a condição de incompressibilidade para u, deve ser válida também a relação

$$\sum_{j=1}^{n} \frac{\partial(\alpha_j u_1)}{\partial x_j} = u_1 \sum_{j=1}^{n} \frac{\partial \alpha_j}{\partial x_j} + \sum_{j=1}^{n} \alpha_j \frac{\partial u_1}{\partial x_j} \tag{19}$$
$$= T(t)[X(\xi) \sum_{j=1}^{n} \frac{\partial \alpha_j}{\partial x_j} + \frac{dX(\xi)}{d\xi} \sum_{j=1}^{n} \alpha_j \frac{\partial \xi}{\partial x_j}] = 0.$$

Como (102) deve ser válida para todo t, então precisamos que seja

$$X(\xi) \sum_{j=1}^{n} \frac{\partial \alpha_j}{\partial x_j} + \frac{dX(\xi)}{d\xi} \sum_{j=1}^{n} \alpha_j \frac{\partial \xi}{\partial x_j} = 0. \tag{20}$$

Quando os α_j são constantes ou dependentes apenas do tempo a condição a ser obedecida para ξ é igual à condição (18) anterior,

$$\sum_{j=1}^{n} \alpha_j \frac{\partial \xi}{\partial x_j} = 0. \tag{21}$$

Observemos que a função $T(t)$ em (10) não deve ter singularidades no caso de se desejar que a velocidade u seja regular, limitada em módulo (norma), não obstante, $T(t)$ singular, infinita para um ou mais valores do tempo t, pode ser considerada como um "marcador" de *blowups*, e assim podemos construir soluções com instantes de *blowup* τ_* bem determinados, à nossa vontade, tais que $T(\tau_*) \to \infty$.

Na ausência de singularidades de $T(t)$ e $X(\xi(x))$, entretanto, desejando apenas velocidades regulares, conclui-se que é possível a uma equação de Navier-Stokes tridimensional (em geral, n-dimensional) *"bem comportada"* ter mais de uma solução para a mesma velocidade inicial. Da especial forma dada à solução $u(x,t)$ em (10), com $T(0) = 0$ ou não, para uma mesma velocidade inicial $u(x,0) = X(\xi(x))T(0)J = u^0(x)$, com $J = (1,1,-2)$, é possível gerar, em princípio, infinitas velocidades diferentes $u(x,t) = X(\xi(x))T(t)J$, para diferentes funções da posição $X(\xi(x))$ e do tempo $T(t)$, que resolvem a equação de Navier-Stokes (1). Se a força externa é zero, isso nos remete de novo à resposta negativa ao 15º problema de Smale [1], como já havíamos visto na seção § 1 anterior pensando apenas na não unicidade da pressão devido ao termo adicional $\theta(t) + q$, onde $q \neq 0$ é uma constante e $\theta(t)$ uma função explícita do tempo.

§ 3 – A não unicidade em dimensão espacial $n = 2$

O que há com as provas de unicidade das soluções das equações de Navier-Stokes em dimensão espacial $n = 2$? O que foi feito na seção anterior com $n = 3$ pode ser aplicado em $n = 2$, com $\xi = x_1 + x_2$, $J = (1,-1)$, portanto também não há unicidade de soluções nas Equações de Navier-Stokes para $n = 2$. Mas este é (ou era) um dos resultados mais bem estabelecidos na teoria matemática dos fluidos, importantíssimo resultado, portanto iremos analisá-lo.

Não sendo possível analisar todas as provas existentes, é possível ao menos entender que tais provas não devem levar em consideração a ausência do termo não linear nas Equações de Navier-Stokes, $\sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} \equiv ((u \cdot \nabla)u)_i, 1 \leq i \leq n$, e foi a esta ausência que recorremos em nosso exemplo.

Semelhantemente a esta causa, também se percebe que diferentes equações do tipo de Navier-Stokes, com ausência de um ou mais termos da respectiva equação completa, e que não obstante tenham a mesma condição inicial $u(x,0) = u^0(x)$, terão provavelmente, no caso geral, diferentes soluções $u(x,t)$ entre elas, e assim não poderá haver unicidade de solução em relação à equação de Navier-Stokes completa, com todos os termos. Se todas apresentassem sempre a mesma e única solução, bastaria para nós resolver somente a mais simples delas, por exemplo, $\nabla p = -\frac{\partial u}{\partial t}$ ou $\nabla p = \nu \nabla^2 u$ (Equação de Poisson se $\nabla p \not\equiv 0$ ou de Laplace se $\nabla p \equiv 0$) ou $\frac{\partial u}{\partial t} = \nu \nabla^2 u$ (Equação do Calor com $\nabla p = 0$), todas com $u(x,0) = u^0(x)$, e conferir se a soma dos demais termos faltantes é igual a zero ao aplicar a solução u obtida na equação reduzida. Se sim, a solução da equação reduzida é também solução da equação completa. Importante exemplo desta ausência são as Equações de Euler, que diferem das Equações de Navier-Stokes pela ausência do operador diferencial nabla aplicado a u, $\nabla^2 u \equiv \Delta u$, devido ao coeficiente de viscosidade ser nulo, $\nu = 0$.

É fácil provar que as três equações acima, assim como a equação $\nabla p + \frac{\partial u}{\partial t} = \nu \nabla^2 u$, não podem realmente ter uma única solução, dada apenas a condição inicial para a velocidade $u(x,0) = u^0(x)$. Pelo contrário, a forma completa das equações de Navier-Stokes, onde supomos que $\sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} \equiv ((u \cdot \nabla)u)_i \not\equiv 0, 1 \leq i \leq n$, tem unicidade de solução para $n = 2$ e em ao menos um pequeno intervalo de tempo não nulo $[0,T]$ para $n = 3$, onde T é conhecido como *blowup time*. Acrescentemos em todas estas equações a condição de incompressibilidade, $\nabla \cdot u = 0$.

Trata-se assim de um interessante problema de Análise Combinatória aplicada à Análise Matemática e Física-Matemática.

§ 4 – Unicidade em dimensão espacial $n = 3$

Verificamos na seção § 2 que o sistema

$$\begin{cases} \nabla p + \frac{\partial u}{\partial t} = \nu \nabla^2 u \\ (u \cdot \nabla)u = 0 \\ \nabla \cdot u = 0 \\ u(x,0) = u^0(x) \end{cases} \tag{22}$$

tem infinitas soluções para a velocidade da forma

$$u(x,t) = X(\xi)T(t)\ (1,1,-2),\ \ \xi = x_1 + x_2 + x_3, \tag{23}$$

com $T(0) = 0$ ou outro valor constante e obedecendo à condição inicial $u^0(x) = X(\xi)T(0)(1,1,-2)$, não obstante existem as provas conhecidas da unicidade de

$$\begin{cases} \nabla p + \frac{\partial u}{\partial t} + (u \cdot \nabla)u = \nu \nabla^2 u \\ \qquad \nabla \cdot u = 0 \\ \quad u(x,0) = u^0(x) \end{cases} \tag{24}$$

em ao menos um pequeno intervalo de tempo $[0,T]$ não nulo, contradizendo o que obtivemos.

Sem ser necessário nos demorarmos nas provas conhecidas, expondo todos os seus detalhes, repetindo suas passagens, é possível constatar em Leray [8], Ladyzhenskaya [9], Kreiss and Lorenz [10], dentre outros, que as provas de existência e unicidade baseiam-se na forma completa das equações de Navier-Stokes, por exemplo (24), e não em uma forma desmembrada das equações de Navier-Stokes, como (22).

§ 5 – Não Unicidade em dimensão espacial $n = 3$, com $\frac{\partial p}{\partial t} = 0$

Não exploramos a condição (4) até o momento, a condição de contorno $u|_{x\in\partial\Omega} = u^{\partial}(x,t)$, uma condição de Dirichlet variável no tempo, nem o fato da pressão não ter variação temporal, $\frac{\partial p}{\partial t} = 0$, conforme proposto no problema original de Smale [1]. De fato são dois grandes complicadores ao problema, mas não impossibilitam a solução da questão. A resposta continua a mesma: negativa.

De (13), fatorando o integrando, separando as variáveis em $X(\xi)$ e $T(t)$, fazendo

$$\nabla^2 X = -X \tag{25}$$

(buscando soluções espacialmente periódicas, combinações de senos e cossenos), retirando o tempo do domínio da pressão e definindo $f = 0$, temos

$$\begin{aligned} p(x) - p_0 &= \int_{\xi_0}^{\xi(x)} \left(\nu \mathrm{T} \nabla^2 X - X \frac{dT}{dt}\right) d\xi \\ &= -\left(\nu\, T + \frac{dT}{dt}\right) \int_{\xi_0}^{\xi(x)} X\, d\xi. \end{aligned} \tag{26}$$

Consideremos p_0 como a pressão na superfície $\xi = \xi_0$, qualquer que seja o instante de tempo $t \geq 0$.

Como a pressão não pode variar no tempo, então o termo que aparece multiplicando a integral deve ser igual a uma constante k, i.e.,

$$\left(\nu\, T + \frac{dT}{dt}\right) = -k, \tag{27}$$

ou

$$\frac{dT}{dt} + \nu\, T + k = 0, \tag{28}$$

uma equação diferencial ordinária linear e de primeira ordem no tempo, cuja solução é da forma

$$T(t) = Ae^{Bt} + Ce^{-Dt} + E, \tag{29}$$

para A, B, C, D, E constantes.

Substituindo (28) em (27) obtemos

$$\begin{cases} B = -\nu \\ D = +\nu \\ E = -\frac{k}{\nu},\ \nu \neq 0 \end{cases} \tag{30}$$

ou seja,

$$T(t) = Fe^{-\nu t} - \frac{k}{\nu}, \nu > 0, \tag{31}$$

para $F = A + C$ uma constante e lembrando que o coeficiente de viscosidade ν não pode ser negativo.

Em $t = 0$, chamando $T(0) = T_0$, temos

$$T(0) = F - \frac{k}{\nu} = T_0, \tag{32}$$

ou

$$F = T_0 + \frac{k}{\nu}, \tag{33}$$

donde

$$T(t) = \left(T_0 + \frac{k}{\nu}\right) e^{-\nu t} - \frac{k}{\nu}, \nu > 0. \tag{34}$$

Daqui se vê que somente o valor de T_0 não é suficiente para encontrar unicidade de solução neste problema, pois há uma infinidade de possibilidades para se escolher a constante k. Incluamos a seguir a condição (4),

$$u|_{x \in \partial\Omega} = u^{\partial}(x, t), \tag{35}$$

onde $\Omega \subseteq \mathbb{R}^3$ é o domínio em $u: \Omega \times [0, \infty) \to \mathbb{R}^3$ e $p: \Omega \to \mathbb{R}$.

Podemos escolher o domínio Ω que seja mais conveniente para a solução deste problema de Smale. Precisamos, em princípio, escolher apenas um único domínio Ω específico tal que se verifique a não unicidade de soluções, sem precisar provar que a mesma conclusão vale também para qualquer região Ω.

Não obstante, constata-se que a possibilidade aqui é bastante ampla, e uma infinidade de regiões $\Omega \subseteq \mathbb{R}^3$ podem ser escolhidas. O valor da função $u^\partial(x,t)$ que devemos escolher é igual ao valor de $u(x,t)$ em $\partial\Omega$, qualquer que seja Ω.

Para $u(x,t)$ dado em (10),

$$u(x_1,x_2,x_3,t) = X(x_1+x_2+x_3)T(t)\,(1,1,-2) = X(\xi)T(t)J, \quad (36)$$
$$\xi(x) = x_1+x_2+x_3,\ J = (1,1,-2),$$

e $X(\xi(x))$ solução de (25), i.e.,

$$X(\xi) = A\, cos\, \xi + B\, sen\, \xi, \quad (37)$$

com A e B constantes reais e $T(t)$ dado em (34), a solução do sistema (1), (2), (3) para a velocidade é

$$u(x,t) = (A\, cos\, \xi + B\, sen\, \xi)\left[\left(T_0 + \frac{k}{\nu}\right)e^{-\nu t} - \frac{k}{\nu}\right]J, \quad (38)$$
$$\xi(x) = x_1+x_2+x_3,\ J = (1,1,-2),\ \nu > 0,$$

valendo

$$u^0(x) = u(x,0) = (A\, cos\, \xi + B\, sen\, \xi)\, T_0 J. \quad (39)$$

Por simplicidade, vamos escolher $A = B = T_0 = k = 1$ em (38), e assim a função $u^\partial(x,t)$ deve ser igual ao valor da solução $u(x,t)$ em $\partial\Omega$, com os parâmetros igualados a 1, i.e., qualquer que seja $\Omega \subseteq \mathbb{R}^3$ teremos

$$u^\partial(x,t) = u(x,t) = (cos\, \xi + sen\, \xi)\left[\left(1 + \frac{1}{\nu}\right)e^{-\nu t} - \frac{1}{\nu}\right]J, \quad (40)$$
$$\xi(x) = x_1+x_2+x_3,\ J = (1,1,-2),\ \nu > 0,$$

valendo

$$u^0(x) = u(x,0) = (cos\, \xi + sen\, \xi)\, J. \quad (41)$$

Vemos que a introdução da condição (4) no sistema (1), (2), (3) pode tornar única a velocidade, conforme (40), mas a pressão continuará não sendo única, como também já acontecia na situação da seção § 1.

Usando (37) e (27) em (26), com $A = B = k = 1$, teremos

$$p(x) - p_0 = +\int_{\xi_0}^{\xi(x)} X\, d\xi = (sen\ \xi - cos\ \xi)|_{\xi_0}^{\xi} \tag{42}$$
$$= (sen\ \xi - cos\ \xi) - (sen\ \xi_0 - cos\ \xi_0),$$
$$\xi(x) = x_1 + x_2 + x_3,$$

onde podemos considerar p_0 como a pressão (constante) na superfície $\xi = \xi_0$.

Vemos assim que, ao contrário da velocidade em (40), a solução acima para a pressão do sistema (1) a (4) tem ainda dois parâmetros livres, p_0 e ξ_0, portanto devemos concluir que a solução (u, p) não é única. Faltam, evidentemente, os valores de p_0 e ξ_0, condições que não fazem parte dos dados do problema.

§ 6 – Inexistência em dimensão espacial $n = 3$

Nesta seção vamos fazer para três dimensões o que já fizemos em duas dimensões em [7], seção § 7.

As equações de Navier-Stokes sem força externa com $n = 3$ são (usando $x \equiv x_1,\ y \equiv x_2$ e $z \equiv x_3$)

$$\begin{cases} \frac{\partial p}{\partial x} + \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + u_3 \frac{\partial u_1}{\partial z} = \nu\, \nabla^2 u_1 \\ \frac{\partial p}{\partial y} + \frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + u_3 \frac{\partial u_2}{\partial z} = \nu\, \nabla^2 u_2 \\ \frac{\partial p}{\partial z} + \frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x} + u_2 \frac{\partial u_3}{\partial y} + u_3 \frac{\partial u_3}{\partial z} = \nu\, \nabla^2 u_3 \end{cases} \tag{43}$$

Podemos dispor o sistema acima de forma parecida com um sistema de equações lineares,

$$\begin{cases} u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + u_3 \frac{\partial u_1}{\partial z} = \nu\, \nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t} \\ u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + u_3 \frac{\partial u_2}{\partial z} = \nu\, \nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t} \\ u_1 \frac{\partial u_3}{\partial x} + u_2 \frac{\partial u_3}{\partial y} + u_3 \frac{\partial u_3}{\partial z} = \nu\, \nabla^2 u_3 - \frac{\partial p}{\partial z} - \frac{\partial u_3}{\partial t} \end{cases} \tag{44}$$

e a seguir em forma de uma equação matricial,

$$\begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z} \\ \frac{\partial u_3}{\partial x} & \frac{\partial u_3}{\partial y} & \frac{\partial u_3}{\partial z} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \nu\, \nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t} \\ \nu\, \nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t} \\ \nu\, \nabla^2 u_3 - \frac{\partial p}{\partial z} - \frac{\partial u_3}{\partial t} \end{pmatrix}. \tag{45}$$

Chamando

$$A = \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z} \\ \frac{\partial u_3}{\partial x} & \frac{\partial u_3}{\partial y} & \frac{\partial u_3}{\partial z} \end{pmatrix}, \tag{46}$$

$$U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \tag{47}$$

$$B = \begin{pmatrix} \nu \nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t} \\ \nu \nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t} \\ \nu \nabla^2 u_3 - \frac{\partial p}{\partial z} - \frac{\partial u_3}{\partial t} \end{pmatrix}, \tag{48}$$

a solução para U da equação (45), $AU = B$, é

$$U = A^{-1}B, \tag{49}$$

que para existir e ter solução única deve-se ter

$$\det A = \frac{\partial u_1}{\partial x}\left(\frac{\partial u_2}{\partial y}\frac{\partial u_3}{\partial z} - \frac{\partial u_2}{\partial z}\frac{\partial u_3}{\partial y}\right) - \frac{\partial u_1}{\partial y}\left(\frac{\partial u_2}{\partial x}\frac{\partial u_3}{\partial z} - \frac{\partial u_2}{\partial z}\frac{\partial u_3}{\partial x}\right) \tag{50}$$

$$+ \frac{\partial u_1}{\partial z}\left(\frac{\partial u_2}{\partial x}\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial y}\frac{\partial u_3}{\partial x}\right) \neq 0,$$

ou seja,

$$\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y}\frac{\partial u_3}{\partial z} + \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial z}\frac{\partial u_3}{\partial x} + \frac{\partial u_1}{\partial z}\frac{\partial u_2}{\partial x}\frac{\partial u_3}{\partial y} \neq \tag{51}$$

$$\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial z}\frac{\partial u_3}{\partial y} + \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}\frac{\partial u_3}{\partial z} + \frac{\partial u_1}{\partial z}\frac{\partial u_2}{\partial y}\frac{\partial u_3}{\partial x},$$

regra que também deve ser obedecida para $t = 0$ (de novo pode nos levar aos casos (C) e (D) de [4], quando o sistema torna-se impossível de ser resolvido).

Se usarmos a condição de incompressibilidade $\nabla \cdot u = 0$,

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} = 0, \tag{52}$$

i.e.,

$$\frac{\partial u_1}{\partial x} = -\left(\frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right), \tag{53}$$

transforma-se a condição (51) em

$$-\left(\frac{\partial u_2}{\partial y}+\frac{\partial u_3}{\partial z}\right)\frac{\partial u_2}{\partial y}\frac{\partial u_3}{\partial z}+\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial z}\frac{\partial u_3}{\partial x}+\frac{\partial u_1}{\partial z}\frac{\partial u_2}{\partial x}\frac{\partial u_3}{\partial y}\neq \tag{54}$$

$$-\left(\frac{\partial u_2}{\partial y}+\frac{\partial u_3}{\partial z}\right)\frac{\partial u_2}{\partial z}\frac{\partial u_3}{\partial y}+\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}\frac{\partial u_3}{\partial z}+\frac{\partial u_1}{\partial z}\frac{\partial u_2}{\partial y}\frac{\partial u_3}{\partial x},$$

ou equivalentemente,

$$+\left(\frac{\partial u_2}{\partial y}+\frac{\partial u_3}{\partial z}\right)\frac{\partial u_2}{\partial z}\frac{\partial u_3}{\partial y}+\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial z}\frac{\partial u_3}{\partial x}+\frac{\partial u_1}{\partial z}\frac{\partial u_2}{\partial x}\frac{\partial u_3}{\partial y}\neq \tag{55}$$

$$+\left(\frac{\partial u_2}{\partial y}+\frac{\partial u_3}{\partial z}\right)\frac{\partial u_2}{\partial y}\frac{\partial u_3}{\partial z}+\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}\frac{\partial u_3}{\partial z}+\frac{\partial u_1}{\partial z}\frac{\partial u_2}{\partial y}\frac{\partial u_3}{\partial x}.$$

Como esta condição deve ser válida para todo t, em $t=0$ deve-se obedecer a

$$+\left(\frac{\partial u_2^0}{\partial y}+\frac{\partial u_3^0}{\partial z}\right)\frac{\partial u_2^0}{\partial z}\frac{\partial u_3^0}{\partial y}+\frac{\partial u_1^0}{\partial y}\frac{\partial u_2^0}{\partial z}\frac{\partial u_3^0}{\partial x}+\frac{\partial u_1^0}{\partial z}\frac{\partial u_2^0}{\partial x}\frac{\partial u_3^0}{\partial y}\neq \tag{56}$$

$$+\left(\frac{\partial u_2^0}{\partial y}+\frac{\partial u_3^0}{\partial z}\right)\frac{\partial u_2^0}{\partial y}\frac{\partial u_3^0}{\partial z}+\frac{\partial u_1^0}{\partial y}\frac{\partial u_2^0}{\partial x}\frac{\partial u_3^0}{\partial z}+\frac{\partial u_1^0}{\partial z}\frac{\partial u_2^0}{\partial y}\frac{\partial u_3^0}{\partial x},$$

i.e.

$$+\left(\frac{\partial u_2^0}{\partial y}+\frac{\partial u_3^0}{\partial z}\right)\frac{\partial u_2^0}{\partial z}\frac{\partial u_3^0}{\partial y}+\frac{\partial u_1^0}{\partial y}\frac{\partial u_2^0}{\partial z}\frac{\partial u_3^0}{\partial x}+\frac{\partial u_1^0}{\partial z}\frac{\partial u_2^0}{\partial x}\frac{\partial u_3^0}{\partial y}\neq \tag{57}$$

$$+\left(\frac{\partial u_2^0}{\partial y}\right)^2\frac{\partial u_3^0}{\partial z}+\left(\frac{\partial u_3^0}{\partial z}\right)^2\frac{\partial u_2^0}{\partial y}+\frac{\partial u_1^0}{\partial y}\frac{\partial u_2^0}{\partial x}\frac{\partial u_3^0}{\partial z}+\frac{\partial u_1^0}{\partial z}\frac{\partial u_2^0}{\partial y}\frac{\partial u_3^0}{\partial x},$$

onde se usou $u(x,y,z,0)=u^0(x,y,z)=(u_1^0(x,y,z),u_2^0(x,y,z),u_3^0(x,y,z))$.

Se a velocidade inicial u^0 for tal que sejam desobedecidas (56)-(57) então ou não haverá solução para o sistema (43) (sistema impossível) ou não haverá uma única solução (sistema indeterminado), tal como na teoria de sistemas lineares. Qualquer que seja um destes casos (sistema impossível ou sistema indeterminado) teremos a resposta negativa ao problema de Smale: não há sempre uma única solução para o sistema (1) a (4), o que já pode ocorrer desde $t=0$.

Definindo

$$U_1=\begin{pmatrix}\nu\nabla^2u_1-\frac{\partial p}{\partial x}-\frac{\partial u_1}{\partial t} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z}\\ \nu\nabla^2u_2-\frac{\partial p}{\partial y}-\frac{\partial u_2}{\partial t} & \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z}\\ \nu\nabla^2u_3-\frac{\partial p}{\partial z}-\frac{\partial u_3}{\partial t} & \frac{\partial u_3}{\partial y} & \frac{\partial u_3}{\partial z}\end{pmatrix}, \tag{58}$$

$$U_2 = \begin{pmatrix} \frac{\partial u_1}{\partial x} & \nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t} & \frac{\partial u_1}{\partial z} \\ \frac{\partial u_2}{\partial x} & \nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t} & \frac{\partial u_2}{\partial z} \\ \frac{\partial u_3}{\partial x} & \nu\nabla^2 u_3 - \frac{\partial p}{\partial z} - \frac{\partial u_3}{\partial t} & \frac{\partial u_3}{\partial z} \end{pmatrix} \tag{59}$$

e

$$U_3 = \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t} \\ \frac{\partial u_1}{\partial y} & \frac{\partial u_2}{\partial y} & \nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t} \\ \frac{\partial u_1}{\partial z} & \frac{\partial u_3}{\partial y} & \nu\nabla^2 u_3 - \frac{\partial p}{\partial z} - \frac{\partial u_3}{\partial t} \end{pmatrix}, \tag{60}$$

a solução para u_1, u_2, u_3 será

$$u_1 = \frac{\det U_1}{\det A}, \tag{61}$$

$$u_2 = \frac{\det U_2}{\det A} \tag{62}$$

e

$$u_3 = \frac{\det U_3}{\det A}. \tag{63}$$

Sendo

$$\begin{aligned} \det U_1 = {} & \left(\nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t}\right)\left(\frac{\partial u_2}{\partial y}\frac{\partial u_3}{\partial z} - \frac{\partial u_2}{\partial z}\frac{\partial u_3}{\partial y}\right) + \\ & + \left(\nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t}\right)\left(\frac{\partial u_1}{\partial z}\frac{\partial u_3}{\partial y} - \frac{\partial u_1}{\partial y}\frac{\partial u_3}{\partial z}\right) + \\ & + \left(\nu\nabla^2 u_3 - \frac{\partial p}{\partial z} - \frac{\partial u_3}{\partial t}\right)\left(\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial z} - \frac{\partial u_1}{\partial z}\frac{\partial u_2}{\partial y}\right), \end{aligned} \tag{64}$$

$$\begin{aligned} \det U_2 = {} & \left(\nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t}\right)\left(\frac{\partial u_2}{\partial z}\frac{\partial u_3}{\partial x} - \frac{\partial u_2}{\partial x}\frac{\partial u_3}{\partial z}\right) + \\ & + \left(\nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t}\right)\left(\frac{\partial u_1}{\partial x}\frac{\partial u_3}{\partial z} - \frac{\partial u_1}{\partial z}\frac{\partial u_3}{\partial x}\right) + \\ & + \left(\nu\nabla^2 u_3 - \frac{\partial p}{\partial z} - \frac{\partial u_3}{\partial t}\right)\left(\frac{\partial u_1}{\partial z}\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial z}\right) \end{aligned} \tag{65}$$

e

$$\det U_3 = \left(\nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t}\right)\left(\frac{\partial u_2}{\partial x}\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial y}\frac{\partial u_3}{\partial x}\right) + \tag{66}$$

$$+\left(\nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t}\right)\left(\frac{\partial u_1}{\partial y}\frac{\partial u_3}{\partial x} - \frac{\partial u_1}{\partial x}\frac{\partial u_3}{\partial y}\right) +$$

$$+\left(\nu\nabla^2 u_3 - \frac{\partial p}{\partial z} - \frac{\partial u_3}{\partial t}\right)\left(\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}\right),$$

com $\det A$ dado em (50), temos então

$$u_1 = \frac{\det U_1}{\det A} = \Big[\left(\nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t}\right)\left(\frac{\partial u_2}{\partial y}\frac{\partial u_3}{\partial z} - \frac{\partial u_2}{\partial z}\frac{\partial u_3}{\partial y}\right) + \tag{67}$$

$$+\left(\nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t}\right)\left(\frac{\partial u_1}{\partial z}\frac{\partial u_3}{\partial y} - \frac{\partial u_1}{\partial y}\frac{\partial u_3}{\partial z}\right) +$$

$$+\left(\nu\nabla^2 u_3 - \frac{\partial p}{\partial z} - \frac{\partial u_3}{\partial t}\right)\left(\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial z} - \frac{\partial u_1}{\partial z}\frac{\partial u_2}{\partial y}\right)\Big] \,/$$

$$\Big[\left(\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y}\frac{\partial u_3}{\partial z} + \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial z}\frac{\partial u_3}{\partial x} + \frac{\partial u_1}{\partial z}\frac{\partial u_2}{\partial x}\frac{\partial u_3}{\partial y}\right) -$$

$$-\left(\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial z}\frac{\partial u_3}{\partial y} + \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}\frac{\partial u_3}{\partial z} + \frac{\partial u_1}{\partial z}\frac{\partial u_2}{\partial y}\frac{\partial u_3}{\partial x}\right)\Big],$$

$$u_2 = \frac{\det U_2}{\det A} = \Big[\left(\nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t}\right)\left(\frac{\partial u_2}{\partial z}\frac{\partial u_3}{\partial x} - \frac{\partial u_2}{\partial x}\frac{\partial u_3}{\partial z}\right) + \tag{68}$$

$$+\left(\nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t}\right)\left(\frac{\partial u_1}{\partial x}\frac{\partial u_3}{\partial z} - \frac{\partial u_1}{\partial z}\frac{\partial u_3}{\partial x}\right) +$$

$$+\left(\nu\nabla^2 u_3 - \frac{\partial p}{\partial z} - \frac{\partial u_3}{\partial t}\right)\left(\frac{\partial u_1}{\partial z}\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial z}\right)\Big] \,/$$

$$\Big[\left(\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y}\frac{\partial u_3}{\partial z} + \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial z}\frac{\partial u_3}{\partial x} + \frac{\partial u_1}{\partial z}\frac{\partial u_2}{\partial x}\frac{\partial u_3}{\partial y}\right) -$$

$$-\left(\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial z}\frac{\partial u_3}{\partial y} + \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}\frac{\partial u_3}{\partial z} + \frac{\partial u_1}{\partial z}\frac{\partial u_2}{\partial y}\frac{\partial u_3}{\partial x}\right)\Big]$$

e

$$u_3 = \frac{\det U_3}{\det A} = \Big[\left(\nu\nabla^2 u_1 - \frac{\partial p}{\partial x} - \frac{\partial u_1}{\partial t}\right)\left(\frac{\partial u_2}{\partial x}\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial y}\frac{\partial u_3}{\partial x}\right) + \tag{69}$$

$$+\left(\nu\nabla^2 u_2 - \frac{\partial p}{\partial y} - \frac{\partial u_2}{\partial t}\right)\left(\frac{\partial u_1}{\partial y}\frac{\partial u_3}{\partial x} - \frac{\partial u_1}{\partial x}\frac{\partial u_3}{\partial y}\right) +$$

$$+\left(\nu\nabla^2 u_3 - \frac{\partial p}{\partial z} - \frac{\partial u_3}{\partial t}\right)\left(\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}\right)\Big] \,/$$

$$\Big[\left(\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y}\frac{\partial u_3}{\partial z} + \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial z}\frac{\partial u_3}{\partial x} + \frac{\partial u_1}{\partial z}\frac{\partial u_2}{\partial x}\frac{\partial u_3}{\partial y}\right) -$$

$$-\left(\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial z}\frac{\partial u_3}{\partial y} + \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}\frac{\partial u_3}{\partial z} + \frac{\partial u_1}{\partial z}\frac{\partial u_2}{\partial y}\frac{\partial u_3}{\partial x}\right)\Big].$$

Se $\det A \neq 0$ o sistema (43) é determinado, se $\det A = \det U_1 = \det U_2 = \det U_3 = 0$ o sistema é indeterminado (infinitas soluções), caso contrário o sistema é impossível ($\det A = 0$ e ao menos uma matrizes U_1, U_2 e U_3 com determinantes diferentes de zero).

É verdade que as soluções (equações) anteriores, equações (67) a (69), são bem mais complicadas que as equações originais em (43), e parece não haver utilidade alguma em resolvê-las.

Mas desta forma complicada se pode chegar com mais certeza à seguinte constatação (já vista em [7] para $n = 2$): as equações de Navier-Stokes (e Euler) têm uma simetria entre as variáveis, tanto as dependentes quanto as independentes. O mesmo também pode ser percebido diretamente em (43).

A simetria neste caso de $n = 3$ é

$$u_1 \mapsto u_2 \tag{70.1}$$

$$u_2 \mapsto u_3 \tag{70.2}$$

$$u_3 \mapsto u_1 \tag{70.3}$$

$$x \mapsto y \tag{70.4}$$

$$y \mapsto z \tag{70.5}$$

$$z \mapsto x \tag{70.6}$$

ficando p e t inalterados:

$$p \leftrightarrow p \tag{71.1}$$

$$t \leftrightarrow t. \tag{71.2}$$

Isso sugere, se não resolve completamente, a questão da solução destas equações. Se as equações em si são simétricas em relação a determinadas transformações, então esperamos que suas soluções também o sejam sob estas transformações. O mesmo método pode ser aplicado também para $n \geq 4$, com a regra (por exemplo)

$$u_i \mapsto u_{i+1},\ u_{n+1} \equiv u_1, \tag{72.1}$$

$$x_i \mapsto x_{i+1},\ x_{n+1} \equiv x_1, \tag{72.2}$$

$$p \leftrightarrow p, \tag{72.3}$$

$$t \leftrightarrow t. \tag{72.4}$$

Nesse caso é preciso que a condição inicial $u(x,0) = u^0(x)$ obedeça também a estas simetrias, mas permanece inalterada a condição de incompressibilidade: $\sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial u_i^0}{\partial x_i} = 0$.

Se fornecemos $u_1(x,y,z,t)$ como dado de entrada no nosso sistema então podemos concluir que as soluções para u_2 e u_3, supostamente simétricas a u_1 pela regra (70) anterior, sejam, respectivamente,

$$u_2(x,y,z,t) = u_1(y,z,x,t), \tag{73.1}$$

$$u_3(x,y,z,t) = u_2(y,z,x,t) = u_1(z,x,y,t), \tag{73.2}$$

i.e., trocamos x por y, y por z e z por x, na solução dada previamente para u_1 e igualamos a u_2 o resultado desta transformação. Repetimos esta regra em u_2 e igualamos a u_3 o resultado da transformação. Restará obter a pressão p ou então, caso ela também tenha sido dada, verificar se as variáveis $u_1,\ u_2,\ u_3,\ p$ realmente satisfazem o sistema original, procedendo a ajustes caso seja necessário.

É de se esperar ainda que p seja simétrica em relação às variáveis x, y e z, ou seja,

$$p(x,y,z,t) = p(y,z,x,t) = p(z,x,y,t). \tag{74}$$

Claro que (73) e (74) admitem implicitamente que temos simetria retangular nas condições iniciais e de contorno do sistema. Uma vez que esta simetria não ocorra, por exemplo, tenhamos outro tipo de simetria, esférica, cilíndrica, ou mesmo simetria nenhuma (caso geral), as igualdades (73) e (74) não têm necessidade de serem satisfeitas. Sendo assim, a solução para o caso em que não há simetria alguma ainda é um problema a resolver, admitindo-se que há ao menos uma solução (quando o sistema é possível; pode-se provar que o sistema sempre é possível, por exemplo, com escolha apropriada de p ou $\partial u/\partial t$).

§ 7 – Conclusão

À pergunta de Smale

"*Do the Navier-Stokes equations on a 3-dimensional domain Ω in* $\mathbb{R}^3$ *have a unique smooth solution for all time?*"

respondemos "Não".

É possível eliminar os termos não lineares em cada uma das equações do sistema (1), obedecendo-se (2) e (3), o que fornecerá infinitas soluções para a velocidade e a pressão, conforme (10) e (13), respectivamente,

$$u(x_1,x_2,x_3,t) = X(x_1+x_2+x_3)T(t)\ (1,1,-2) = X(\xi)T(t)J, \tag{75}$$

$$\xi(x) = x_1 + x_2 + x_3,\ J = (1, 1, -2),$$

e

$$p(x,t) - p_0(t) = \int_{\xi_0}^{\xi(x)} \left(\nu \mathrm{T}\nabla^2 X - X\frac{dT}{dt} + f\right) d\xi, \tag{76}$$

considerando que a pressão pode variar no tempo e temos força externa. Claro que devemos escolher para p_0, X, T, f funções *smooth* (C^∞) em seus respectivos domínios. Como resultado teremos também $u, p \in C^\infty$.

Eliminando a variação temporal da pressão, fazendo $f = 0$ e incluindo-se a condição (4), velocidade $u^\partial(x,t)$ na fronteira $\partial\Omega$, restringimos a infinidade de velocidades, tornando-a única, dada em (40),

$$u^\partial(x,t) = u(x,t) = (cos\ \xi + sen\ \xi)\left[\left(1 + \frac{1}{\nu}\right)e^{-\nu t} - \frac{1}{\nu}\right]J, \tag{77}$$
$$\xi(x) = x_1 + x_2 + x_3,\ J = (1, 1, -2),\ \nu > 0,$$

valendo

$$u^0(x) = u(x, 0) = (cos\ \xi + sen\ \xi)\, J, \tag{78}$$

e onde igualamos a 1 todos os parâmetros livres, $A = B = T_0 = k = 1$, mas ainda temos infinitas soluções possíveis para a pressão, de acordo com (42),

$$\begin{aligned} p(x) - p_0 &= +\int_{\xi_0}^{\xi(x)} X\, d\xi = (sen\ \xi - cos\ \xi)|_{\xi_0}^{\xi} \\ &= (sen\ \xi - cos\ \xi) - (sen\ \xi_0 - cos\ \xi_0), \\ \xi(x) &= x_1 + x_2 + x_3, \end{aligned} \tag{79}$$

onde consideramos p_0 como a pressão (constante) na superfície $\xi = \xi_0$. Para garantir a unicidade deveriam ser fornecidos os valores de p_0 e ξ_0, condições que não fazem parte dos dados do problema.

Interessante perceber que não é apenas a não unicidade de soluções que encontramos, mas também verificamos na seção § 6 anterior a possibilidade de não existir solução alguma para o sistema, em especial em $t = 0$, caso neste instante o determinante contendo as derivadas $\frac{\partial u_i}{\partial x_j}$ seja igual a zero e um ou dois dos determinantes $\det U_1, \det U_2, \det U_3$ também sejam iguais a zero. Estes determinantes $\det U_i$ dependem de $\frac{\partial p}{\partial x_i}$ e $\frac{\partial u_i}{\partial t}$, portanto não poderão ser quaisquer valores da pressão e velocidade que tornarão o sistema univocamente bem determinado (de fato, esta afirmação é válida para qualquer que seja o valor de t). Vimos situação semelhante na equação (7) da seção § 1, relacionada à pressão ser uma função potencial para φ, i.e., φ ser uma função gradiente[5]. Na prática, entretanto, poderemos sempre obter um sistema possível, por exemplo, com adequada escolha da pressão.

Dedicado a Steve Smale!

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11 – A naive solution for Navier-Stokes equations

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Abstract – We seek some attempt solutions for the system of Navier-Stokes equations for spatial dimensions $n = 2$ and $n = 3$. These solutions have the principal objective to provide a better numerical evaluation of the exact analytical solution, thus contributing to the solution not only from a theoretical mathematical problem, but from a practical problem worldwide.

Keywords – Navier-Stokes equations, numerical solutions, exact solutions, equivalent systems, Millennium Problem, existence, smoothness.

1 – Introduction

The reading the last pages of chapter 10 of the book by Ian Stewart, "Seventeen Equations that Changed the World" [1], reminded me once again of the importance of the Navier-Stokes equations, especially of its solutions. A sense of urgency proved necessary for this issue. It is not equal to seek proof of the Riemann hypothesis, which although it is one of the most difficult problems of mathematics does not seem to bring greater immediate consequences to the world.

The problem of the Navier-Stokes equations described in the millennium problems[2] was solved for the case (C), the breakdown of the solutions[3], [4], based on the acceptance of known theorems of existence and uniqueness solutions, theorems I believed fail only when the nonlinear terms are equal to zero, or not have all the terms, although I recognize that the cases more interesting and useful to solve are the cases (A) and (B), the proof of existence and smoothness of their solutions for all initial velocity $u^0(x)$ obeying determinate conditions.

The world is running a serious heating problem, either by natural or human causes. The more likely they are combined causes, of course. The northern hemisphere is heating up more (much more...) that the southern hemisphere, so we cannot rule out the human influence in this heat. Evidently the northern hemisphere is the most industrialized hemisphere of the world, which produces more heat due to their machines, and thus would be more likely to contribute to this warming.

The problem of global warming is not only the increase in temperature, the feeling of discomfort, but also in the disasters that it is able to produce, as the melting ice of the poles, the corresponding increase in sea levels, as well as torrential rains, storms, fires and the most destructive hurricanes.

According Ian Stewart in the mencioned book, two climate vital aspects are the atmosphere and the oceans. Both are fluid, and both can be treated using the Navier-Stokes equation. The secrets of the climate system are closed in the Navier-Stokes equation. He said, referring to a research council document in physical sciences and engineering (EPSRC – Engineering & Physical Sciences Research Council, from United Kingdom), published in 2010: "The secrets of the climate system are closed in the Navier-Stokes equation, but it is too complex to be solved directly". Instead, researchers of climate models are using numerical methods to calculate the fluid flow at the point of a three-dimensional grid covering the globe from the depths of the oceans to the highest points of the atmosphere. The horizontal grid spacing is 100 km; anything less makes your computation impractical. Faster computers will not serve much, then the best way forward is to think harder. Mathematicians are developing more efficient means to numerically solve the Navier-Stokes equation.

Then that's it. The purpose of this paper is to find a solution to the system of Navier-Stokes equations, given the initial condition $u(x,0)=u^0(x),\ x\in\mathbb{R}^n, n=2$ and $n=3$, for both the cases that must be obeyed the equation of incompressibility, $\nabla\cdot u=\nabla\cdot u^0=0$, as also for the general case, any values of $\nabla\cdot u$ and $\nabla\cdot u^0$. When $\nabla\cdot u\neq 0$ must be added the term $\frac{1}{3}\,\nu\nabla(\nabla\cdot u)$ to $\nu\nabla^2 u$ on the right side of the Navier-Stokes equations[5], which for simplicity will be omitted here. Obviously this method can be used for the numerical solution of these equations, which I hope to have your accuracy greatly increased (1 m or less instead of 100 km would be excellent). A grid with width cell 100 km is absolutely unreliable.

2 – Solutions for $n=2$

The system of Navier-Stokes equations in spatial dimension $n=2$ is

(2.1)
$$\begin{cases}\dfrac{\partial p}{\partial x}+\dfrac{\partial u_1}{\partial t}+u_1\dfrac{\partial u_1}{\partial x}+u_2\dfrac{\partial u_1}{\partial y}=\nu\nabla^2 u_1+f_1\\[2mm]\dfrac{\partial p}{\partial y}+\dfrac{\partial u_2}{\partial t}+u_1\dfrac{\partial u_2}{\partial x}+u_2\dfrac{\partial u_2}{\partial y}=\nu\nabla^2 u_2+f_2\end{cases}$$

or in vectorial form

(2.2)
$$\nabla p+\frac{\partial u}{\partial t}+(u\cdot\nabla)u=\nu\nabla^2 u+f,$$

where $u(x,y,t)=\big(u_1(x,y,t),u_2(x,y,t)\big),\ u:\mathbb{R}^2\times[0,\infty)\to\mathbb{R}^2$, is the velocity of the fluid, of components u_1,u_2, p is the pressure, $p:\mathbb{R}^2\times[0,\infty)\to\mathbb{R}$, and $f(x,y,t)=\big(f_1(x,y,t),f_2(x,y,t)\big),\ f:\mathbb{R}^2\times[0,\infty)\to\mathbb{R}^2$, is the density of external force applied in the fluid in point (x,y) and at the instant of time t, for example, gravity force per mass unity, with $x,y,t\in\mathbb{R},\ t\geq 0$. The coefficient $\nu\geq 0$ is the viscosity coefficient, and in the special case that $\nu = 0$ we have the Euler equations. $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ is the nabla operator and $\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \equiv \Delta$ is the Laplacian operator. We are using fluid mass density $\rho = 1$.

If u_1 and u_2 are solutions of system (1) then are valid the following equalities:

$$u_2 = \frac{\nu \nabla^2 u_1 + f_1 - \left(\frac{\partial p}{\partial x} + \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x}\right)}{\frac{\partial u_1}{\partial y}}, \text{ if } \frac{\partial u_1}{\partial y} \neq 0, \tag{2.3}$$

and

$$u_1 = \frac{\nu \nabla^2 u_2 + f_2 - \left(\frac{\partial p}{\partial y} + \frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial y}\right)}{\frac{\partial u_2}{\partial x}}, \text{ if } \frac{\partial u_2}{\partial x} \neq 0. \tag{2.4}$$

The equation (2.3) says that u_2 is a function of u_1, as well as the equation (2.4) says that u_1 is a function of u_2. Therefore, if we have the correct value of u_1 we can get the value of u_2, and vice versa, need for this too that the pressure can be obtained. The equations (2.3) and (2.4) can not contradict each other, i.e., the obtaining u_2 given u_1 in (2.3) must be verified next by the use of the equation (2.4), confirming it, and vice versa. If the pressure p is not a given function for the problem, both equations (2.3) and (2.4) need be solved to the complete obtainment of p. Thus, in principle, the velocity and pressure can be obtained completely following this method, since that $\frac{\partial u_1}{\partial y} \frac{\partial u_2}{\partial x} \neq 0$. In this case, the systems (2.3)–(2.4) and (2.1) are equivalent.

The solutions (2.3) and (2.4) are valid for all $t \geq 0$ on condition that $\frac{\partial u_1}{\partial y} \frac{\partial u_2}{\partial x} \neq 0$, and in this case, in $t = 0$, defining $f(x, y, 0) = f^0(x, y)$ and $p(x, y, 0) = p^0(x, y)$, we come to

$$u_2^0 = \frac{\nu \nabla^2 u_1^0 + f_1^0 - \left(\frac{\partial p^0}{\partial x} + \frac{\partial u_1}{\partial t}|_{t=0} + u_1^0 \frac{\partial u_1^0}{\partial x}\right)}{\frac{\partial u_1^0}{\partial y}}, \text{ if } \frac{\partial u_1^0}{\partial y} \neq 0, \tag{2.5}$$

and

$$u_1^0 = \frac{\nu \nabla^2 u_2^0 + f_2^0 - \left(\frac{\partial p^0}{\partial y} + \frac{\partial u_2}{\partial t}|_{t=0} + u_2^0 \frac{\partial u_2^0}{\partial y}\right)}{\frac{\partial u_2^0}{\partial x}}, \text{ if } \frac{\partial u_2^0}{\partial x} \neq 0, \tag{2.6}$$

i.e., u_1^0 and u_2^0 are related by (2.5) and (2.6), beyond the incompressibility condition, $\nabla \cdot u^0 = 0$, if this is a condition imposed.

The equations (2.5) and (2.6) can be used to calculate $\frac{\partial u_1}{\partial t}|_{t=0}$ and $\frac{\partial u_2}{\partial t}|_{t=0}$, supposing that the pressure or its respective spatial derivatives are provided at least at time $t = 0$.

For other values of $t, t > 0$, through the value of $\frac{\partial u}{\partial t}$, held fixed position (x, y), it is possible to calculate the value of $u(x, y, t)$, obviously by integrating with respect to time the local acceleration $\frac{\partial u}{\partial t}$, i.e.,

(2.7) $$u = \int \frac{\partial u}{\partial t} dt + v(x, y),$$

where $v(x, y)$ may be encountered by given initial conditions.

Numerically, we have

(2.8) $$u^{T+\Delta T} = u^T + \frac{\partial u}{\partial t}|_{t=T} \Delta T,$$

where u^T is the fluid velocity in the position (x, y) at time $t = T$. ΔT is a positive not null small constant, the increment in time to each step calculation for u^T.

Using (2.1) in (2.8) comes

(2.9) $$u_1^{T+\Delta T} = u_1^T + \left(\nu \nabla^2 u_1^T + f_1^T - \frac{\partial p^T}{\partial x} - u_1^T \frac{\partial u_1^T}{\partial x} - u_2^T \frac{\partial u_1^T}{\partial y}\right) \Delta T,$$

(2.10) $$u_2^{T+\Delta T} = u_2^T + \left(\nu \nabla^2 u_2^T + f_2^T - \frac{\partial p^T}{\partial y} - u_1^T \frac{\partial u_2^T}{\partial x} - u_2^T \frac{\partial u_2^T}{\partial y}\right) \Delta T,$$

where f^T and p^T are the external force and pressure, respectively, in the position (x, y) at time $t = T$, supposing given $p^T \equiv p(x, y, T)$. Numerically and algorithmically, we need to use the approximations (among other that knows in the literature about numerical methods[6])

(2.11) $$\frac{\partial u_1^T}{\partial x} \approx \frac{u_1^T(x+\Delta x,y,T) - u_1^T(x,y,T)}{\Delta x},$$

(2.12) $$\frac{\partial u_1^T}{\partial y} \approx \frac{u_1^T(x,y+\Delta y,T) - u_1^T(x,y,T)}{\Delta y},$$

(2.13) $$\frac{\partial u_2^T}{\partial x} \approx \frac{u_2^T(x+\Delta x,y,T) - u_2^T(x,y,T)}{\Delta x},$$

(2.14) $$\frac{\partial u_2^T}{\partial y} \approx \frac{u_2^T(x,y+\Delta y,T) - u_2^T(x,y,T)}{\Delta y},$$

$$\nabla^2 u_1^T \approx \frac{u_1^T(x+2\Delta x,y,T)-2u_1^T(x+\Delta x,y,T)+u_1^T(x,y,T)}{(\Delta x)^2}+ \tag{2.15}$$

$$+\frac{u_1^T(x,y+2\Delta y,T)-2u_1^T(x,y+\Delta y,T)+u_1^T(x,y,T)}{(\Delta y)^2},$$

$$\nabla^2 u_2^T \approx \frac{u_2^T(x+2\Delta x,y,T)-2u_2^T(x+\Delta x,y,T)+u_2^T(x,y,T)}{(\Delta x)^2}+ \tag{2.16}$$

$$+\frac{u_2^T(x,y+2\Delta y,T)-2u_2^T(x,y+\Delta y,T)+u_2^T(x,y,T)}{(\Delta y)^2},$$

where $\Delta x \times \Delta y$ is the grid cell size.

This numerical-algorithmic approach, which resulted in the equations (2.9) to (2.16), it shows that we can calculate approximately the system solution (2.1) from $t = 0$ up to any $t = T_{max}$, and the same method can be used in $n = 3$. When greater T_{max} value, however, the greater the accumulation of numerical errors to the correct result. It will be very convenient if it is possible to obtain an exact solution (the great dream) to this problem, at least in certain situations, eliminating thus to the maximum the occurrence of numerical errors. Our naive solution, or better, our first naive attempt solution, will be described to follow.

The smaller the value of T, the closest correct value of u are the results obtained with (2.9) and (2.10). Therefore, considering t a small value, in the first order approximation in time the solution to the u components will be

$$u_1 = u_1^0 + \left(\nu\nabla^2 u_1^0 + f_1 - \frac{\partial p}{\partial x} - u_1^0\frac{\partial u_1^0}{\partial x} - u_2^0\frac{\partial u_1^0}{\partial y}\right)t, \tag{2.17}$$

$$u_2 = u_2^0 + \left(\nu\nabla^2 u_2^0 + f_2 - \frac{\partial p}{\partial y} - u_1^0\frac{\partial u_2^0}{\partial x} - u_2^0\frac{\partial u_2^0}{\partial y}\right)t, \tag{2.18}$$

which shows the possibility of infinite solutions to velocity, given only the initial velocity, since each different pressure can, in principle, imply a different velocity. Unfortunately, in general the above solution is not limited to the increased time, and therefore in general there is not here a case of velocity belonging to Schwartz space, space of fast decreasing functions. This time t in (2.17) and (2.18) corresponds exactly to the ΔT value that appears in (2.9) and (2.10).

Defining $x_1 := x, x_2 := y$, for an arbitrary value of t, we can try a solution to the system (2.1) in the form

$$u_i = u_i^0 + X_i\left(u_1^0, u_2^0, f_i, \frac{\partial p}{\partial x_i}\right)T_i(t), \tag{2.19}$$

with

$$T_i(0) = 0,\ T_i{}'(0) = 1, \tag{2.20}$$

in special

$$(2.21) \qquad X_i = \nu\nabla^2 u_i^0 + f_i - \frac{\partial p}{\partial x_i} - u_1^0 \frac{\partial u_i^0}{\partial x} - u_2^0 \frac{\partial u_i^0}{\partial y},$$

or else, for example,

$$(2.22) \qquad u_i = u_i^0 + X_i(u_1^0, u_2^0)t + \int \left(f_i - \frac{\partial p}{\partial x_i}\right) dt + v_i(x, y),$$

$$(2.23) \qquad X_i = \nu\nabla^2 u_i^0 - u_1^0 \frac{\partial u_i^0}{\partial x} - u_2^0 \frac{\partial u_i^0}{\partial y},$$

solutions based on (2.17) and (2.18), with

$$(2.24) \qquad \int \left(f_i - \frac{\partial p}{\partial x_i}\right) dt|_{t=0} + v_i(x, y) = 0.$$

Differentiating (2.22) in relation to time, obviously, we obtain

$$(2.25) \qquad \frac{\partial u_i}{\partial t} = X_i(u_1^0, u_2^0) + f_i - \frac{\partial p}{\partial x_i},$$

or, using (2.23),

$$(2.26) \qquad \frac{\partial u_i}{\partial t} = \nu\nabla^2 u_i^0 - u_1^0 \frac{\partial u_i^0}{\partial x} - u_2^0 \frac{\partial u_i^0}{\partial y} + f_i - \frac{\partial p}{\partial x_i}.$$

To the equation (2.26) to be equivalent to the system (2.1) for all u_i we need to have

$$(2.27) \qquad \nu\nabla^2 u_i - u_1 \frac{\partial u_i}{\partial x} - u_2 \frac{\partial u_i}{\partial y} = \nu\nabla^2 u_i^0 - u_1^0 \frac{\partial u_i^0}{\partial x} - u_2^0 \frac{\partial u_i^0}{\partial y},$$

therefore, trying

$$(2.28) \qquad u_i(x, y, t) = u_i^0(x, y) + w_i(t),\ w_i(0) = 0,$$

and substituting (2.28) in (2.27), it is necessary that

$$(2.29) \qquad w_1(t) \frac{\partial u_i^0}{\partial x} + w_2(t) \frac{\partial u_i^0}{\partial y} = 0.$$

The trivial solutions of (2.29) are $w_1(t) = w_2(t) = 0$ and $u_i^0 = cte$. A more general condition is

$$(2.30) \qquad \frac{w_1(t)}{w_2(t)} = -\frac{\partial u_i^0/\partial y}{\partial u_i^0/\partial x} = k,\ k \in \mathbb{R}^*,\ i = 1, 2.$$

Well, the solution (2.28) there is not the same form that (2.22)–(2.23), except if

(2.31) $$\begin{cases} f_i - \frac{\partial p}{\partial x_i} = v_i(x,y) = 0 \\ w_i(t) = k_i t,\ k_i \in \mathbb{R}^* \\ X_i = \nu \nabla^2 u_i^0 - u_1^0 \frac{\partial u_i^0}{\partial x} - u_2^0 \frac{\partial u_i^0}{\partial y} = k_i,\ k_i \in \mathbb{R}^* \end{cases}$$

and, according (2.30),

(2.32) $$\frac{\partial u_i^0}{\partial y} = -k \frac{\partial u_i^0}{\partial x},\ k = \frac{k_1}{k_2},\ k, k_1, k_2 \in \mathbb{R}^*.$$

For this reason, the attempt solution (2.22)–(2.23) correctly solved the system (2.1) for some initial velocities, in special when (2.31) and (2.32) are obeyed. Another case of solution when (2.22)–(2.23) is valid, using trivial solution of (2.29), is

(2.33) $$\begin{cases} f_i - \frac{\partial p}{\partial x_i} = v_i(x,y) = 0 \\ u_i^0 = u_i = cte. \end{cases}$$

The dependence of f in relation to p, related in (2.31) and (2.33), or

(2.34) $$\nabla p = f,$$

shows that it's necessary f be a gradient function, and thus p is a potential function for f (see, for example, [7]). An example for f is a constant gravity acceleration, like $f = (0, -g)$, assuming a two-dimensional world, and in this case we have $p = -gy$.

For more generic initial velocity, the form given by (2.28) is our next attempt solution,

(2.35) $$u_i(x,y,t) = u_i^0(x,y) + w_i(t),\ w_i(0) = 0.$$

Applying (2.35) in (2.1) comes

(2.36) $$\frac{\partial p}{\partial x_i} + \frac{d}{dt} w_i(t) + u_1^0 \frac{\partial u_i^0}{\partial x} + u_2^0 \frac{\partial u_i^0}{\partial y} + w_i(t) \left[\frac{\partial u_i^0}{\partial x} + \frac{\partial u_i^0}{\partial y}\right] =$$
$$= \nu \nabla^2 u_i^0 + f_i,$$

using $x_1 := x,\ x_2 := y$.

A consistent initial velocity also needs to be (2.1) solution, for $t = 0$. In $t = 0$ the equation (2.36) is equivalent to

(2.37) $$\frac{\partial p^0}{\partial x_i} + w_i'(0) + u_1^0 \frac{\partial u_i^0}{\partial x} + u_2^0 \frac{\partial u_i^0}{\partial y} = \nu \nabla^2 u_i^0 + f_i^0,$$

using $w_i(0) = 0$, so

$$(2.38) \qquad u_1^0 \frac{\partial u_i^0}{\partial x} + u_2^0 \frac{\partial u_i^0}{\partial y} = \nu \nabla^2 u_i^0 + f_i^0 - \frac{\partial p^0}{\partial x_i} - w_i{}'(0),$$

the superior symbol 0 meaning the respective function value at time $t = 0$.

Substituting (2.38) in (2.36) we obtain

$$(2.39) \qquad \left(\frac{\partial p}{\partial x_i} - \frac{\partial p^0}{\partial x_i}\right) + \left(w_i'(t) - w_i'(0)\right) + \left[w_1(t)\frac{\partial u_i^0}{\partial x} + w_2(t)\frac{\partial u_i^0}{\partial y}\right] = \\ = f_i(x,y,t) - f_i^0(x,y),$$

a equality that allow us to solve the system (2.1) in many situations, for any u^0 (or better, $\forall u^0 \in C^2(\mathbb{R}^2)$), making u_i^0 (and therefore u_i) a function of p, w_j, f_i, or by contrary, making p a function of u_i^0, w_j, f_i, being $u_i(x,y,t) = u_i^0(x,y) + w_i(t)$, according (2.35). But for this reason we cannot to accept any external force and pressure in the system, or model, except when (2.39) is true and the pressure can be calculated. Between numerically solve the system (2.39) or (2.1) seems to me that (2.39) is faster to solve, in special when the pressure is given, $p \in C^1(\mathbb{R}^2 \times [0,\infty))$, and the velocity is the unique unknown variable.

The next and last attempt solution is

$$(2.40) \qquad u_i(x,y,t) = u_i^0(x,y)\, w_i(t), \ \ w_i(0) = 1,$$

where $u_i: \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}$, $u_i^0: \mathbb{R}^2 \to \mathbb{R}$, $w_i: [0,\infty) \to \mathbb{R}$, $i = 1, 2$.

Repeating the steps from (2.35) to (2.39) with (2.40), applying (2.40) in (2.1) we come to

$$(2.41) \qquad \frac{\partial p}{\partial x_i} + u_i^0 \frac{d}{dt} w_i + w_1 w_i u_1^0 \frac{\partial u_i^0}{\partial x} + w_2 w_i u_2^0 \frac{\partial u_i^0}{\partial y} = \\ = \frac{\partial p}{\partial x_i} + u_i^0 w_i' + w_i \left[w_1 u_1^0 \frac{\partial u_i^0}{\partial x} + w_2 u_2^0 \frac{\partial u_i^0}{\partial y}\right] = \nu w_i \nabla^2 u_i^0 + f_i.$$

As we have said, a consistent initial velocity also needs to be (2.1) solution, for $t = 0$. In $t = 0$ the equation (2.41) is equivalent to

$$(2.42) \qquad \frac{\partial p^0}{\partial x_i} + u_i^0 w_i'^0 + w_1^0 w_i^0 u_1^0 \frac{\partial u_i^0}{\partial x} + w_2^0 w_i^0 u_2^0 \frac{\partial u_i^0}{\partial y} = \\ = \nu w_i^0 \nabla^2 u_i^0 + f_i^0,$$

defining $w_i'^0 = \frac{dw_i}{dt}|_{t=0}$ and $w_i^0 = w_i(0) = 1$, so

$$(2.43) \qquad w_1^0 w_i^0 u_1^0 \frac{\partial u_i^0}{\partial x} + w_2^0 w_i^0 u_2^0 \frac{\partial u_i^0}{\partial y} - \nu w_i^0 \nabla^2 u_i^0 =$$

$$= \left[u_1^0 \frac{\partial u_i^0}{\partial x} + u_2^0 \frac{\partial u_i^0}{\partial y} - \nu \nabla^2 u_i^0\right] =$$
$$= f_i^0 - \frac{\partial p^0}{\partial x_i} - u_i^0 w_i'^0.$$

Supposing $w_1 = w_2 = w$ and therefore $w_1^0 = w_2^0 = w^0 = 1$, $w_1' = w_2' = w'$, $w_1'^0 = w_2'^0 = w'^0$, we have from (2.41) and (2.43), respectively,

(2.44) $$\frac{\partial p}{\partial x_i} + u_i^0\, w' + w^2 \left[u_1^0 \frac{\partial u_i^0}{\partial x} + u_2^0 \frac{\partial u_i^0}{\partial y}\right] = \nu w \nabla^2 u_i^0 + f_i$$

and

(2.45) $$\left[u_1^0 \frac{\partial u_i^0}{\partial x} + u_2^0 \frac{\partial u_i^0}{\partial y}\right] = \nu \nabla^2 u_i^0 + f_i^0 - \frac{\partial p^0}{\partial x_i} - u_i^0 w'^0.$$

Taking the factor $\left[u_1^0 \frac{\partial u_i^0}{\partial x} + u_2^0 \frac{\partial u_i^0}{\partial y}\right]$ in (2.45) and leading it in (2.44) we obtain

(2.46) $$\left(\frac{\partial p}{\partial x_i} - \alpha \frac{\partial p^0}{\partial x_i}\right) = \nu\,(w - \alpha)\nabla^2 u_i^0 - u_i^0(w' - \alpha w'^0) +$$
$$+\left(f_i - \alpha f_i^0\right),$$

with $\alpha = w^2(t) \neq 0$. This relation (2.46) shows us that there are many possibilities to solve the system of Navier-Stokes equations, for an infinite set of initial velocities, external forces and pressure, thereby eliminating the non-linear term.

The integration of (2.46) is

(2.47) $$p - \alpha p^0 = \int_L \; S \cdot dl,$$
$$S = \nu\,(w - \alpha)\nabla^2 u^0 - u^0(w' - \alpha w'^0) + (f - \alpha f^0),$$

where L is any continuous path linking a point (x_0, y_0) to (x, y), supposing that the integrand S is a gradient field[7], without singularities.

3 – Solutions for $n = 3$

Similar to what we saw in section 2 for $n = 2$, now we solve the Navier-Stokes equations for spatial dimension $n = 3$ and fluid mass density $\rho = 1$. As we know, it can be put in the form of a system of three nonlinear partial differential equations, as follows:

$$(3.1) \qquad \begin{cases} \frac{\partial p}{\partial x} + \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + u_3 \frac{\partial u_1}{\partial z} = \nu \nabla^2 u_1 + f_1 \\ \frac{\partial p}{\partial y} + \frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + u_3 \frac{\partial u_2}{\partial z} = \nu \nabla^2 u_2 + f_2 \\ \frac{\partial p}{\partial z} + \frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x} + u_2 \frac{\partial u_3}{\partial y} + u_3 \frac{\partial u_3}{\partial z} = \nu \nabla^2 u_3 + f_3 \end{cases}$$

where $u(x,y,z,t) = \big(u_1(x,y,z,t), u_2(x,y,z,t), u_3(x,y,z,t)\big)$, $u: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$, is the velocity of the fluid, of components u_1, u_2, u_3, p is the pressure, $p: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}$, and $f(x,y,z,t) = \big(f_1(x,y,z,t), f_2(x,y,z,t), f_3(x,y,z,t)\big)$, $f: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$, is the density of external force applied in the fluid in point (x,y,z) and at the instant of time t, for example, gravity force per mass unity, with $x,y,z,t \in \mathbb{R}$, $t \geq 0$. The coefficient $\nu \geq 0$ is the viscosity coefficient, and in the special case that $\nu = 0$ we have the Euler equations. $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ is the nabla operator and $\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \equiv \Delta$ is the Laplacian operator.

Writing u_1 as a function of u_2 and u_3 we have by the system (3.1) above,

$$(3.2) \qquad u_1 = \frac{\nu \nabla^2 u_2 + f_2 - \left(\frac{\partial p}{\partial y} + \frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial y} + u_3 \frac{\partial u_2}{\partial z}\right)}{\frac{\partial u_2}{\partial x}}, \text{ if } \frac{\partial u_2}{\partial x} \neq 0,$$

$$(3.3) \qquad u_1 = \frac{\nu \nabla^2 u_3 + f_3 - \left(\frac{\partial p}{\partial z} + \frac{\partial u_3}{\partial t} + u_2 \frac{\partial u_3}{\partial y} + u_3 \frac{\partial u_3}{\partial z}\right)}{\frac{\partial u_3}{\partial x}}, \text{ if } \frac{\partial u_3}{\partial x} \neq 0,$$

$$(3.4) \qquad \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + u_3 \frac{\partial u_1}{\partial z} = \nu \nabla^2 u_1 + f_1 - \frac{\partial p}{\partial x},$$

therefore valid system when $\frac{\partial u_2}{\partial x} \frac{\partial u_3}{\partial x} \neq 0$.

Similarly to u_1, we obtain the following equations for u_2 and u_3, in index notation, defining $x_1 := x$, $x_2 := y$, $x_3 := z$, and index 4 = index 1, index 5 = index 2, with $1 \leq j \leq 3$,

$$(3.5) \qquad u_i = \frac{\nu \nabla^2 u_j + f_j - \left(\frac{\partial p}{\partial x_j} + \frac{\partial u_j}{\partial t} + u_{i+1} \frac{\partial u_j}{\partial x_{i+1}} + u_{i+2} \frac{\partial u_j}{\partial x_{i+2}}\right)}{\frac{\partial u_j}{\partial x_i}}, \text{ if } \frac{\partial u_j}{\partial x_i} \neq 0,$$

$$(3.6) \qquad \frac{\partial u_i}{\partial t} + u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y} + u_3 \frac{\partial u_i}{\partial z} = \nu \nabla^2 u_i + f_i - \frac{\partial p}{\partial x_i},$$

therefore valid systems when $\frac{\partial u_{i+1}}{\partial x_i} \frac{\partial u_{i+2}}{\partial x_i} \neq 0$, for $1 \leq i \leq 3$.

All solutions obtained in (3.5) can not contradict each other, as well as (3.6) must be true for each i.

The solutions (3.5) are valid for all $t \geq 0$ on condition that $\frac{\partial u_{i+1}}{\partial x_i}\frac{\partial u_{i+2}}{\partial x_i} \neq 0$, for $1 \leq i \leq 3$, and in this case, in $t = 0$, defining $f(x, y, z, 0) = f^0(x, y, z)$ and $p(x, y, z, 0) = p^0(x, y, z)$ and using index notation, we come to

$$(3.7) \qquad u_i^0 = \frac{\nu\nabla^2 u_j^0 + f_j^0 - \left(\frac{\partial p^0}{\partial x_j} + \frac{\partial u_j}{\partial t}|_{t=0} + u_{i+1}^0 \frac{\partial u_j^0}{\partial x_{i+1}} + u_{i+2}^0 \frac{\partial u_j^0}{\partial x_{i+2}}\right)}{\frac{\partial u_j^0}{\partial x_i}}, 1 \leq j \leq 3,$$

where the superior index 0 means the respective value function at time $t = 0$. The equation (3.7) shows that the sum $\frac{\partial p^0}{\partial x_j} + \frac{\partial u_j}{\partial t}|_{t=0}$ cannot have any arbitrary value, independently of u_i^0 relation (3.7), contradicting it.

Numerically we can solve (3.1) through following iteration algorithm, just like we do for $n = 2$, for each natural i in $1 \leq i \leq 3$:

$$(3.8) \qquad u_i^{T+\Delta T} = u_i^T + \left(\nu\nabla^2 u_i^T + f_i^T - \frac{\partial p^T}{\partial x} - \sum_{j=1}^n u_j^T \frac{\partial u_i^T}{\partial x_j}\right)\Delta T,$$

where u^T, f^T and p^T are the velocity, external force and pressure, respectively, in the position (x, y, z) at time $t = T$, supposing given $p^T \equiv p(x, y, z, T)$. ΔT is a positive not null small constant, the increment in time to each step calculation for u^T.

Again, we need to use the approximations (among other that knows in the literature containing numerical methods[6])

$$(3.9) \qquad \frac{\partial u_1^T}{\partial x} \approx \frac{u_1^T(x+\Delta x,y,z,T) - u_1^T(x,y,z,T)}{\Delta x},$$

$$(3.10) \qquad \frac{\partial u_1^T}{\partial y} \approx \frac{u_1^T(x,y+\Delta y,z,T) - u_1^T(x,y,z,T)}{\Delta y},$$

$$(3.11) \qquad \frac{\partial u_1^T}{\partial z} \approx \frac{u_1^T(x,y,z+\Delta z,T) - u_1^T(x,y,z,T)}{\Delta z},$$

$$(3.12) \qquad \frac{\partial u_2^T}{\partial x} \approx \frac{u_2^T(x+\Delta x,y,z,T) - u_2^T(x,y,z,T)}{\Delta x},$$

$$(3.13) \qquad \frac{\partial u_2^T}{\partial y} \approx \frac{u_2^T(x,y+\Delta y,z,T) - u_2^T(x,y,z,T)}{\Delta y},$$

$$(3.14) \qquad \frac{\partial u_2^T}{\partial z} \approx \frac{u_2^T(x,y,z+\Delta z,T) - u_2^T(x,y,z,T)}{\Delta z},$$

$$(3.15) \qquad \frac{\partial u_3^T}{\partial x} \approx \frac{u_3^T(x+\Delta x,y,z,T) - u_3^T(x,y,z,T)}{\Delta x},$$

(3.16) $$\frac{\partial u_3^T}{\partial y} \approx \frac{u_3^T(x,y+\Delta y,z,T) - u_3^T(x,y,z,T)}{\Delta y},$$

(3.17) $$\frac{\partial u_3^T}{\partial z} \approx \frac{u_3^T(x,y,z+\Delta z,T) - u_3^T(x,y,z,T)}{\Delta z},$$

(3.18) $$\nabla^2 u_1^T \approx \frac{u_1^T(x+2\Delta x,y,z,T) - 2u_1^T(x+\Delta x,y,z,T) + u_1^T(x,y,z,T)}{(\Delta x)^2} +$$
$$+ \frac{u_1^T(x,y+2\Delta y,z,T) - 2u_1^T(x,y+\Delta y,z,T) + u_1^T(x,y,z,T)}{(\Delta y)^2} +$$
$$+ \frac{u_1^T(x,y,z+2\Delta z,T) - 2u_1^T(x,y,z+\Delta z,T) + u_1^T(x,y,z,T)}{(\Delta z)^2},$$

(3.19) $$\nabla^2 u_2^T \approx \frac{u_2^T(x+2\Delta x,y,z,T) - 2u_2^T(x+\Delta x,y,z,T) + u_2^T(x,y,z,T)}{(\Delta x)^2} +$$
$$+ \frac{u_2^T(x,y+2\Delta y,z,T) - 2u_2^T(x,y+\Delta y,z,T) + u_2^T(x,y,z,T)}{(\Delta y)^2} +$$
$$+ \frac{u_2^T(x,y,z+2\Delta z,T) - 2u_2^T(x,y,z+\Delta z,T) + u_2^T(x,y,z,T)}{(\Delta z)^2},$$

(3.20) $$\nabla^2 u_3^T \approx \frac{u_3^T(x+2\Delta x,y,z,T) - 2u_3^T(x+\Delta x,y,z,T) + u_3^T(x,y,z,T)}{(\Delta x)^2} +$$
$$+ \frac{u_3^T(x,y+2\Delta y,z,T) - 2u_3^T(x,y+\Delta y,z,T) + u_3^T(x,y,z,T)}{(\Delta y)^2} +$$
$$+ \frac{u_3^T(x,y,z+2\Delta z,T) - 2u_3^T(x,y,z+\Delta z,T) + u_3^T(x,y,z,T)}{(\Delta z)^2},$$

where $\Delta x \times \Delta y \times \Delta z$ is the three-dimensional grid cell size.

The greater the value of T, the greater the number of times that need to iterate the solution given in (3.8), more numeric errors are added to the correct solution of system (3.1), is therefore highly desirable to find an exact solution for (3.1).

All attempt solutions seen for the case $n = 2$ can be used for $n = 3$, with obviously adaptations. The simplest (and naive) of these solutions is the similar one to (2.35),

(3.21) $$u_i(x, y, z, t) = u_i^0(x, y, z) + w_i(t),\ \ w_i(0) = 0,$$

or

(3.22) $$u(x, y, z, t) = u^0(x, y, z) + w(t),\ \ w(0) = 0,$$

whose direct application in (3.1) and more the correspondent use for $t=0$ leads to the similar condition (2.39) seen previously, i.e.,

$$(3.23)\qquad \left(\frac{\partial p}{\partial x_i}-\frac{\partial p^0}{\partial x_i}\right)+\left(w_i{}'(t)-w_i{}'(0)\right)+ \\ +\left[w_1\frac{\partial u_i^0}{\partial x}+w_2\frac{\partial u_i^0}{\partial y}+w_3\frac{\partial u_i^0}{\partial z}\right]=f_i(x,y,z,t)-f_i{}^0(x,y,z).$$

As we have said for two dimensions, this equality allow us to solve the system (3.1) in many situations, for any u^0 (say, $\forall u^0\in C^2(\mathbb{R}^3)$), making u_i^0 (and u_i) a function of p,w_j,f_i, or by contrary, making p a function of u_i^0,w_j,f_i, being $u_i(x,y,z,t)=u_i^0(x,y,z)+w_i(t)$, according (3.21). For this reason, when (3.21) is valid we cannot to accept any external force and pressure in the system, or model, except when (3.23) is true and the pressure can be calculated. Again, between numerically solve the system (3.23) or (3.1) seems to me that (3.23) is faster to solve, in special when the pressure is given, $p\in C^1(\mathbb{R}^3\times[0,\infty))$, and the velocity is the unique unknown variable. By default, however, the initial velocity is the given function and the pressure is an unknown variable to be calculated, and in this manner it is necessary that

$$(3.24)\qquad S=\left(f(x,y,z,t)-f^0(x,y,z)\right)-\left(w'(t)-w'(0)\right)- \\ -\left(w_1\frac{\partial u_i^0}{\partial x}+w_2\frac{\partial u_i^0}{\partial y}+w_3\frac{\partial u_i^0}{\partial z}\right)_{1\le i\le 3}$$

is a gradient vector function[7]. In this general case we have

$$(3.25)\qquad p-p^0=\int_L S\cdot dl,$$

where L is any continuous path linking a point (x_0,y_0,z_0) to (x,y,z).

In the special case when $S=(S_1,\ S_2,\ S_3)$ is equal to zero or an explicit function of time and $S(0)=0$, the solution for (3.23) is

$$(3.26)\qquad p-p^0=[S_1(t)(x-x_0)+S_2(t)(y-y_0)+S_3(t)(z-z_0)]+ \\ +\theta(t),$$

where $\theta(t)$ is a well behaved (or physically reasonable) generic time function with $\theta(0)=0$, and $p^0(x,y,z)=p(x,y,z,0)$.

The solutions for u, (3.22), and p, (3.25), are not unique, due to infinities different possibilities to construct $w(t)$, $w(0)=0$. Beyond this, the pressure may be unlimited, when $S(t)\neq 0$, due to linear term $[S_1(t)(x-x_0)+S_2(t)(y-y_0)+S_3(t)(z-z_0)]$, although we can choose $u^0(x,y,z)$ and $w(t)$ that limit the velocity $u(x,y,z,t)$.

The next and last attempt solution for $n = 3$ is

(3.27) $$u_i(x, y, z, t) = u_i^0(x, y, z)\, w(t),\ \ w(0) = 1,$$

where $u_i: \mathbb{R}^3 \times [0, \infty) \to \mathbb{R},\ u_i^0: \mathbb{R}^3 \to \mathbb{R},\ \ w: [0, \infty) \to \mathbb{R},\ \ 1 \leq i \leq 3.$

Applying (3.27) in (3.1), following similarly the steps from (2.40) to (2.46), we come to

(3.28) $$\left(\frac{\partial p}{\partial x_i} - \alpha \frac{\partial p^0}{\partial x_i}\right) = \nu\, (w - \alpha)\nabla^2 u_i^0 - u_i^0(w' - \alpha w'^0) + \\ +\left(f_i - \alpha f_i^0\right),$$

with $\alpha = w^2(t) \neq 0.$ Note that this form is no longer necessary to worry about the non-linear term, although the correspondent to (2.45) is implicitly valid for all $i = 1, 2, 3$, of course:

(3.29) $$\left[u_1^0 \frac{\partial u_i^0}{\partial x} + u_2^0 \frac{\partial u_i^0}{\partial y} + u_3^0 \frac{\partial u_i^0}{\partial z}\right] = \nu \nabla^2 u_i^0 + f_i^0 - \frac{\partial p^0}{\partial x_i} - u_i^0 w'^0.$$

The integration of (3.27) is

(3.30) $$p - \alpha p^0 = \int_L\ S \cdot dl,$$
$$S = \nu\, (w - \alpha)\nabla^2 u^0 - u^0(w' - \alpha w'^0) + (f - \alpha f^0),$$

where L is any continuous path linking a point (x_0, y_0, z_0) to (x, y, z), supposing again that the integrand S is a gradient field[7], without singularities.

This solution is not always possible, when S is not a gradient field, but it is easily soluble when u^0 is equal to zero and the external force also is $f = f^0 = 0$. In this situation we have $u = 0$ and

(3.31) $$p = w^2(t) p^0 + \theta(t),\ \ w(0) = 1,\ \ \theta(0) = 0,$$

again showing the non-uniqueness of the solution for the pressure. $\theta(t)$ is our well behaved (physically reasonable) generic time function.

4 – Conclusion

We do not solve exactly the Navier-Stokes equations in the general case, given any initial velocity, nor proved that this is possible, but we developed some attempt solutions for some initial velocities. Particularly, if we know exactly the value of one ($n = 2$) or two ($n = 3$) velocity components and the pressure we can find the exact value of the component we have not initially, according to equations (2.3), (2.4) and (3.5). In special, for $1 \leq i, j \leq 3$, the exact solution that we seek is

(4.1) $$u_i = \frac{\nu\nabla^2 u_j + f_j - \left(\frac{\partial p}{\partial x_j} + \frac{\partial u_j}{\partial t} + u_{i+1}\frac{\partial u_j}{\partial x_{i+1}} + u_{i+2}\frac{\partial u_j}{\partial x_{i+2}}\right)}{\frac{\partial u_j}{\partial x_i}}, \text{ if } \frac{\partial u_j}{\partial x_i} \neq 0.$$

Nothing easier than this, although there can be no contradiction, of course. For example, for $n = 3$ the equation (3.6) must continue to be satisfied, i.e.,

(4.2) $$\frac{\partial u_i}{\partial t} + u_1\frac{\partial u_i}{\partial x} + u_2\frac{\partial u_i}{\partial y} + u_3\frac{\partial u_i}{\partial z} = \nu\nabla^2 u_i + f_i - \frac{\partial p}{\partial x_i}.$$

Of the numerical point of view, I think that solve the system (3.23),

(4.3) $$\left(\frac{\partial p}{\partial x_i} - \frac{\partial p^0}{\partial x_i}\right) + \left(w_i{}'(t) - w_i{}'(0)\right) + \\ + \left[w_1\frac{\partial u_i^0}{\partial x} + w_2\frac{\partial u_i^0}{\partial y} + w_3\frac{\partial u_i^0}{\partial z}\right] = f_i(x,y,z,t) - f_i^0(x,y,z),$$

$1 \leq i \leq 3$, is faster than (3.1), and they are equivalent systems, assuming the validity of

(4.4) $$u_i(x,y,z,t) = u_i^0(x,y,z) + w_i(t),\ w_i(0) = 0.$$

The solution above shows that the velocity u can vary in a same point from the initial velocity u^0 to any other value, adding a convenient time function $w(t)$.

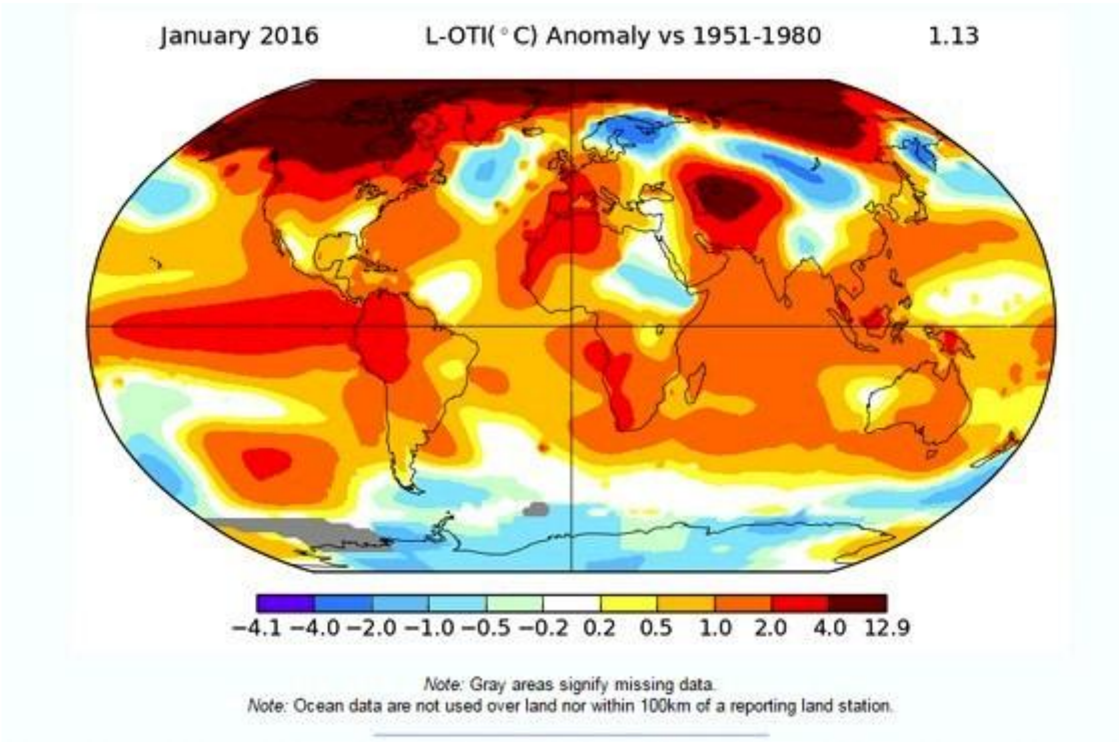
More than this, we reaches to an important conclusion: the solutions of the Navier-Stokes equations may not be unique, even for two spatial dimensions, even for three spatial dimensions, even with all equations with all terms and even for any very small time $t > 0$, and indeed for any value spatial dimension n. We lack at least initial conditions for the pressure, among other requirements, at least in the cases which we analyze. The possibility of infinite solutions, however, even for cases in which all terms are present, leads us to conclude on the need to provide more equations to the models that claim to accurately simulate the atmospheric conditions or fluids in general terms, from simplest cases to the most complex one, or else build more complete Navier-Stokes equations, containing more dependent variable, initial and boundary conditions. What we can see, the same initial velocity can generate both an eternal calm as a giant seaquake.

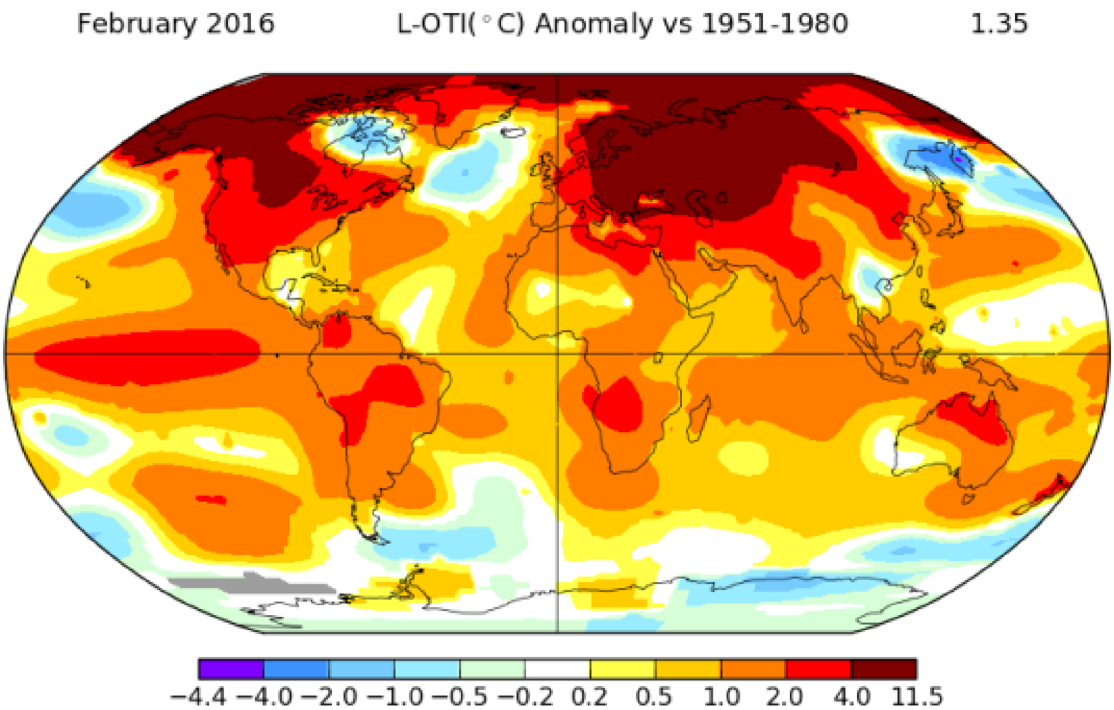
Of course the velocity of a storm, hurricane or a tsunami does not need to be regular, limited, continuous, infinitely differentiable, well behaved and belonging to the Schwartz space, nor obey to the incompressibility condition. This gives us enough freedom to work with these equations.

I think that, in practical terms, the external force can act as a pressure or velocity controller, since it is not only due to the uncontrollable nature, but can also be conveniently constructed by engineering. This is a clear example of Applied Mathematical.

More naive than these solutions is this author…

To world's stability…





Images source: NASA (see http://data.giss.nasa.gov/gistemp/maps/)

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12 – Solution for Navier-Stokes Equations – Lagrangian and Eulerian Descriptions

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Abstract – We find an exact solution for the system of Navier-Stokes equations, supposing that there is some solution, following the Eulerian and Lagrangian descriptions, for spatial dimension n = 3. As we had seen in other previous articles, it is possible that there are infinite solutions for pressure and velocity, given only the condition of initial velocity.

Keywords – Navier-Stokes equations, velocity, pressure, Eulerian description, Lagrangian description, formulation, classical mechanics, Newtonian mechanics, Newton's law, second law of Newton, equivalent systems, exact solutions, Millennium Problem, existence, smoothness, Bernoulli's law, Turbulence Theory, Theory of Perturbations, Numerical Methods, Computational Fluid Dynamics.

§ 1

Essentially the Navier-Stokes equations relating the velocity u and pressure p suffered by a volume element dV at position (x, y, z) and time t. In the formulation or description Eulerian the position (x, y, z) is fixed in time, running different volume elements of fluid in this same position, while the time varies. In the Lagrangian formulation the position (x, y, z) refers to the instantaneous position of a specific volume element $dV = dx\, dy\, dz$ at time t, and this position varies with the movement of this same element dV.

Basically, the deduction of the Navier-Stokes equations is a classical mechanics problem, a problem of Newtonian mechanics, which use the 2nd law of Newton $F = ma$, force is equal to mass multiplied by acceleration. We all know that the force described in Newton's law may have different expressions, varying only in time or also with the position, or with the distance to the source, varying with the body's velocity, etc. Each specific problem must to define how the forces involved in the system are applied and what they mean. I suggest consulting the classic Landau & Lifshitz[1] or Prandtl's book[2] for a more detailed description of the deduction of these equations. Note that the deduction by Landau & Lifshitz [1] contain more parameters than the shown in the references [2] and [3].

In spatial dimension $n = 3$, the Navier-Stokes equations can be put in the form of a system of three nonlinear partial differential equations, as follows:

$$(1)\quad \begin{cases} \frac{\partial p}{\partial x} + \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + u_3 \frac{\partial u_1}{\partial z} = \nu \nabla^2 u_1 + \frac{1}{3} \nu \nabla_1 (\nabla \cdot u) + f_1 \\ \frac{\partial p}{\partial y} + \frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + u_3 \frac{\partial u_2}{\partial z} = \nu \nabla^2 u_2 + \frac{1}{3} \nu \nabla_2 (\nabla \cdot u) + f_2 \\ \frac{\partial p}{\partial z} + \frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x} + u_2 \frac{\partial u_3}{\partial y} + u_3 \frac{\partial u_3}{\partial z} = \nu \nabla^2 u_3 + \frac{1}{3} \nu \nabla_3 (\nabla \cdot u) + f_3 \end{cases}$$

where $u(x,y,z,t) = \big(u_1(x,y,z,t), u_2(x,y,z,t), u_3(x,y,z,t)\big)$, $u: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$, is the velocity of the fluid, of components u_1, u_2, u_3, p is the pressure, $p: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}$, and $f(x,y,z,t) = \big(f_1(x,y,z,t), f_2(x,y,z,t), f_3(x,y,z,t)\big)$, $f: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$, is the density of external force applied in the fluid in point (x,y,z) and at the instant of time t, for example, gravity force per mass unity, with $x,y,z,t \in \mathbb{R}$, $t \geq 0$. The coefficient $\nu \geq 0$ is the viscosity coefficient, and in the special case that $\nu = 0$ we have the Euler equations. $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ is the nabla operator and $\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \equiv \Delta$ is the Laplacian operator.

The non-linear terms $u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y} + u_3 \frac{\partial u_i}{\partial z}$, $1 \leq i \leq 3$, are a natural consequence of the Eulerian formulation of motion, and corresponds to part of the total derivative of velocity with respect to time of a volume element dV in the fluid, i.e., its acceleration. If $u = (u_1(x,y,z,t), u_2(x,y,z,t), u_3(x,y,z,t))$ and these x,y,z also vary in time, $x = x(t)$, $y = y(t)$, $z = z(t)$, then, by the chain rule,

$$(2)\qquad \frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}.$$

Defining $\frac{dx}{dt} = u_1$, $\frac{dy}{dt} = u_2$, $\frac{dz}{dt} = u_3$, comes

$$(3)\qquad \frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} u_1 + \frac{\partial u}{\partial y} u_2 + \frac{\partial u}{\partial z} u_3,$$

and therefore

$$(4)\qquad \frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y} + u_3 \frac{\partial u_i}{\partial z}, 1 \leq i \leq 3,$$

which contain the non-linear terms that appear in (1).

Numerically, searching a computational result, i.e., in practical terms, there can be no difference between the Eulerian and Lagrangian formulations for the evaluate of $\frac{Du}{Dt}$ (or $\frac{du}{dt}$, it is the same physical and mathematical entity). Only conceptually and formally there is difference in the two approaches. I agree, however, that you first consider (x,y,z) variable in time (Lagrangian formulation) and then consider (x,y,z) fixed (Eulerian formulation), seems to be subject to criticism. In our present deduction, starting from Navier-Stokes equations in

Eulerian description, implicitly with a solution (u, p), next the pressure, and its corresponding gradient, they travel with the volume element $dV = dx\, dy\, dz$, i.e., obeys to the Lagrangian description of motion, as well as the external force f, in order to avoid contradictions. The velocity u also will obey to the Lagrangian description, and it is representing the velocity of a generic volume element dV over time, initially at position (x_0, y_0, z_0) and with initial velocity $u^0 = u(0) = const.$, $u = u(t)$. Done the solution in Lagrangian description, the solution for pressure in Eulerian description will be given explicitly and at end a solution in function of the initial data.

Following this idea, the system (1) above can be transformed in

$$(5)\qquad \begin{cases} \frac{1}{u_1}\frac{\partial p}{\partial t} + \frac{Du_1}{Dt} = \nu\ (\nabla^2 u_1)|_t + \frac{1}{3}\nu(\nabla_1(\nabla \cdot u))|_t + f_1|_t \\ \frac{1}{u_2}\frac{\partial p}{\partial t} + \frac{Du_2}{Dt} = \nu\ (\nabla^2 u_2)|_t\ + \frac{1}{3}\nu(\nabla_2(\nabla \cdot u))|_t + f_2|_t \\ \frac{1}{u_3}\frac{\partial p}{\partial t} + \frac{Du_3}{Dt} = \nu\ (\nabla^2 u_3)|_t + \frac{1}{3}\nu(\nabla_3(\nabla \cdot u))|_t + f_3|_t \end{cases}$$

thus (1) and (5) are equivalent systems, according (4) validity, since that the partial derivatives of the pressure and velocities were correctly transformed to the variable time, using $\partial x = u_1 \partial t,\ \partial y = u_2 \partial t,\ \partial z = u_3 \partial t$. The nabla and Laplacian operators are considered calculated in Lagrangian formulation, i.e., in the variable time. Likewise for the calculation of $\frac{Du}{Dt}$, following (4), and external force f, using $x = x(t), y = y(t), z = z(t)$. The integration of the system (5) shows that anyone of its equations can be used for solve it, and the results must be equals each other. Then this is a condition to the occurrence of solutions. In the sequence the procedure in more details for obtaining the pressure in Lagrangian formulation, a time dependent function.

Given $u = u(x, y, z, t) \in C^\infty(\mathbb{R}^3 \times [0, \infty))$ obeying the initial conditions and an integrable vector function f, the system's solution (1) for p, using the condensed notation given by (4), is

$$(6)\qquad \begin{aligned} &p = \int_L\ S \cdot dl + \theta(t), \\ &S = \nu\ \nabla^2 u + \frac{1}{3}\nu\ \nabla(\nabla \cdot u) + f - \frac{Du}{Dt}, \end{aligned}$$

where L is any continuous path linking a point (x_0, y_0, z_0) to (x, y, z) and $\theta(t)$ is a generic time function, physically and mathematically reasonable, for example with $\theta(0) = 0$. We are supposing that the vector S is a gradient vector function $(\nabla \times S = 0,\ S = \nabla p,\ p$ potential function of $S)$.

In Eulerian description and in special case when the integrand S in (6) is a constant vector or a dependent function only on the time variable, we come to

(7) $$p = p^0 + S_1(t)\,(x - x_0) + S_2(t)\,(y - y_0) + S_3(t)\,(z - z_0),$$
$$S_i(t) = \nu\,\nabla^2 u_i + \tfrac{1}{3}\nu\,\nabla_i(\nabla \cdot u) + f_i - \tfrac{Du_i}{Dt},$$

where $p^0 = p^0(t)$ is the pressure in the point (x_0, y_0, z_0) at time t.

When the variables x, y, z in (6) as well as f and u are in Lagrangian description, representing a motion over time of a hypothetical volume element dV or particle of fluid, we need eliminate the dependence of the position substituting in (6)

(8) $$dl = (dx,\ dy,\ dz) = (u_1 dt,\ u_2 dt,\ u_3 dt)$$

and integrating over time. The result is

(9) $$p(t) = p^0 + \int_0^t \sum_{i=1}^3 S_i(t)\,u_i(t)\,dt,$$

$p^0 = p(0) = const.$

This calculation can be more facilitated making $u_i \frac{Du_i}{Dt} dt = u_i du_i$ and $\int_0^t u_i \frac{Du_i}{Dt} dt = \int_{u_i^0}^{u_i} u_i du_i = \frac{1}{2}\left(u_i^2 - u_i^{0\,2}\right)$, so (9) is equal to

(10) $$p(t) = p^0 - \tfrac{1}{2}\sum_{i=1}^3\left(u_i^2 - u_i^{0\,2}\right) + \int_0^t \sum_{i=1}^3 R_i(t)\,u_i(t)\,dt,$$
$$R_i(t) = \nu(\nabla^2 u_i)|_t + \tfrac{1}{3}\nu(\nabla_\mathrm{i}(\nabla \cdot u))|_t + f_i|_t,$$

i.e.,

(11) $$p(t) = p^0 - \tfrac{1}{2}(u^2 - u^{0\,2}) + \int_0^t R \cdot u\,dt,$$
$$R = \nu(\nabla^2 u)|_t + \tfrac{1}{3}\nu(\nabla(\nabla \cdot u))|_t + f|_t,$$

$p, p^0 \in \mathbb{R}, u, u^0, f, R \in \mathbb{R}^3, u = (u_1, u_2, u_3)(t),\ u^0 = (u_1^0, u_2^0, u_3^0) = u(0),$ $f = (f_1, f_2, f_3)(t),$ in Lagrangian description. $u^2 = u \cdot u$ and $u^{0\,2} = u^0 \cdot u^0$ are the square modules of the respective vectors u and u^0.

When $f = 0$ and $\nu = 0$ (or most in general $R = 0$) it is simply

(12) $$p = p^0 - \tfrac{1}{2}(u^2 - u^{0\,2}),$$

which then can be considered an exact solution for Euler equations in Lagrangian description, and similarly to Bernoulli's law without external force (gravity, in special).

Unfortunately, in Eulerian description, neither

(13) $$p(x,y,z,t) = p^0(x,y,z) - \frac{1}{2}(u^2 - u^{0\,2}) + \int_L R \cdot dl,$$

$p^0(x,y,z) = p(x,y,z,0),\ u^0 = u^0(x,y,z) = u(x,y,z,0)$, nor

(14) $$p(x,y,z,t) = p^0(t) - \frac{1}{2}(u^2 - u^{0\,2}) + \int_L R \cdot dl,$$

$p^0(t) = p(x_0,y_0,z_0,t),\ u^0 = u^0(t) = u(x_0,y_0,z_0,t)$, solve (1) for all cases of velocities, both formulations supposing $R = \nu\nabla^2 u + \frac{1}{3}\nu\nabla(\nabla\cdot u) + f$ a gradient vector function ($\nabla \times R = 0,\ R = \nabla\phi,\ \phi$ potential function of R).

For example, for $R = 0$ the solution (14) is valid only when

(15) $$\frac{\partial p}{\partial x_i} = -\sum_{j=1}^{3} u_j \frac{\partial u_j}{\partial x_i} = -\left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}\right),$$

i.e.,

(16) $$\frac{\partial u_i}{\partial t} = \sum_{j=1}^{3} u_j \left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right).$$

How to return to the Eulerian formulation if only was obtained a complete solution in the Lagrangian formulation? As well as we can choose any convenient velocity $u(t) = (u_1(t),\ u_2(t),\ u_3(t))$ to calculate the pressure (11) that complies with the initial conditions (Lagrangian formulation), we also can choose appropriates $u(x,y,z,t)$ (Eulerian formulation) and $x(t), y(t), z(t)$ to the velocities and positions of the system and taking the corresponding inverse functions in the obtained solution. This choose is not completely free because will be necessary to calculate a system of ordinary differential equations to obtain the correct set of $x(t), y(t), z(t)$, such that

(17) $$\begin{cases} \frac{dx}{dt} = u_1(x,y,z,t) \\ \frac{dy}{dt} = u_2(x,y,z,t) \\ \frac{dz}{dt} = u_3(x,y,z,t) \end{cases}$$

Nevertheless, this yet can save lots calculation time.

It will be necessary find solutions of (17) such that it is always possible to make any point (x,y,z) of the velocity domain can be achieved for each time t, introducing for this initial positions (x_0,y_0,z_0) conveniently calculated according to (17). Thus we will have velocities and pressures that, in principle, can be calculated for any position and time, independently of one another, not only for a single position for each time. For different values of (x,y,z) and t we will, in the general case, obtain the velocity and pressure of different volume elements dV, starting from different initial positions (x_0, y_0, z_0).

We can escape the need to solve (17), but admitting its validity and the corresponding existence of solution, previously choosing differentiable functions $x = x(t), y = y(t), z = z(t)$ and then calculating directly the solution for velocity in the Lagrangian formulation,

$$(18) \qquad \begin{cases} u_1(t) = \frac{dx}{dt} \\ u_2(t) = \frac{dy}{dt} \\ u_3(t) = \frac{dz}{dt} \end{cases}$$

hereafter calculating $\frac{\partial u_i}{\partial x_j}, \frac{\partial^2 u_i}{\partial x_j^2}$ and the differential operations $\nabla \cdot u, \nabla(\nabla \cdot u)$ and $\nabla^2 u$ through of the transformations

$$(19.1) \qquad \frac{\partial u_i}{\partial x_j} = \frac{\partial u_i/\partial t}{\partial x_j/\partial t} = \frac{1}{u_j}\frac{\partial u_i}{\partial t}$$

$$(19.2) \qquad \nabla \cdot u = \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j} = \sum_{j=1}^{3} \frac{1}{u_j}\frac{\partial u_j}{\partial t}$$

$$(19.3) \qquad \nabla_i(\nabla \cdot u) = \frac{\partial/\partial t}{\partial x_i/\partial t} \sum_{j=1}^{3} \frac{1}{u_j}\frac{\partial u_j}{\partial t} = \\ = \frac{1}{u_i} \sum_{j=1}^{3} \frac{1}{u_j}\left[-\frac{1}{u_j}\left(\frac{\partial u_i}{\partial t}\right)\left(\frac{\partial u_j}{\partial t}\right) + \frac{\partial^2 u_i}{\partial t^2}\right]$$

and

$$(20.1) \qquad \frac{\partial^2 u_i}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) = \frac{\partial/\partial t}{\partial x_j/\partial t}\left(\frac{1}{u_j}\frac{\partial u_i}{\partial t}\right) = \\ = \frac{1}{u_j^2}\left[-\frac{1}{u_j}\left(\frac{\partial u_i}{\partial t}\right)\left(\frac{\partial u_j}{\partial t}\right) + \frac{\partial^2 u_i}{\partial t^2}\right]$$

$$(20.2) \qquad \nabla^2 u_i = \sum_{j=1}^{3} \frac{\partial^2 u_i}{\partial x_j^2} = \sum_{j=1}^{3} \frac{1}{u_j^2}\left[-\frac{1}{u_j}\left(\frac{\partial u_i}{\partial t}\right)\left(\frac{\partial u_j}{\partial t}\right) + \frac{\partial^2 u_i}{\partial t^2}\right],$$

and finally calculating the pressure in (9) or (11), with $\frac{Du_i}{Dt} \equiv \frac{du_i}{dt}$, supposing finites the limits in equations (19) and (20) when $u_j \to 0$. Note that perhaps the denominators appearing in (19) and (20) explaining the occurrence of *blowup time* reported in the literature[3], when the limits are not finites.

Concluding, answering the question, in the result of pressure in Lagrangian formulation given by (9) or (11), conveniently transforming the initial position

$(x_0,\ y_0,\ z_0)$ as function of a generic position (x, y, z) and time t, we will have a correct value of the pressure in Eulerian formulation. The same is valid for the velocity in Lagrangian formulation, if the correspondent Eulerian formulation was not previously obtained.

§ 2

It is worth mentioning that the Navier-Stokes equations in the standard Lagrangian format, traditional one, are different than previously deduced.

Based on [5] the Navier-Stokes equations without external force and with mass density $\rho = 1$ are

(21.1) $$\frac{\partial^2 X_i}{\partial t^2} = -\sum_{j=1}^{3} \frac{\partial A_j}{\partial x_i}\frac{\partial p}{\partial a_j} + \\ + \nu \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\left(\frac{\partial^2 A_l}{\partial x_k \partial x_k}\frac{\partial u_i}{\partial a_l} + \frac{\partial A_j}{\partial x_k}\frac{\partial A_l}{\partial x_k}\frac{\partial^2 u_i}{\partial a_j \partial a_l}\right),$$

(21.2) $$\frac{\partial A_j}{\partial x_i} \equiv \frac{\partial}{\partial x_i} X_j(x_n, t)|_{x_n = X_n(a_m, s|t)},$$

where a_m is the label given to the fluid particle at time s. Its position and velocity at time t are, respectively, $X_n(a_m, s|t)$ and $u_n(a_m, s|t)$.

The significant difference between (21) and (5) is that our pressure (5) is varying only with time, as the initial position is a constant for each particle, not variable. In (21) the pressure varies with the initial position (label) and there is a summation on the three coordinates. We did in (5) $\partial x_i = u_i \partial t$. The nabla operator has also a very difficult expression in the traditional Lagrangian formulation, a triple summation varying on positions (functions of time, evidently) and initial positions.

§ 3

Without passing through the Lagrangian formulation, given a velocity $u(x, y, z, t)$ at least two times differentiable with respect to spatial coordinates and one respect to time and an integrable external force $f(x, y, z, t)$, perhaps a better expression for the solution of the equation (1) is, in fact,

(22) $$p(x, y, z, t) = \sum_{i=1}^{3} \int_{P_i^0}^{P_i} S_i dx_i + \theta(t), \\ S_i = -\left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}\right) + \nu(\nabla^2 u_i) + \frac{1}{3}\nu\big(\nabla_i(\nabla \cdot u)\big) + f_i,$$

supposing possible the integrations and that the vector $S = -\left[\frac{\partial u}{\partial t} + (u \cdot \nabla)u\right] + \nu\nabla^2 u + \frac{1}{3}\nu\nabla(\nabla \cdot u) + f$ is a gradient function. This is the development of the solution (6) for the specific path L going parallely (or perpendicularly) to axes X, Y

and Z from $(x_1^0, x_2^0, x_3^0) \equiv (x_0, y_0, z_0)$ to $(x_1, x_2, x_3) \equiv (x, y, z)$, since that the solution (6) is valid for any piecewise smooth path L. We can choose $P_1^0 = (x_0, y_0, z_0)$, $P_2^0 = (x, y_0, z_0)$, $P_3^0 = (x, y, z_0)$ for the origin points and $P_1 = (x, y_0, z_0)$, $P_2 = (x, y, z_0)$, $P_3 = (x, y, z)$ for the destination points. $\theta(t)$ is a generic time function, physically and mathematically reasonable, for example with $\theta(0) = 0$ or adjustable for some given condition. Again we have seen that the system of Navier-Stokes equations has no unique solution, only given initial conditions, supposing that there is some solution. We can choose different velocities that have the same initial velocity and also result, in general, in different pressures.

The remark given for system (5), when used here, leads us to the following conclusion: the integration of the system (1), confronting with (5), shows that anyone of its equations can be used for solve it, and the results must be equals each other. Then again this is a condition to the occurrence of solutions, which shows to us the possibility of existence of breakdown solutions, as will become clearer in §5.

§ 4

Another way to solve (1) with $f = 0$ seems to me to be the best of all, for its extreme ease of calculation, also without we need to resort to Lagrangian formulation and its conceptual difficulties. If $u(x, y, z, 0) = u^0(x, y, z)$ is the initial velocity of the system, valid solution in $t = 0$, then $u(x, y, z, t) = u^0(x + t, y + t, z + t)$ is a solution for velocity in $t \geq 0$, a non-unique solution, where specifically there is the additional initial condition

(23) $$\frac{\partial u_i}{\partial t}\Big|_{t=0} = \sum_{j=1}^{3} \frac{\partial u_i^0}{\partial x_j}.$$

Similarly, $p(x, y, z, t) = p^0(x + t, y + t, z + t)$ is the correspondent solution for pressure in $t \geq 0$, being $p^0(x, y, z)$ the initial condition for pressure. See reference [6] for a proof of this theorem.

The velocities $u^0(x + t, y, z)$, $u^0(x, y + t, z)$ and $u^0(x, y, z + t)$ are also solutions, and respectively also the pressures $p^0(x + t, y, z)$, $p^0(x, y + t, z)$ and $p^0(x, y, z + t)$, each one with its respective additional initial condition

(24) $$\frac{\partial u_i}{\partial t}\Big|_{t=0} = \frac{\partial u_i^0}{\partial x} \quad \text{or} \quad \frac{\partial u_i}{\partial t}\Big|_{t=0} = \frac{\partial u_i^0}{\partial y} \quad \text{or} \quad \frac{\partial u_i}{\partial t}\Big|_{t=0} = \frac{\partial u_i^0}{\partial z},$$

whose proof requires only a small adaptation of [6], for the particular index which occurs the transformation $x_j \longmapsto x_j + t$.

Other solutions may be searched, without external force, for example in the kind $u(x, y, z, t) = u^0\big(x + T_1(t),\ y + T_2(t),\ z + T_3(t)\big)$, $T_i(0) = 0$, and therefore $p(x, y, z, t) = p^0\big(x + T_1(t),\ y + T_2(t),\ z + T_3(t)\big)$, supposing $T_i(t)$ smooth.

§ 5

This article would not be complete without mentioning the potential flows. When there is a potential function ϕ such that $u = \nabla\phi$ then $\nabla \times u = 0$, i.e., the velocity is an irrotational field. When the incompressibility condition is required, i.e., $\nabla \cdot u = 0$, the velocity is solenoidal, and if the field is also irrotational then $\nabla^2 u = \nabla(\nabla \cdot u) - \nabla \times (\nabla \times u) = 0$, i.e., the Navier-Stokes equations are reduced to Euler's equations and the velocity-potential ϕ must satisfied the Laplace's equation, $\nabla^2\phi = 0$, as well as the velocity.

According Courant[7] (p.241), for $n = 2$ the "general solution" of the potential equation (or Laplace's equation) is the real part of any analytic function of the complex variable $x + iy$. For $n = 3$ one can also easily obtain solutions which depend on arbitrary functions. For example, let $f(w, t)$ be analytic in the complex variable w for fixed real t. Then, for arbitrary values of t, both the real and imaginary parts of the function

(25) $$u = f(z + ix\cos t + iy\sin t\,, t)$$

of the real variables x, y, z are solutions of the equation $\nabla^2 u = 0$. Further solutions may be obtained by superposition:

(26) $$u = \int_a^b f(z + ix\cos t + iy\sin t\,, t)dt.$$

For example, if we set

(27) $$f(w, t) = w^n e^{iht},$$

where n and h are integers, and integrate from $-\pi$ to $+\pi$, we get homogeneous polynomials

(28) $$u = \int_{-\pi}^{\pi} (z + ix\cos t + iy\sin t)^n\, e^{iht} dt$$

in x, y, z, following example given by Courant. Introducing polar coordinates $z = r\cos\theta, x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi$, we obtain

(29) $$\begin{aligned} u &= 2r^n e^{ih\phi} \int_0^{\pi} (\cos\theta + i\sin\theta\cos t)^n \cos ht \;\, dt \\ &= r^n e^{ih\phi} P_{n,h}(\cos\theta), \end{aligned}$$

where $P_{n,h}(\cos\theta)$ are the associated Legendre functions.

On the other hand, according Tokaty[8], Lagrange[9] came to the conclusion that Euler´s equations could be solved only for two specific conditions: (1) for potential (irrotational) flows, and (2) for non-potential (rotational) but steady flows. The external force in [9] is considered with potential, $f = \nabla V$, and the fluid is incompressible.

Lagrange also proved, as well as Laplace (*Mécanique Céleste*), Poisson (*Traité de Méchanique*), Cauchy (*Mémoire sur la Théorie des Ondes*) and Stokes (*On the Friction of Fluids in Motion and the Equilibrium and Motion of Elastic Solids*), that if the differential of the fluid's velocity $u_1 dx + u_2 dy + u_3 dz$ is a differential exact in some instant of time (for example, in $t = 0$) then it is also for all time ($t \geq 0$) of this movement on the same conditions. This means that a potential flow is always potential flow, since $t = 0$. Then, from the previous paragraph, if the initial velocity have not an exact differential (i.e., if the initial velocity is not a gradient function, irrotational, with potential) and the external force have potential then the Euler equations have no solution in this case of incompressible and potential flows, for non-steady flows.

And what happens with respect to Navier-Stokes equations, which is the major problem?

For stationary (say, steady) flows, where $\frac{\partial u}{\partial t} \equiv 0$ and $u = u^0$ for all $t \geq 0$, the condition for existence of solution (obtaining the pressure) is that

(30) $$\frac{\partial S_i}{\partial x_j} = \frac{\partial S_j}{\partial x_i}$$

for all pair (i, j), $1 \leq i, j \leq 3$, defining

(31) $$S_i = \nu \nabla^2 u_i^0 - \sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j} + f_i,$$

where $f \equiv f^0$ is the stationary external force. This is a common condition for existence of solution for a system $\nabla p = S$, representing the stationary Navier-Stokes equations, that is $\nabla \times S = 0$.

For non-stationary flows it is knows that the Lagrange's theorem, as well as the Kelvin's circulation theorem, is not valid for Navier-Stokes equations, but here it is implied that $\nu \nabla^2 u \neq 0$, the general case.

The vorticity $\omega = \nabla \times u \neq 0$ is generated at solid boundaries[10], thus without boundaries ($\Omega = \mathbb{R}^3$) no generation of vorticity, and without vorticity there is potential flow and vanishes the Laplacian of velocity if $\nabla \cdot u = 0$, then it is possible again the validity of Lagrange's theorem in an unlimited domain without boundaries and with both smooth and irrotational initial velocities and external forces, for incompressible fluids.

Regardless, the general condition for existence of solution for pressure in $t = 0$ is (30), for all pair (i, j), $1 \leq i, j \leq 3$, substituting (31) by

$$S_i = \nu\nabla^2 u_i^0 - \left(\frac{\partial u_i}{\partial t}\Big|_{t=0} + \sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j}\right) + f_i^0, \tag{32}$$

$f_i^0 = f_i(x_1, x_2, x_3, 0)$.

It is important the application of (30), (31) and (32) in Turbulence Theory and Theory of Perturbations. According Landau & Lifshitz[1], in the chapter III (Turbulence), article § 26 (Stability of steady flow), of his famous book on Fluids Mechanics,

For any problem of viscous flow under given steady conditions there must in principle exist an exact steady solution of the equations of fluid dynamics. These solutions formally exist for all Reynolds numbers. Yet not every solution of the equations of motion, even if it is exact, can actually occur in Nature. Those which do must not only the equations of fluid dynamics, but also be stable. Any small perturbations which arise must decrease in the course of time. If, on the contrary, the small perturbations which inevitably occur in the flow tend to increase with time, the flow is unstable and cannot actually exist.

But it is not true that any given initial velocity, yet small in module, or large velocity, is according with relation (30), or further approximations following perturbative methods, with S_i given by (31) or (32), and for this reason we sometimes (or yet oftentimes) cannot obtain the necessary solution to the pressure or else we obtain a wrong solution. The same is said about the Numerical Methods in Computational Fluid Dynamics. For any time $t \geq 0$ need be valid the relation (30) with

$$S_i = \nu\nabla^2 u_i - \left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}\right) + f_i. \tag{33}$$

§ 6

I think that this is better than nothing... It is no longer true that the Navier-Stokes and Euler equations do not have a general solution (when there is some).

Apply some of these methods to the famous 6th Millenium Problem[3] on existence and smoothness of the Navier-Stokes equations is not so difficult at the same time also it is not absolutely trivial. It takes some time. I hope to do it soon. On the other hand, apply these methods to the case $n = 2$ or $\nu = 0$ (Euler equation) is almost immediate.

To Leonard Euler, in memorian,
the greatest mathematician of all time.
309th anniversary of his birth,
April-15-1707-2016.

Last update: August-27-2016.

Euler, and mathematical community,
forgive me for my mistakes…
This subject is very difficult!

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13 – Solution for Euler Equations – Lagrangian and Eulerian Descriptions

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Abstract – We find an exact solution for the system of Euler equations, supposing that there is some solution, following the Eulerian and Lagrangian descriptions, for spatial dimension $\mathrm{n} = 3$. As we had seen in other previous articles, it is possible that there are infinite solutions for pressure and velocity, given only the condition of initial velocity.

Keywords – Euler equations, velocity, pressure, Eulerian description, Lagrangian description, formulation, classical mechanics, Newtonian mechanics, Newton's law, second law of Newton, equivalent systems, exact solutions, Bernoulli's law.

§ 1

Essentially the Euler (and Navier-Stokes) equations relating the velocity u and pressure p suffered by a volume element dV at position (x, y, z) and time t. In the formulation or description Eulerian the position (x, y, z) is fixed in time, running different volume elements of fluid in this same position, while the time varies. In the Lagrangian formulation the position (x, y, z) refers to the instantaneous position of a specific volume element $dV = dx\, dy\, dz$ at time t, and this position varies with the movement of this same element dV.

Basically, the deduction of the Euler equations is a classical mechanics problem, a problem of Newtonian mechanics, which use the 2nd law of Newton $F = ma$, force is equal to mass multiplied by acceleration. We all know that the force described in Newton's law may have different expressions, varying only in time or also with the position, or with the distance to the source, varying with the body's velocity, etc. Each specific problem must to define how the forces involved in the system are applied and what they mean. I suggest consulting the classic Landau & Lifshitz[1] or Prandtl's book[2] for a more detailed description of the deduction of these equations (including Navier-Stokes equations).

In spatial dimension $n = 3$, the Euler equations can be put in the form of a system of three nonlinear partial differential equations, as follows:

$$(1) \quad \begin{cases} \frac{\partial p}{\partial x} + \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + u_3 \frac{\partial u_1}{\partial z} = f_1 \\ \frac{\partial p}{\partial y} + \frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + u_3 \frac{\partial u_2}{\partial z} = f_2 \\ \frac{\partial p}{\partial z} + \frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x} + u_2 \frac{\partial u_3}{\partial y} + u_3 \frac{\partial u_3}{\partial z} = f_3 \end{cases}$$

where $u(x,y,z,t) = \big(u_1(x,y,z,t), u_2(x,y,z,t), u_3(x,y,z,t)\big)$, $u: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$, is the velocity of the fluid, of components u_1, u_2, u_3, p is the pressure, $p: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}$, and $f(x,y,z,t) = \big(f_1(x,y,z,t), f_2(x,y,z,t), f_3(x,y,z,t)\big)$, $f: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$, is the density of external force applied in the fluid in point (x,y,z) and at the instant of time t, for example, gravity force per mass unity, with $x,y,z,t \in \mathbb{R}$, $t \geq 0$. $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ is the nabla operator and $\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \equiv \Delta$ is the Laplacian operator.

The non-linear terms $u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y} + u_3 \frac{\partial u_i}{\partial z}$, $1 \leq i \leq 3$, are a natural consequence of the Eulerian formulation of motion, and corresponds to part of the total derivative of velocity with respect to time of a volume element dV in the fluid, i.e., its acceleration. If $u = (u_1(x,y,z,t), u_2(x,y,z,t), u_3(x,y,z,t))$ and these x, y, z also vary in time, $x = x(t)$, $y = y(t)$, $z = z(t)$, then, by the chain rule,

$$(2) \qquad \frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}.$$

Defining $\frac{dx}{dt} = u_1$, $\frac{dy}{dt} = u_2$, $\frac{dz}{dt} = u_3$, comes

$$(3) \qquad \frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} u_1 + \frac{\partial u}{\partial y} u_2 + \frac{\partial u}{\partial z} u_3,$$

and therefore

$$(4) \qquad \frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y} + u_3 \frac{\partial u_i}{\partial z}, 1 \leq i \leq 3,$$

which contain the non-linear terms that appear in (1).

Numerically, searching a computational result, i.e., in practical terms, there can be no difference between the Eulerian and Lagrangian formulations for the evaluation of $\frac{Du}{Dt}$ (or $\frac{du}{dt}$, it is the same physical and mathematical entity). Only conceptually and formally there is difference in the two approaches. I agree, however, that you first consider (x,y,z) variable in time (Lagrangian formulation) and then consider (x,y,z) fixed (Eulerian formulation), seems to be subject to criticism. In our present deduction, starting from Euler equations in Eulerian description, implicitly with a solution (u,p), next the pressure, and its corresponding gradient, they travel with the volume element $dV = dx\, dy\, dz$, i.e., obeys to the Lagrangian description of motion, as well as the external force f, in order to avoid contradictions. The velocity u also will obey to the Lagrangian description, and it is representing the velocity of a generic volume element dV over time, initially at position (x_0, y_0, z_0) and with initial velocity $u^0 = u(0) = const.$, $u = u(t)$. Done the solution in Lagrangian description, the solution for pressure in Eulerian description will be given explicitly.

Following this idea, the system (1) above can be transformed into

$$(5) \qquad \begin{cases} \frac{1}{u_1}\frac{\partial p}{\partial t} + \frac{Du_1}{Dt} = f_1 \\ \frac{1}{u_2}\frac{\partial p}{\partial t} + \frac{Du_2}{Dt} = f_2 \\ \frac{1}{u_3}\frac{\partial p}{\partial t} + \frac{Du_3}{Dt} = f_3 \end{cases}$$

thus (1) and (5) are equivalent systems, according (4) validity, since that the partial derivatives of the pressure and velocities were correctly transformed to the variable time, using $\partial x = u_1 \partial t,\ \partial y = u_2 \partial t,\ \partial z = u_3 \partial t$. Likewise for the calculation of $\frac{Du}{Dt}$, according (4), and external force f, using $x = x(t), y = y(t), z = z(t)$. The integration of the system (5) shows that anyone of its equations can be used for solve it, and the results must be equals each other. Then this is a condition to the occurrence of solutions. In the sequence the procedure in more details for obtaining the pressure in Lagrangian formulation, a time dependent function, starting by solution for pressure in Eulerian description.

Given $u = u(x, y, z, t) \in C^1(\mathbb{R}^3 \times [0, \infty))$ obeying the initial conditions and a vector function f (both when in Eulerian description) such that the difference $f - \frac{Du}{Dt}$ is gradient, the system's solution (1) for p, using the condensed notation given by (4), is

$$(6) \qquad p = \int_L \left(f - \frac{Du}{Dt}\right) \cdot dl + \theta(t),$$

where L is any continuous path linking a point (x_0, y_0, z_0) to (x, y, z) and $\theta(t)$ is a generic time function, physically and mathematically reasonable, for example with $\theta(0) = 0$.

In Eulerian description and in special case when $f - \frac{Du}{Dt}$ is a constant vector or a dependent function only on the time variable, we come to

$$(7) \qquad \begin{aligned} &p = p^0 + S_1(t)\,(x - x_0) + S_2(t)\,(y - y_0) + S_3(t)\,(z - z_0), \\ &S_i(t) = f_i - \frac{Du_i}{Dt}, \end{aligned}$$

where $p^0 = p^0(t)$ is the pressure in the point (x_0, y_0, z_0) at time t.

When the variables x, y, z in (6) as well as f and u are in Lagrangian description, representing a motion over time of a hypothetical volume element dV or particle of fluid, we need eliminate the dependence of the position using in (6)

(8) $$dl = (dx,\ dy,\ dz) = (u_1 dt,\ u_2 dt,\ u_3 dt)$$

and integrating over time. The result is

(9) $$p(t) = p^0 + \int_0^t \sum_{i=1}^3 S_i(t)\, u_i(t)\, dt,$$

$$p^0 = p(0) = const.$$

This expression can be more facilitated making $u_i \frac{Du_i}{Dt} dt = u_i du_i$ and $\int_0^t u_i \frac{Du_i}{Dt} dt = \int_{u_i^0}^{u_i} u_i du_i = \frac{1}{2}\left(u_i^2 - u_i^{0\,2}\right)$, so (9) is equal to

(10) $$p(t) = p^0 - \frac{1}{2}\sum_{i=1}^3\left(u_i^2 - u_i^{0\,2}\right) + \int_0^t \sum_{i=1}^3 f_i(t)\, u_i(t)\, dt,$$

i.e.,

(11) $$p(t) = p^0 - \frac{1}{2}(u^2 - u^{0\,2}) + \int_0^t f \cdot u\, dt,$$

$p, p^0 \in \mathbb{R}, u, u^0, f \in \mathbb{R}^3, u = (u_1, u_2, u_3)(t),\ u^0 = (u_1^0, u_2^0, u_3^0) = u(0),$ $f = (f_1, f_2, f_3)(t)$, in Lagrangian description. $u^2 = u \cdot u$ and $u^{0\,2} = u^0 \cdot u^0$ are the square modules of the respective vectors u and u^0.

When $f = 0$ the solution (11) is simply

(12) $$p = p^0 - \frac{1}{2}(u^2 - u^{0\,2}),$$

which then can be considered an exact solution for Euler equations in Lagrangian description, and similarly to Bernoulli's law without external force (gravity, in special).

Unfortunately, in Eulerian description, neither

(13) $$p(x, y, z, t) = p^0(x, y, z) - \frac{1}{2}(u^2 - u^{0\,2}) + \int_L\ f \cdot dl,$$

$p^0(x, y, z) = p(x, y, z, 0),\ u^0 = u^0(x, y, z) = u(x, y, z, 0)$, nor

(14) $$p(x, y, z, t) = p^0(t) - \frac{1}{2}(u^2 - u^{0\,2}) + \int_L\ f \cdot dl,$$

$p^0(t) = p(x_0, y_0, z_0, t)$, $u^0 = u^0(t) = u(x_0, y_0, z_0, t)$, solve (1) for all cases of velocities, both formulations supposing f a gradient vector function ($\nabla \times f = 0$, $f = \nabla\phi$, ϕ potential function of f).

For example, for $f = 0$ the solution (14) is valid only when

(15) $$\frac{\partial p}{\partial x_i} = -\sum_{j=1}^{3} u_j \frac{\partial u_j}{\partial x_i} = -\left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}\right),$$

i.e.,

(16) $$\frac{\partial u_i}{\partial t} = \sum_{j=1}^{3} u_j \left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right).$$

How to return to the Eulerian formulation if only was obtained a complete solution in the Lagrangian formulation? As well as we can choose any convenient velocity $u(t) = (u_1(t),\ u_2(t),\ u_3(t))$ to calculate the pressure (11) that complies with the initial conditions (Lagrangian formulation), we also can choose appropriates $u(x, y, z, t)$ (Eulerian formulation) and $x(t), y(t), z(t)$ to the velocities and positions of the system and taking the corresponding inverse functions in the obtained solution. This choose is not completely free because will be necessary to calculate a system of ordinary differential equations to obtain the correct set of $x(t), y(t), z(t)$, such that

(17) $$\begin{cases} \frac{dx}{dt} = u_1(x, y, z, t) \\ \frac{dy}{dt} = u_2(x, y, z, t) \\ \frac{dz}{dt} = u_3(x, y, z, t) \end{cases}$$

Nevertheless, this yet can save lots calculation time.

It will be necessary find solutions of (17) such that it is always possible to make any point (x, y, z) of the velocity domain can be achieved for each time t, introducing for this initial positions (x_0, y_0, z_0) conveniently calculated according to (17). Thus we will have velocities and pressures that, in principle, can be calculated for any position and time, independently of one another, not only for a single position for each time. For different values of (x, y, z) and t we will, in the general case, obtain the velocity and pressure of different volume elements dV, starting from different initial positions $(x_0,\ y_0,\ z_0)$.

We can escape the need to solve (17), but admitting its validity and the corresponding existence of solution, previously choosing differentiable functions $x = x(t), y = y(t), z = z(t)$ and then calculating directly the solution for velocity in the Lagrangian formulation,

(18) $$\begin{cases} u_1(t) = \frac{dx}{dt} \\ u_2(t) = \frac{dy}{dt} \\ u_3(t) = \frac{dz}{dt} \end{cases}$$

Concluding, answering the question, in the result of pressure in Lagrangian formulation given by (9) or (11), conveniently transforming the initial position $(x_0,\ y_0,\ z_0)$ as function of a generic position (x, y, z) and time t, we will have a correct value of the pressure in Eulerian formulation, since that keeping the same essential original significance. The same is valid for the velocity in Lagrangian formulation, if the correspondent Eulerian formulation was not previously obtained.

§ 2

It is worth mentioning that the Euler equations in the standard Lagrangian format, traditional one, are different than previously deduced.

Based on [5] the Euler equations without external force and with mass density $\rho = 1$ are

(19.1) $$\frac{\partial^2 X_i}{\partial t^2} = -\sum_{j=1}^{3} \frac{\partial A_j}{\partial x_i} \frac{\partial p}{\partial a_j},$$

(19.2) $$\frac{\partial A_j}{\partial x_i} \equiv \frac{\partial}{\partial x_i} X_j(x_n, t)|_{x_n = X_n(a_m, s|t)},$$

where a_m is the label given to the fluid particle at time s. Its position and velocity at time t are, respectively, $X_n(a_m, s|t)$ and $u_n(a_m, s|t)$.

The significant difference between (19) and (5) is that our pressure (5) is varying only with time, as the initial position is a constant for each particle, not variable. In (19) the pressure varies with the initial position (label) and there is a summation on the three coordinates. We did in (5) $\partial x_i = u_i \partial t$.

§ 3

Without passing through the Lagrangian formulation, given a differentiable velocity $u(x, y, z, t)$ and an integrable external force $f(x, y, z, t)$, perhaps a better expression for the solution of the equation (1) is, in fact,

(20) $$p = \sum_{i=1}^{3} \int_{P_i^0}^{P_i} \left[-\left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} \right) + f_i \right] dx_i + \theta(t),$$

supposing possible the integrations and that the vector $S = -[\frac{\partial u}{\partial t} + (u \cdot \nabla)u] + f$ is a gradient function. This is the development of the solution (6) for the specific path

L going parallely (or perpendicularly) to axes X, Y and Z from $(x_1^0, x_2^0, x_3^0) \equiv (x_0, y_0, z_0)$ to $(x_1, x_2, x_3) \equiv (x, y, z)$, since that the solution (6) is valid for any piecewise smooth path L. We choose $P_1^0 = (x_0, y_0, z_0)$, $P_2^0 = (x, y_0, z_0)$, $P_3^0 = (x, y, z_0)$ and $P_1 = (x, y_0, z_0)$, $P_2 = (x, y, z_0)$, $P_3 = (x, y, z)$. $\theta(t)$ is a generic time function, physically and mathematically reasonable, for example with $\theta(0) = 0$ or adjustable for some given condition. Again we have seen that the system of Euler equations has no unique solution, only given initial conditions, supposing that there is some solution. We can choose different velocities that have the same initial velocity and also result, in general, in different pressures.

The remark given for system (5), when used here, leads us to the following conclusion: the integration of the system (1), confronting with (5), shows that anyone of its equations can be used for solve it, and the results must be equals each other. Then again this is a condition to the occurrence of solutions, which shows to us the possibility of existence of breakdown solutions, as will become clearer in §5.

§ 4

Another way to solve (1) with $f = 0$ seems to me to be the best of all, for its extreme ease of calculation, also without we need to resort to Lagrangian formulation and its conceptual difficulties. If $u(x, y, z, 0) = u^0(x, y, z)$ is the initial velocity of the system, valid solution in $t = 0$, then $u(x, y, z, t) = u^0(x + t, y + t, z + t)$ is a solution for velocity in $t \geq 0$, a non-unique solution. Similarly, $p(x, y, z, t) = p^0(x + t, y + t, z + t)$ is the correspondent solution for pressure in $t \geq 0$, being $p^0(x, y, z)$ the initial condition for pressure. The velocities $u^0(x + t, y, z)$, $u^0(x, y + t, z)$ and $u^0(x, y, z + t)$ are also solutions, and respectively also the pressures $p^0(x + t, y, z)$, $p^0(x, y + t, z)$ and $p^0(x, y, z + t)$. Other solutions may be searched, for example in the kind $u(x, y, z, t) = u^0\big(x + T_1(t), y + T_2(t), z + T_3(t)\big)$, $T_i(0) = 0$, and therefore $p(x, y, z, t) = p^0\big(x + T_1(t), y + T_2(t), z + T_3(t)\big)$. See [6].

§ 5

This article would not be complete without mentioning the potential flows. When there is a potential function ϕ such that $u = \nabla\phi$ then $\nabla \times u = 0$, i.e., the velocity is an irrotational field. When the incompressibility condition is required, i.e., $\nabla \cdot u = 0$, the velocity is solenoidal, and if the field is also irrotational then $\nabla^2 u = \nabla(\nabla \cdot u) - \nabla \times (\nabla \times u) = 0$, i.e., the Navier-Stokes equations are reduced to Euler's equations and the velocity-potential ϕ must satisfied the Laplace's equation, $\nabla^2\phi = 0$, as well as the velocity.

According Courant[7] (p.241), for $n = 2$ the "general solution" of the potential equation (or Laplace's equation) is the real part of any analytic function of the complex variable $x + iy$. For $n = 3$ one can also easily obtain solutions which depend on arbitrary functions. For example, let $f(w, t)$ be analytic in the complex variable w for fixed real t. Then, for arbitrary values of t, both the real and imaginary parts of the function

$$(21) \qquad u = f(z + ix\cos t + iy\sin t\,, t)$$

of the real variables x, y, z are solutions of the equation $\nabla^2 u = 0$. Further solutions may be obtained by superposition:

$$(22) \qquad u = \int_a^b f(z + ix\cos t + iy\sin t\,, t)dt.$$

For example, if we set

$$(23) \qquad f(w,t) = w^n e^{iht},$$

where n and h are integers, and integrate from $-\pi$ to $+\pi$, we get homogeneous polynomials

$$(24) \qquad u = \int_{-\pi}^{\pi} (z + ix\cos t + iy\sin t)^n\, e^{iht} dt$$

in x, y, z, following example given by Courant. Introducing polar coordinates $z = r\cos\theta, x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi$, we obtain

$$(25) \qquad \begin{aligned} u &= 2r^n e^{ih\phi} \int_0^{\pi} (\cos\theta + i\sin\theta\cos t)^n \cos ht\ \ dt \\ &= r^n e^{ih\phi} P_{n,h}(\cos\theta), \end{aligned}$$

where $P_{n,h}(\cos\theta)$ are the associated Legendre functions.

On the other hand, according to Tokaty[8], Lagrange[9] came to the conclusion that Euler´s equations could be solved only for two specific conditions: (1) for potential (irrotational) flows, and (2) for non-potential (rotational) but steady flows. The external force in [9] is considered with potential, $f = \nabla V$, and the fluid is incompressible.

Lagrange also proved, as well as Laplace (*Mécanique Céleste*), Poisson (*Traité de Méchanique*), Cauchy (*Mémoire sur la Théorie des Ondes*) and Stokes (*On the Friction of Fluids in Motion and the Equilibrium and Motion of Elastic Solids*), that if the differential of the fluid's velocity $u_1dx + u_2dy + u_3dz$ is a differential exact in some instant of time (for example, in $t = 0$) then it is also for all time ($t \geq 0$) of this movement on the same conditions. This means that a potential flow is always potential flow, since $t = 0$. Then, from the previous paragraph, if the initial velocity have not an exact differential (i.e., if the initial velocity is not a gradient function, irrotational, with potential) and the external force have potential then the Euler's equations have no solution in this case of incompressible and potential flows, for non-steady flows.

For steady flows, where $\frac{\partial u}{\partial t} \equiv 0$ and $u = u^0$ for all $t \geq 0$, the condition for existence of solution (obtaining the pressure) is that

(26) $$\frac{\partial S_i}{\partial x_j} = \frac{\partial S_j}{\partial x_i}$$

for all pair (i, j), $1 \leq i, j \leq 3$, defining

(27) $$S_i = f_i - \sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j},$$

where $f \equiv f^0$ is the stationary external force. This is a common condition for existence of solution for a system $\nabla p = S$, representing the stationary Euler's equations, that is $\nabla \times S = 0$.

§ 6

I think that this is better than nothing… It is no longer true that the Euler equations do not have a general solution (when there is some).

Apply some of these methods to the Navier-Stokes equations and to the famous 6th Millennium Problem[4] on existence and smoothness of the Navier-Stokes equations apparently is not so difficult at the same time also it is not absolutely trivial. It takes some time. I hope to do it soon. On the other hand, apply these methods to the case $n = 2$ is almost immediate.

To Leonard Euler, in memorian,
the greatest mathematician of all time.
He was brilliant, great intuitive genius.

Euler, and mathematical community,
forgive me for my mistakes…
This subject is very difficult!

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14 – Two Theorems on Solutions in Eulerian Description

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Abstract – We present two proofs of theorems needed to the major work we are doing on existence and breakdown solutions of the Navier-Stokes equations for incompressible case without external force in n = 3 spatial dimensions.
Keywords – Navier-Stokes equations, velocity, pressure, Eulerian description, Lagrangian description, formulation, equivalent equations, exact solutions, existence, inexistence.

Let $u^0(x,y,z)$ and $p^0(x,y,z)$ be respectively the initial velocity and initial pressure of the three-dimensional incompressible ($\nabla\cdot u=\nabla\cdot u^0=0$) Navier-Stokes equations without external force and with mass density equal to 1,

$$(1)\qquad \frac{\partial p(X,t)}{\partial x_i}+\frac{\partial u_i(X,t)}{\partial t}+\sum_{j=1}^{3}u_j(X,t)\frac{\partial u_i(X,t)}{\partial x_j}=\nu\nabla^2 u_i(X,t),$$

$1\le i\le 3,\ X=(x_1,x_2,x_3)\in\mathbb{R}^3,\ x_1\equiv x,\ x_2\equiv y,\ x_3\equiv z,\ x_i,t\ \in\mathbb{R},\ t\ge 0.$

Then in $t=0$ is valid, for each integer i belongs to $1\le i\le 3$,

$$(2)\qquad \frac{\partial p^0(X)}{\partial x_i}+\frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0}+\sum_{j=1}^{3}u_j^0(X)\frac{\partial u_i^0(X)}{\partial x_j}=\nu\nabla^2 u_i^0(X).$$

Supposing that $u(x,y,z,t)=u^0(x+t,y+t,z+t)$ and $p(x,y,z,t)=p^0(x+t,y+t,z+t)$ is a solution (u,p) for (1), we have

$$(3)\qquad \frac{\partial p^0(\xi)}{\partial x_i}+\frac{\partial u_i^0(\xi)}{\partial t}+\sum_{j=1}^{3}u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial x_j}=\nu\nabla^2 u_i^0(\xi),$$

where $\xi=(\xi_1,\xi_2,\xi_3)$ and $\xi_i=\xi_i(X,t)=x_i+t,1\le i\le 3$.

For $t=0$ the equations (2) and (3) are equals, because in $t=0$ we have $\xi_i=x_i$ and therefore $\xi=(\xi_1,\xi_2,\xi_3)=(x_1,x_2,x_3)=X$.

For $t>0$, if (2) is valid for any $X=(x,y,z)\in\mathbb{R}^3$ then (3) is valid for any $\xi\in\mathbb{R}^3$ substituting $x\mapsto\xi_1=x+t,\ y\mapsto\xi_2=y+t,\ z\mapsto\xi_3=z+t,\ x,y,z\in\mathbb{R},t\ge 0$, so $u(x,y,z,t)=u^0(x+t,y+t,z+t)$ and $p(x,y,z,t)=p^0(x+t,y+t,z+t)$, i.e., $u(X,t)=u^0(\xi)$ and $p(X,t)=p^0(\xi)$, solve equation (3) and therefore the Navier-Stokes equation (1).

The initial motivation to prove it is as follows. Let $A(x),B(x),C(x)$ and $D(x)$ functions such that is always valid, for any $x\in\mathbb{R}$, the relation

(4) $$A(x) + B(x) + C(x) = D(x).$$

Then, as $(x+t) \in \mathbb{R}, x, t \in \mathbb{R}, t \geq 0$, need be valid too the relation

(5) $$A(x+t) + B(x+t) + C(x+t) = D(x+t).$$

The same argument can be used for functions of two and three spatial dimensions (or better, for n spatial dimensions), for example, $\forall x, y, z, t \in \mathbb{R}, t \geq 0$,

(6) $$\begin{aligned} &A_i(x,y,z) + B_i(x,y,z) + C_i(x,y,z) = D_i(x,y,z) \\ &\Longrightarrow A_i(x+t,\ y+t,\ z+t) + B_i(x+t,\ y+t,\ z+t) + \\ &\quad + C_i(x+t,\ y+t,\ z+t) = D_i(x+t,\ y+t,\ z+t). \end{aligned}$$

Applying the previous relation (6) to the Navier-Stokes equations (2) for $t = 0$, if

(7.1) $$A_i(x,y,z) = \frac{\partial p^0(X)}{\partial x_i},$$

(7.2) $$B_i(x,y,z) = \frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0},$$

(7.3) $$C_i(x,y,z) = \sum_{j=1}^{3} u_j^0(X)\frac{\partial u_i^0(X)}{\partial x_j},$$

(7.4) $$D_i(x,y,z) = \nu\nabla^2 u_i^0(X),$$

(7.5) $$A_i(x,y,z) + B_i(x,y,z) + C_i(x,y,z) = D_i(x,y,z),$$

$X = (x,y,z)$, then, using $\xi = \xi(X,t) = (x+t, y+t, z+t)$, need be valid too the equalities

(8.1) $$A_i(x+t,\ y+t,\ z+t) = \frac{\partial p^0(\xi)}{\partial x_i},$$

(8.2) $$B_i(x+t,\ y+t,\ z+t) = \left(\frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0}\right)(\xi),$$

(8.3) $$C_i(x+t,\ y+t,\ z+t) = \sum_{j=1}^{3} u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial x_j},$$

(8.4) $$D_i(x+t,\ y+t,\ z+t) = \nu\nabla^2 u_i^0(\xi),$$

(8.5) $$\begin{aligned} &A_i(x+t,\ y+t,\ z+t) + B_i(x+t,\ y+t,\ z+t) + \\ &+ C_i(x+t,\ y+t,\ z+t) = D_i(x+t,\ y+t,\ z+t). \end{aligned}$$

The expression $(\frac{\partial u_i(X,t)}{\partial t}|_{t=0})(\xi)$ in (8.2) means that first is calculated the value of $\frac{\partial u_i(X,t)}{\partial t}$, next we assign the value $t=0$ in this result and then we substitute $x \mapsto \xi_1 = x+t,\ y \mapsto \xi_2 = y+t,\ z \mapsto \xi_3 = z+t$, i.e., $X \mapsto \xi$.

Note that the right side of the relations (8.1) to (8.4) corresponds to each parcel of the Navier-Stokes equations (8.5) with the solution (u,p) such that

(9.1) $\qquad u(X,t) = u^0(\xi)$,
(9.2) $\qquad p(X,t) = p^0(\xi)$,

$X=(x,y,z),\ \xi=\xi(X,t)=(x+t,\ y+t,\ z+t)$, then (9) is a solution for (1) if $u^0(X)$ and $p^0(X)$ are initial conditions.

We will now prove that if the variables (9.1) and (9.2) solve (1) for $t \geq 0$ then $u^0(x,y,z)$ and $p^0(x,y,z)$ solve (1) for $t=0$, i.e., then both $u^0(x,y,z)$ and $p^0(x,y,z)$ solve (2). This is an important result of this paper. We'll use the chain rule[1].

Proof: Starting from (1), the three-dimensional incompressible Navier-Stokes equations, where $\nabla \cdot u = \nabla \cdot u^0 = 0$,

(10) $$\frac{\partial p(X,t)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t} + \sum_{j=1}^{3} u_j(X,t)\frac{\partial u_i(X,t)}{\partial x_j} = \nu\nabla^2 u_i(X,t),$$

$1 \leq i \leq 3,\ X=(x,y,z)$, if a solution (u,p) for them is (9), i.e.,

(11.1) $\qquad u(X,t) = u^0(\xi)$,
(11.2) $\qquad p(X,t) = p^0(\xi)$,

$\xi=\xi(X,t)=(x+t,\ y+t,\ z+t)$, then we have, according (3),

(12) $$\frac{\partial p^0(\xi)}{\partial x_i} + \frac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^{3} u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial x_j} = \nu\nabla^2 u_i^0(\xi).$$

How $\xi_i = x_i + t$ then $\frac{\partial \xi_i}{\partial x_i} = \frac{\partial \xi_i}{\partial t} = 1$ and $\frac{\partial \xi_i}{\partial x_j} = 0$ if $i \neq j$, so using the chain rule[1] we have, for each parcel in (10) and (12),

(13.1) $$\frac{\partial p(X,t)}{\partial x_i} = \frac{\partial p^0(\xi)}{\partial x_i} = \sum_{j=1}^{3}\frac{\partial p^0(\xi)}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_i} = \frac{\partial p^0(\xi)}{\partial \xi_i}$$

(13.2) $$\frac{\partial u_i(X,t)}{\partial t} = \frac{\partial u_i^0(\xi)}{\partial t} = \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j}\frac{\partial \xi_j}{\partial t} = \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j}$$

(13.3) $$u_j(X,t)\frac{\partial u_i(X,t)}{\partial x_j} = u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial x_j} = u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_j} =$$

$$= u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial \xi_j}$$

(13.4) $$\nabla^2 u_i(X,t) = \nabla^2 u_i^0(\xi) = \left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\frac{\partial}{\partial x_3}\right) u_i^0(\xi) =$$
$$= \sum_{j=1}^{3}\left(\frac{\partial}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_j}\frac{\partial}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_j}\right) u_i^0(\xi) = \sum_{j=1}^{3}\left(\frac{\partial}{\partial \xi_j}\frac{\partial}{\partial \xi_j}\right) u_i^0(\xi) =$$
$$= \nabla_\xi^2 u_i^0(\xi)$$

Adding the parcels (13), with (13.3) for each integer $j = 1, 2, 3$ and the multiplication of (13.4) by viscosity coefficient ν, we come to

(14) $$\frac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j} + \sum_{j=1}^{3} u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial \xi_j} = \nu\nabla_\xi^2 u_i^0(\xi),$$

which is equivalent to previous Navier-Stokes equations (10) and (12) with the solution (11), although it is not a conventional Navier-Stokes equation because the time derivative disappears, i.e.,

(15) $$\frac{\partial u_i(X,t)}{\partial t} \mapsto \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j}.$$

Note that the right side of (15) is not $\frac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j}$, because here u_i^0 is, initially, a function only of $\xi = (\xi_1, \xi_2, \xi_3)$, not including t, but each ξ_i is a function of t and for this reason here is $\frac{\partial u_i(X,t)}{\partial t} = \frac{\partial u_i^0(\xi)}{\partial t} = \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j}\frac{\partial \xi_j}{\partial t} = \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j}$, with $\xi_j = x_j + t$, $\frac{\partial \xi_j}{\partial t} = 1$.

In $t = 0$, when $\xi_i = x_i$, the equation (14) became

(16) $$\frac{\partial p^0(X)}{\partial x_i} + \sum_{j=1}^{3}\frac{\partial u_i^0(X)}{\partial x_j} + \sum_{j=1}^{3} u_j^0(X)\frac{\partial u_i^0(X)}{\partial x_j} = \nu\nabla^2 u_i^0(X).$$

If this equation is equivalent to (2) then

(17) $$\frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0} = \sum_{j=1}^{3}\frac{\partial u_i^0(X)}{\partial x_j},$$

which is thereby a good manner of define or choose the temporal derivative of velocity at $t = 0$ when the solution for velocity is $u(X,t) = u^0(\xi)$.

Similarly, for $t > 0$ we have

(18) $$\frac{\partial u_i(X,t)}{\partial t} = \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j},$$

$X = (x, y, z),\ \xi = (\xi_1, \xi_2, \xi_3),\ \xi_i = \xi_i(X, t) = x_i + t,\ 1 \le i \le 3.$

Concluding, assuming that (9), identical to (11), is a solution for (1), identical to (10), we come to (16) for $t = 0$, which is equivalent to (2) with the additional initial condition (17) and it has a solution $(u^0(X), p^0(X))$. This is what we wanted to prove. □

Next, we will prove the opposite way of the previous demonstration: if $u^0(x, y, z)$ and $p^0(x, y, z)$ solve (1) for $t = 0$, i.e., if both $u^0(x, y, z)$ and $p^0(x, y, z)$ solve (2), then the variables (u, p) given in (9.1) and (9.2) solve (1) for $t \ge 0$. This is the fundamental result of this paper. The proof basically follows what we write from beginning of this paper until the equations (9), increasing the transformations (13) and the conditions (17) and (18). We'll use the chain rule[1] again.

Proof: If $u^0(x, y, z)$ and $p^0(x, y, z)$ solve the three-dimensional incompressible $(\nabla \cdot u = \nabla \cdot u^0 = 0)$ Navier-Stokes equations

(19) $$\frac{\partial p(X,t)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t} + \sum_{j=1}^{3} u_j(X,t) \frac{\partial u_i(X,t)}{\partial x_j} = \nu \nabla^2 u_i(X,t)$$

for $t = 0$, with $1 \le i \le 3$, $X = (x_1, x_2, x_3) \in \mathbb{R}^3$, $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$, $x_i, t \in \mathbb{R}$, $t \ge 0$, then in $t = 0$ is valid, for each integer i belongs to $1 \le i \le 3$,

(20) $$\frac{\partial p^0(X)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0} + \sum_{j=1}^{3} u_j^0(X) \frac{\partial u_i^0(X)}{\partial x_j} = \nu \nabla^2 u_i^0(X).$$

Supposing that $u(x, y, z, t) = u^0(x + t, y + t, z + t)$ and $p(x, y, z, t) = p^0(x + t, y + t, z + t)$ is a solution (u, p) for (19), we have

(21) $$\frac{\partial p^0(\xi)}{\partial x_i} + \frac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^{3} u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial x_j} = \nu \nabla^2 u_i^0(\xi),$$

using $\xi = (\xi_1, \xi_2, \xi_3)$ and $\xi_i = \xi_i(X, t) = x_i + t$, $1 \le i \le 3$.

For $t = 0$ the equations (20) and (21) are equals, because in $t = 0$ we have $\xi_i = x_i$ and therefore $\xi = (\xi_1, \xi_2, \xi_3) = (x_1, x_2, x_3) = X$.

For $t > 0$, if (20) is valid for any $X = (x, y, z) \in \mathbb{R}^3$ then (21) is valid for any $\xi \in \mathbb{R}^3$ substituting $x \mapsto \xi_1 = x + t$, $y \mapsto \xi_2 = y + t$, $z \mapsto \xi_3 = z + t$, $x, y, z \in \mathbb{R}, t \ge 0$, according transformations (22) below, so $u(x, y, z, t) = u^0(x + t, y + t, z + t)$ and $p(x, y, z, t) = p^0(x + t, y + t, z + t)$, i.e., $u(X, t) = u^0(\xi)$ and $p(X, t) = p^0(\xi)$, solve equation (21) and therefore the Navier-Stokes equation (19).

How $\xi_i = x_i + t$ then $\frac{\partial \xi_i}{\partial x_i} = \frac{\partial \xi_i}{\partial t} = 1$ and $\frac{\partial \xi_i}{\partial x_j} = 0$ if $i \ne j$, so using the chain rule[1] we have, for each parcel in (21), (22.1) $$\frac{\partial p^0(\xi)}{\partial x_i} = \frac{\partial p(\xi(X,t))}{\partial x_i} = \sum_{j=1}^{3} \frac{\partial p^0(\xi)}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_i} = \frac{\partial p^0(\xi)}{\partial \xi_i}$$

(22.2) $$\frac{\partial u_i^0(\xi)}{\partial t} = \frac{\partial u_i(\xi(X,t))}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} \frac{\partial \xi_j}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j}$$

(22.3) $$u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial x_j} = u_j(\xi(X,t)) \frac{\partial u_i(\xi(X,t))}{\partial x_j} = u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_j} =$$
$$= u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial \xi_j}$$

(22.4) $$\nabla^2 u_i^0(\xi) = \nabla^2 u_i(\xi(X,t)) = \sum_{j=1}^{3} \left(\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j}\right) u_i^0(\xi(X,t)) =$$
$$= \sum_{j=1}^{3} \left(\frac{\partial}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_j} \frac{\partial}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_j}\right) u_i^0(\xi) = \sum_{j=1}^{3} \left(\frac{\partial}{\partial \xi_j} \frac{\partial}{\partial \xi_j}\right) u_i^0(\xi) =$$
$$= \nabla_\xi^2 u_i^0(\xi)$$

The equation (21) transformed through by (22) gives

(23) $$\frac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} + \sum_{j=1}^{3} u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial \xi_j} = \nu \nabla_\xi^2 u_i^0(\xi),$$

that is, we transform $X \longmapsto \xi$ and from $\xi_i = x_i + t$ we have $\frac{\partial \xi_i}{\partial x_i} = 1$ and therefore $\partial x_i = \partial \xi_i$.

The unexpected transformation is

(24) $$\frac{\partial u_i^0(\xi)}{\partial t} = \frac{\partial u_i(\xi(X,t))}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j},$$

making (23) not be in the form of a standard Navier-Stokes equation. In $t = 0$ the transformation (24) becomes

(25) $$\frac{\partial u_i^0(\xi)}{\partial t}\Big|_{t=0} = \frac{\partial u_i(\xi(X,t))}{\partial t}\Big|_{t=0} = \frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0} = \sum_{j=1}^{3} \frac{\partial u_i^0(X)}{\partial x_j},$$

$\xi_j = x_j$, $\xi = X$, for $t = 0$, thus we need to assume the additional initial condition

(26) $$\frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0} = \sum_{j=1}^{3} \frac{\partial u_i^0(X)}{\partial x_j}$$

when the solution for Navier-Stokes equation (1), identical to (19), is given by (9), i.e.,

(27.1) $$u(X,t) = u^0(\xi),$$
(27.2) $$p(X,t) = p^0(\xi),$$

$X = (x, y, z),\ \xi = \xi(X, t) = (x + t,\ y + t,\ z + t)$.

Concluding, if $(u^0(X), p^0(X))$ solve (2), identical to (20), substituting in (20) the transformation $X \mapsto \xi,\ X = (x, y, z),\ \xi = (\xi_1, \xi_2, \xi_3),\ \xi_i = x_i + t$, we come to (23),

(28) $$\frac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} + \sum_{j=1}^{3} u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial \xi_j} = \nu \nabla_\xi^2 u_i^0(\xi),$$

assuming the additional initial condition (26)

(29) $$\frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0} = \sum_{j=1}^{3} \frac{\partial u_i^0(X)}{\partial x_j}$$

due to transformation (24),

(30) $$\frac{\partial u_i^0(\xi)}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j}.$$

Using (30) in (28) we come to

(31) $$\frac{\partial p^0(\xi)}{\partial \xi_i} + \frac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^{3} u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial \xi_j} = \nu \nabla_\xi^2 u_i^0(\xi),$$

the Navier-Stokes equations with the solution $(u^0(\xi), p^0(\xi))$, i.e., $(u(X,t), p(X,t))$, according (27), identical to (9).

Using (27) and $\partial \xi_i = \partial x_i$ in (31) we come finally to

(32) $$\frac{\partial p(X,t)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t} + \sum_{j=1}^{3} u_j(X,t) \frac{\partial u_i(X,t)}{\partial x_j} = \nu \nabla_X^2 u_i(X,t),$$

the Navier-Stokes equations (1) with the solution $(u(X,t), p(X,t))$. This is what we wanted to prove. □

What we see in the two previous proofs can be applied, with the obvious adaptations, to solutions of the form

(33.1) $u(X,t) = u^0(\xi)$,
(33.2) $p(X,t) = p^0(\xi)$,

$X = (x, y, z),\ \xi = (\xi_1, \xi_2, \xi_3),\ \xi_i = x_i + T_i(t),\ T_i(0) = 0,\ 1 \le i \le 3$,

with the conditions

(34) $$\frac{\partial u_i(X,t)}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} \frac{\partial \xi_j}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} T_j'(t),$$

and

(35) $$\frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} T_j'(0) = \sum_{j=1}^{3} \frac{\partial u_i^0(X)}{\partial x_j} T_j'(0),$$

being the functions $T_i(t)$ differentiable of class $C^1([0,\infty))$. In this case the equations (23) and (28) are

(36) $$\frac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} T_j'(t) + \sum_{j=1}^{3} u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial \xi_j} =$$

$$= \frac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} \left[T_j'(t) + u_j^0(\xi)\right] = \nu \nabla_\xi^2 u_i^0(\xi).$$

Note that the equation (34) implies

(37) $$u_i(X,t) = u_i^0(X) + \int_0^t \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} T_j^{'}(t)\, dt =$$
$$= u_i^0(\xi_1, \xi_2, \xi_3) = u_i^0\big(x_1 + T_1(t),\ x_2 + T_2(t),\ x_3 + T_3(t)\big),$$

that must be true for all differentiable function $u_i^0(\xi)$ with $\xi_i = x_i + T_i(t)$, $T_i(t)$ differentiable, $T_i(0) = 0$, $1 \le i \le 3$.

It is clear that in the Eulerian description[2] the computational and analytical challenges will be, more than solving the Navier-Stokes equations for $t > 0$, solve these equations for $t = 0$, the initial instant. Unfortunately, it is not for all pair of values (u^0, p^0) that exists solution to the equation (28) and related equations, so or u^0 is a function of p^0, or p^0 is a function of u^0, or both u^0 and p^0 are functions of another functions, for example, a potential function ϕ such that $u^0 = \nabla\phi(t=0)$, $u = \nabla\phi$, resulting in the known Bernoulli's law. From Lagrangian description[2] yet there is a great problem: the collision of particles of the fluid, which always are forgotten or overlooked. The adoption of the continuity equation do not solves it.

NOTE: A few days ago I realized the possibility of proving the invariance of waves and Maxwell's equations with respect to Galilean transformations with this method, without the need to introduce the famous coefficient $\sqrt{1-\frac{v^2}{c^2}}$ of Einstein, including the Schrodinger equation. This also seems to be able to reach to the General Relativity.

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15 – Notes on Uniqueness Solutions of Navier-Stokes Equations

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Abstract: § 1: *remembering the need of imposed the boundary condition $u(x,t)=0$ at infinity to ensure uniqueness solutions to the Navier-Stokes equations. This section is historical only.* § 2*: verifying that for potential and incompressible flows there is no uniqueness solutions when the velocity is equal to zero at infinity. More than this, when the velocity is equal to zero at infinity for all $t \geq 0$ there is no uniqueness solutions, in general case. Exceptions when $u^0 = 0$.* § 3: *non-uniqueness in time for incompressible and potential flows, if $u \neq 0$.* § 4: *a more general solution of Euler and Navier-Stokes equations for incompressible and irrotational (potential) flows, given the initial velocity.* § 5: *Solution for Euler and Navier-Stokes equations using Taylor's series of powers of t around $t=0$.*

§ 1

Recently I wrote a paper named "A Naive Solution for Navier-Stokes Equations"[1] where I concluded that it is possible does not exist the uniqueness of solutions in these equations for $n=3$, even with all terms and for any $t>0$.

This conclusion inhibited me to publish officially my other article "Three Examples of Unbounded Energy for $t>0$"[2], also a very important paper.

This distressful and no way out situation disappears when we impose the boundary condition $\lim_{|x|\to\infty} u(x,t)=0$, which guarantees the desired uniqueness of solutions at least in a finite and not null time interval $[0,T]$. Possibly others boundary conditions also arrive at the uniqueness, but null velocity at infinite may imply a minimum volume of $|u|^2$ and the respective total kinetic energy.

Thus is necessary do some changes in the expressions of external forces, pressures and velocities used in [2] to establish again the breakdown solution in [3], due to occurrence of unbounded energy $\int_{\mathbb{R}^3} |u|^2 dx \to \infty$ in $t>0$. In special, a general example, for $1 \leq i \leq 3$ and $\nabla\cdot u = \nabla\cdot u^0 = \nabla\cdot v = 0$, is

$$u_i(x,t) = u_i^0(x)e^{-t} + v_i(x)e^{-t}(1-e^{-t}),\ u, u^0, v, x \in \mathbb{R}^3,$$

$$u_i^0(x) \in S(\mathbb{R}^3),\ v_i(x) \in C^\infty(\mathbb{R}^3),\ v \notin L^2(\mathbb{R}^3),\ \lim_{|x|\to\infty} v(x) = 0,$$

$$p \in C^\infty\big(\mathbb{R}^3 \times [0,\infty)\big),$$

$$f_i = \left(\frac{\partial p}{\partial x_i} + \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} - \nu\, \nabla^2 u_i\right) \in S\big(\mathbb{R}^3 \times [0,\infty)\big).$$

The conditions (4) for initial velocity and (5) for external force, conforming description given in [3],

(4) $|\partial_x^\alpha u^0(x)| \le C_{\alpha K}(1+|x|)^{-K}\colon \mathbb{R}^3,\ \forall \alpha, K$

(5) $|\partial_x^\alpha \partial_t^m f(x,t)| \le C_{\alpha m K}(1+|x|+t)^{-K}\colon \mathbb{R}^3 \times [0,\infty)\ , \forall \alpha, m, K$

is a kind of *straitjacket,* and for me do not seem good conditions to make possible physically reasonable solutions, rather only restricts the solutions to a very limited and very artificial set of possibilities. If it were possible to the external force be in the set $C^\infty\big(\mathbb{R}^3 \times [0,\infty)\big)$, such as the velocity and pressure in $t > 0$, even being only limited functions and equals zero as $|x| \to \infty$, instead Schwartz Space, the possible solutions will be much more interesting and realistic.

July-03-2016

§ 2

As we know, when $\nabla \times u = 0$ exist a potential function ϕ such that $u = \nabla\phi$. When $\nabla \times u = 0$ and $\nabla \cdot u = 0$ then $\nabla^2\phi = 0$ and $\nabla^2 u = 0$, therefore the Navier-Stokes equations are reduced to Euler's equations and the solutions for velocity are given by Laplace's equation, they are harmonic functions, i.e.,

$$\nabla^2 u = \nabla(\nabla \cdot u) - \nabla \times (\nabla \times u) = (\nabla^2 u_1, \nabla^2 u_2, \nabla^2 u_3) = 0$$

and

$$u = \nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right), \nabla \cdot u = 0 \Longrightarrow \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) = 0.$$

It is clear that there is no uniqueness solutions in all cases, in special when the velocity is both irrotational and incompressible, even if the velocity vanishes at infinity. Defining $\phi(x,t) = \phi^0(x)T(t)$, $T(0) = 1$, then we have $u = \nabla\phi = T(t)\nabla\phi^0 = T(t)u^0(x)$ and so there are endless possibilities for constructing u given u^0, because there are endless possibilities for constructing $T(t)$ with $T(0) = 1$, even if $\lim_{|x|\to\infty} u = T(t) \lim_{|x|\to\infty} u^0 = 0$.

According proof in my other paper [4], if $u(x,y,z,0) = u^0(x,y,z)$ is the initial velocity of the system, valid solution in $t = 0$, then $u(x,y,z,t) = u^0(x+t, y+t, z+t)$ is a solution for velocity in $t \ge 0$. Similarly, $p(x,y,z,t) = p^0(x+t, y+t, z+t)$ is the correspondent solution for pressure in $t \ge 0$, being $p^0(x,y,z)$ the initial condition for pressure. More than this, the velocities $u^0(x+t,y,z)$, $u^0(x,y+t,z)$ and $u^0(x,y,z+t)$ are also solutions, and respectively also the pressures $p^0(x+t,y,z)$, $p^0(x,y+t,z)$ and $p^0(x,y,z+t)$. That is, when the velocity is equal to zero at infinity for all $t \geq 0$ there is no uniqueness solutions, in general case. Apparently, an additional complication if the uniqueness condition is required.

Exception to the two previous paragraphs when $u^0 = 0$.

July-19-2016

§ 3

In line with previous date, if $\nabla \cdot u = 0$ and $\nabla \times u = 0$ then $\nabla^2 u = 0$. For $u = (u_1, u_2, u_3)$ and $w = (w_1, w_2, w_3)$, defining $w_i = A(t)u_i + B_i(t), 1 \leq i \leq 3$, we will have $\nabla \cdot w = 0$, $\nabla \times w = 0$ and $\nabla^2 w = 0$.

If $u = \nabla\phi$ solves the Navier-Stokes equations then

$$\nabla p + \frac{\partial u}{\partial t} + (u \cdot \nabla)u = \nu \nabla^2 u$$

$$\nabla p + \nabla\left(\frac{\partial \phi}{\partial t}\right) + (\nabla \times u) \times u + \frac{1}{2}\nabla|u|^2 =$$
$$= \nu\big(\nabla(\nabla \cdot u) - \nabla \times (\nabla \times u)\big)$$

$$\nabla p + \nabla\left(\frac{\partial \phi}{\partial t}\right) + \nabla\left(\frac{1}{2}|u|^2\right) = 0$$

$$\nabla\left(p + \frac{\partial \phi}{\partial t} + \frac{1}{2}|u|^2\right) = 0$$

$$p + \frac{\partial \phi}{\partial t} + \frac{1}{2}|u|^2 = \theta(t),$$

which is the Bernoulli's law without external force.

With a gradient external force $f = \nabla U$ we will have

$$p + \frac{\partial \phi}{\partial t} + \frac{1}{2}|u|^2 = U + \theta(t).$$

For w defined as above, substituting $u \mapsto w$ in the Navier-Stokes equations comes

$$p + \frac{\partial \varphi}{\partial t} + \frac{1}{2}|w|^2 = U + \theta(t),$$

where $\varphi = A(t)\phi + B_1 x + B_2 y + B_3 z$, and p is the new pressure for the velocity $w = A(t)u + B(t), B = (B_1, B_2, B_3)$.

If $A(0)=1$ and $B_i(0)=0,\ 1\leq i\leq 3$, then u and w obey the same initial condition and both solve the Navier-Stokes (and Euler) equations and they are incompressible and potential flows. In this case, there is no uniqueness solution, for $A(t)\not\equiv 1$ or $B(t)\not\equiv 0$, i.e., $u\not\equiv w$.

Imposing the boundary condition at infinity $u|_{r\to\infty}=0, r=\sqrt{x^2+y^2+z^2}$, the velocity $w=A(t)u$ obey the same boundary condition, for $A(0)=1,\ A(t)\not\equiv 1$ finite for all $t\geq 0$, i.e. $w(x,y,z,t)=A(t)u(x,y,z,t)$ and $u(x,y,z,t)$ obey the same initial and boundary conditions, so there is no uniqueness solutions for Navier-Stokes (and Euler) equations in this case of incompressible and potential flows with velocity zero at infinity, if $u\neq 0$.

July-30-2016
August-15-2016

§ 4

Other class of solutions for velocity is built through of the transformations $x_i\mapsto \alpha(t)x_i+ct,\ 1\leq i\leq 3,\ \alpha(t),c\in\mathbb{R},\ \alpha(t)\neq 0,\ \alpha(0)=1$, in the parameters of the initial velocity, i.e.,

(4.1) $$\boldsymbol{u}(x,y,z,t)=A(t)\boldsymbol{u^0}(\alpha x+ct,\alpha y+ct,\alpha z+ct)+\boldsymbol{B}(t),$$

because if

(4.2.1) $$\nabla^2\boldsymbol{u^0}(x,y,z)=\nabla^2[A(t)\boldsymbol{u^0}(x,y,z)+\boldsymbol{B}(t)]=\boldsymbol{0}$$

(4.2.2) $$\nabla\cdot\boldsymbol{u^0}(x,y,z)=\nabla\cdot[A(t)\boldsymbol{u^0}(x,y,z)+\boldsymbol{B}(t)]=0$$

(4.2.3) $$\nabla\times\boldsymbol{u^0}(x,y,z)=\nabla\times[A(t)\boldsymbol{u^0}(x,y,z)+\boldsymbol{B}(t)]=\boldsymbol{0}$$

then also

(4.3.1) $$\nabla^2[A(t)\boldsymbol{u^0}(\alpha x+ct,\alpha y+ct,\alpha z+ct)+\boldsymbol{B}(t)]=\boldsymbol{0}$$

(4.3.2) $$\nabla\cdot[A(t)\boldsymbol{u^0}(\alpha x+ct,\alpha y+ct,\alpha z+ct)+\boldsymbol{B}(t)]=0$$

(4.3.3) $$\nabla\times[A(t)\boldsymbol{u^0}(\alpha x+ct,\alpha y+ct,\alpha z+ct)+\boldsymbol{B}(t)]=\boldsymbol{0}$$

α a function of time, that is, the velocity (4.1) with $A(0)=1,\ \boldsymbol{B}(0)=\boldsymbol{0},\ \alpha(0)=1$, $\alpha(t)\neq 0$, is a solution for Euler (and Navier-Stokes) equations with initial velocity $\boldsymbol{u^0}(x,y,z)$, a general solution for incompressible and irrotational (potential) flows, in the case of conservative external forces. The respective pressure is again given by the Bernoulli's law, (4.4) $$p + \frac{\partial \phi}{\partial t} + \frac{1}{2}|\boldsymbol{u}|^2 = U + \theta(t),$$

$\boldsymbol{u} = \nabla\phi$, $\boldsymbol{f} = \nabla U$, also without uniqueness solution due to $\theta(t)$ and $\boldsymbol{u}$.

August-18-2016

§ 5

In [5] we got to a great result, the complete solution for Euler and Navier-Stokes equations for incompressible fluids, using the expansion of the velocity u in a Taylor's series of powers of time around $t = 0$. This work was a natural evolution of [6], which in turn is a consequence of [4]. In [4] the external force is equal to zero, and in [6] the external force need be conservative, derived from a potential. In [5] a very general condition is accepted, since that the initial velocity, external force and pressure belong to C^∞, and of general way they can be expressed in Taylor's series of time t.

This expansion of the velocity in a Taylor´s series in relation to time around $t = 0$, considering x, y, z as constant, for $1 \le i \le 3$, is

(5.1) $$u_i = u_i|_{t=0} + \frac{\partial u_i}{\partial t}|_{t=0}\, t + \frac{\partial^2 u_i}{\partial t^2}|_{t=0}\, \frac{t^2}{2} + \frac{\partial^3 u_i}{\partial t^3}|_{t=0}\, \frac{t^3}{6} + \cdots + \frac{\partial^k u_i}{\partial t^k}|_{t=0}\, \frac{t^k}{k!} + \cdots$$

or

(5.2) $$u_i = u_i^0 + \sum_{k=1}^{\infty} \frac{\partial^k u_i}{\partial t^k}|_{t=0}\, \frac{t^k}{k!}.$$

For the calculation of $\frac{\partial u_i}{\partial t}, \frac{\partial^2 u_i}{\partial t^2}, \frac{\partial^3 u_i}{\partial t^3}, \ldots$ we use the values that are obtained directly from the Navier-Stokes equations and its derivatives in relation to time, i.e.,

(5.3) $$\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} + \nu\, \nabla^2 u_i + f_i,$$

(5.4) $$\frac{\partial^2 u_i}{\partial t^2} = -\frac{\partial^2 p}{\partial t\, \partial x_i} - \sum_{j=1}^{3} \left(\frac{\partial u_j}{\partial t} \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} \right) + \nu\, \nabla^2 \frac{\partial u_i}{\partial t} + \frac{\partial f_i}{\partial t},$$

and using induction we come to

(5.5) $$\frac{\partial^k u_i}{\partial t^k} = -\frac{\partial^k p}{\partial t^{k-1}\, \partial x_i} - \sum_{j=1}^{3} N_j^{k-1} + \nu\, \nabla^2 \frac{\partial^{k-1} u_i}{\partial t^{k-1}} + \frac{\partial^{k-1} f_i}{\partial t^{k-1}},$$
$$N_j^{k-1} = \frac{\partial}{\partial t} N_j^{k-2} = \sum_{l=0}^{k-1} \binom{k-1}{l} \partial_t^{k-1-l} u_j\, \frac{\partial}{\partial x_j} \partial_t^l u_i,$$

$$\partial_t^0 u_n = u_n,\ \partial_t^m u_n = \frac{\partial^m u_n}{\partial t^m},\ \binom{k-1}{l} = \frac{(k-1)!}{(k-1-l)!\, l!}.$$

In (5.1) and (5.2) it is necessary to know the values of the derivatives $\frac{\partial u_i}{\partial t}, \frac{\partial^2 u_i}{\partial t^2}, \dots, \frac{\partial^k u_i}{\partial t^k}$ in $t = 0$ then we must to calculate, from (5.3) to (5.5),

(5.6) $$\frac{\partial u_i}{\partial t}\Big|_{t=0} = -\frac{\partial p^0}{\partial x_i} - \sum_{j=1}^3 u_j^0 \frac{\partial u_i^0}{\partial x_j} + \nu\, \nabla^2 u_i^0 + f_i^0,$$

the superior index 0 meaning the value of the respective function at $t = 0$,

(5.7) $$\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0} = -\frac{\partial^2 p}{\partial t\, \partial x_i}\Big|_{t=0} - \sum_{j=1}^3 N_j^1\Big|_{t=0} +$$
$$+\nu\, \nabla^2 \frac{\partial u_i}{\partial t}\Big|_{t=0} + \frac{\partial f_i}{\partial t}\Big|_{t=0},$$
$$N_j^1\Big|_{t=0} = \sum_{j=1}^3 \left(\frac{\partial u_j}{\partial t}\Big|_{t=0} \frac{\partial u_i^0}{\partial x_j} + u_j^0 \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t}\Big|_{t=0}\right),$$

and of generic form,

(5.8) $$\frac{\partial^k u_i}{\partial t^k}\Big|_{t=0} = -\frac{\partial^k p}{\partial t^{k-1}\, \partial x_i}\Big|_{t=0} - \sum_{j=1}^3 N_j^{k-1}\Big|_{t=0} +$$
$$+\nu\, \nabla^2 \frac{\partial^{k-1} u_i}{\partial t^{k-1}}\Big|_{t=0} + \frac{\partial^{k-1} f_i}{\partial t^{k-1}}\Big|_{t=0},$$
$$N_j^{k-1}\Big|_{t=0} = \sum_{l=0}^{k-1} \binom{k-1}{l} \partial_t^{k-1-l} u_j\Big|_{t=0} \frac{\partial}{\partial x_j} \partial_t^l u_i\Big|_{t=0},$$
$$\partial_t^0 u_n\Big|_{t=0} = u_n^0,\ \partial_t^m u_n\Big|_{t=0} = \frac{\partial^m u_n}{\partial t^m}\Big|_{t=0}.$$

This solution is an explicit representation of the non-uniqueness solution of the Euler's $(\nu = 0)$ and Navier-Stokes equations because it is possible choose any smooth pressure to be a solution for the problem, in special in the cases without boundary conditions, where the domain of the position (x, y, z) is the whole space $\mathbb{R}^3$ and be of the class C^∞ is a necessary condition for the velocity, in each point of space and time.

The presented method can be implemented in other equations, of course, for example, heat equation, Schrödinger equation, wave equation and many others. In linear equations the facility of operations seems to be great.

September-07-2016

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16 – Breakdown of Euler Equations – New Approach

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Abstract – The solution for the problem of Breakdown of Euler Equations, like the Millenium Problem for Navier-Stokes equations.

§ 1

Motived by the 6th Millenium Problem, relative to the solution of the Navier-Stokes equations or prove of the inexistence of solutions, obeying certain conditions, I wrote this paper for solve this problem substituting Navier-Stokes by Euler equations, since that these same questions are unsolved for Euler equations, although these last are not on the Clay Institute's list of prize problems.[1] The natural sequence of this paper is the correspondent to Navier-Stokes equations.

In his famous *Méchanique Analitique* (1788), using the notions of total or complete differential and exact differential, and creating the concept of velocity-potential, for an external force with potential (a gradient or conservative external force, which also can be a force equal to zero) Lagrange came to the conclusion that Euler´s equations could be solved only for two specific conditions: (1) for potential (irrotational) flows, and (2) for non-potential (rotational) but steady flows.[2],[3] In Lagrange[3], pp. 536-542, the pressure is represented as λ, the external force components as X, Y, Z, the velocity components as p, q, r, the rectangular coordinates as x, y, z and time as t. The velocity-potential is φ and the force-potential is V.

The solution for pressure obtained by Lagrange for incompressible fluids in potential flow case was

$$\lambda = V + \frac{d\varphi}{dt} + \frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 + \frac{1}{2}\left(\frac{d\varphi}{dy}\right)^2 + \frac{1}{2}\left(\frac{d\varphi}{dz}\right)^2,$$

and an arbitrary function of t could be added here because this variable is treated in the integration as a constant, which is nothing more nor less that the Bernoulli's law, except by the signs of λ and V (the use by Lagrange of d is as our ∂, means partial derivative).

The determination of φ will depend upon equation (continuity equation, the incompressibility condition)

$$\frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz} = 0,$$

in which after substitution of the expressions $\frac{d\varphi}{dx}, \frac{d\varphi}{dy}, \frac{d\varphi}{dz}$ for p, q, r becomes

$$\frac{d^2\varphi}{dx^2} + \frac{d^2\varphi}{dy^2} + \frac{d^2\varphi}{dz^2} = 0,$$

that is the Laplace's equation.

Thus, conclude Lagrange, all the remaining difficulty will now lie in the integration of this last equation.

Of course that it is possible describe a fluid movement without potential flow and conservative forces, simply by setting the external force as

$$(1.1) \qquad \boldsymbol{f} = \nabla p + \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u},$$

given any pressure p and velocity u, both differentiable functions of class C and C^2, respectively, velocity with potential or no, obeying the incompressibility condition or no, but we do not need this kind of force here.

In the present paper we are interested only in conservative external forces, i.e., with potential, including zero, and the validity of incompressible flow condition, which require for the solution of Euler equations a potential velocity for non-steady flows.

I think that the deduction used by Lagrange in Euler's equations can be implemented also in Navier-Stokes equations, and we will come to $\nabla^2\boldsymbol{u} = \boldsymbol{0}$. I am hopeful to prove this in next article, concluding this subject. Really, today, 08-12-2016, my answer to the problem of breakdown of Navier-Stokes equations is as follow: given an initial velocity $\boldsymbol{u^0}$ which is potential flow and a not null and not conservative external force, in special both belonging to the Schwartz Space, there is no solution $(\boldsymbol{u}, p)$ for Navier-Stokes equations, velocities $\boldsymbol{u}$ and $\boldsymbol{u^0}$ obeying the incompressibility condition or not, i.e., satisfying the Laplace's equation or not, which is not exactly equal to Lagrange's proof. My *prototype* of external force is

$$(1.2) \qquad \boldsymbol{f} = \boldsymbol{g} + \left[(\boldsymbol{u^0} \cdot \nabla)\boldsymbol{u^0} - \nu\left(\nabla^2\boldsymbol{u^0} + \tfrac{1}{3}\nabla(\nabla \cdot \boldsymbol{u^0})\right)\right] e^{-t},$$

where $\boldsymbol{g}$ is non gradient and decreases exponentially in the time.

§ 2

When $\nabla \times \boldsymbol{u} = 0$ then exist a potential function ϕ such that $\boldsymbol{u} = \nabla\phi$. When $\nabla \times \boldsymbol{u} = \boldsymbol{0}$ and $\nabla \cdot \boldsymbol{u} = 0$ then $\nabla^2\phi = 0$ and $\nabla^2\boldsymbol{u} = \boldsymbol{0}$, therefore the Navier-Stokes equations are reduced to Euler's equations and the solutions for velocity are given by Laplace's equation, they are harmonic functions, i.e., (2.1) $$\nabla^2\boldsymbol{u} = \nabla(\nabla\cdot\boldsymbol{u}) - \nabla\times(\nabla\times\boldsymbol{u}) = (\nabla^2 u, \nabla^2 v, \nabla^2 w) = \boldsymbol{0}$$

and

(2.2) $$\boldsymbol{u} = \nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right), \nabla\cdot\boldsymbol{u} = 0 \Longrightarrow \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) = 0.$$

It is clear that there is no uniqueness solution in all cases, in special when the velocity is both irrotational and incompressible, even if the velocity vanishes at infinity. Defining $\phi(x,y,z,t) = \phi^0(x,y,z)T(t)$, $T(0) = 1$, $T(t) \not\equiv 1$, then we have $\boldsymbol{u} = \nabla\phi = T(t)\nabla\phi^0 = T(t)\boldsymbol{u}^{\boldsymbol{0}}(x,y,z)$ and so there are endless possibilities for constructing $\boldsymbol{u}$ given $\boldsymbol{u}^{\boldsymbol{0}}$, because there are endless possibilities for constructing $T(t)$ with $T(0) = 1$, even if $\lim_{r\to\infty}\boldsymbol{u} = T(t)\lim_{r\to\infty}\boldsymbol{u}^{\boldsymbol{0}} = \boldsymbol{0}$, where $r = \sqrt{x^2+y^2+z^2}$. Exception if the initial velocity is identically null, when for the previous reasoning the velocity is $\boldsymbol{u} = \boldsymbol{0}$ unique.

A more long way to see this is for example as follow. If $\nabla\cdot\boldsymbol{u} = 0$ and $\nabla\times\boldsymbol{u} = \boldsymbol{0}$ then $\nabla^2\boldsymbol{u} = \boldsymbol{0}$. For $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{w} = (w_1, w_2, w_3)$, defining $w_i = A(t)u_i + B_i(t), 1 \le i \le 3$, we will have $\nabla\cdot\boldsymbol{w} = 0$, $\nabla\times\boldsymbol{w} = \boldsymbol{0}$ and $\nabla^2\boldsymbol{w} = \boldsymbol{0}$.

If $\boldsymbol{u} = \nabla\phi$ solves the Navier-Stokes equations then, from

(2.3.1) $$\nabla p + \frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nu\nabla^2\boldsymbol{u}$$

(2.3.2) $$\nabla p + \nabla\left(\frac{\partial\phi}{\partial t}\right) + (\nabla\times\boldsymbol{u})\times\boldsymbol{u} + \frac{1}{2}\nabla|\boldsymbol{u}|^2 = \\ = \nu\big(\nabla(\nabla\cdot\boldsymbol{u}) - \nabla\times(\nabla\times\boldsymbol{u})\big)$$

(2.3.3) $$\nabla p + \nabla\left(\frac{\partial\phi}{\partial t}\right) + \nabla\left(\frac{1}{2}|\boldsymbol{u}|^2\right) = \boldsymbol{0}$$

(2.3.4) $$\nabla\left(p + \frac{\partial\phi}{\partial t} + \frac{1}{2}|\boldsymbol{u}|^2\right) = \boldsymbol{0},$$

we obtain

(2.4) $$p + \frac{\partial\phi}{\partial t} + \frac{1}{2}|\boldsymbol{u}|^2 = \theta(t),$$

which is the Bernoulli's law without external force.

With a gradient external force $\boldsymbol{f} = \nabla U$ we will have

(2.5) $$p + \frac{\partial\phi}{\partial t} + \frac{1}{2}|\boldsymbol{u}|^2 = U + \theta(t).$$

For $\boldsymbol{w}$ defined as above, substituting $\boldsymbol{u} \mapsto \boldsymbol{w}$ in the Navier-Stokes equations (2.3.1) comes

$$p + \frac{\partial \varphi}{\partial t} + \frac{1}{2}|\boldsymbol{w}|^2 = U + \theta(t), \tag{2.6}$$

where $\varphi = A(t)\phi + B_1(t)x + B_2(t)y + B_3(t)z$, and p is the new pressure for the velocity $\boldsymbol{w} = A(t)\boldsymbol{u} + \boldsymbol{B}(t), \boldsymbol{B} = (B_1, B_2, B_3)$.

If $A(0) = 1$ and $B_i(0) = 0,\ 1 \leq i \leq 3$, then $\boldsymbol{u}$ and $\boldsymbol{w}$ obey the same initial condition and both solve the Navier-Stokes (and Euler) equations and they are incompressible and potential flows. Thus, in this case, there is no uniqueness solution, for $A(t) \not\equiv 1$ or $\boldsymbol{B}(t) \not\equiv \boldsymbol{0}$, i.e., $\boldsymbol{u} \not\equiv \boldsymbol{w}$.

Imposing the boundary condition at infinity $\boldsymbol{u}|_{r\to\infty} = \boldsymbol{0}, r = \sqrt{x^2 + y^2 + z^2}$, the velocity $\boldsymbol{w} = A(t)\boldsymbol{u}$ obey the same boundary condition, for $A(0) = 1,\ A(t) \not\equiv 1$ finite for all $t \geq 0$, i.e. $\boldsymbol{w}(x,y,z,t) = A(t)\boldsymbol{u}(x,y,z,t)$ and $\boldsymbol{u}(x,y,z,t)$ obey the same initial and boundary conditions, so there is no uniqueness solutions for Navier-Stokes (and Euler) equations in this case of incompressible and potential flows with velocity zero at infinity, if $\boldsymbol{u} \neq \boldsymbol{0}$.

§ 3

Sobolev[4] (pp. 12, 13, 18, 19) is very assured to affirm that *the problem of the motion of an incompressible fluid is equivalent to that of finding an unknown function* V *(the velocity-potential) such that*

$$\boldsymbol{v} = grad\ V, \qquad v_x = \frac{\partial V}{\partial x}, \ \ v_y = \frac{\partial V}{\partial y}, \ \ v_z = \frac{\partial V}{\partial z}.$$

Continuing his citation, *substituting these expressions for the velocity components in the continuity equation, we get*

$$\varrho\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) = 0$$

or

$$\nabla^2 V = 0. \tag{1.17}$$

(...)

Later we shall write down the complete set of equations of motion for a fluid and we shall show that any function V *which satisfies (1.17) does indeed describe a possible motion of the fluid. Thus to solve a problem of fluid motion it suffices to know to find the requisite solutions of equation (1.17).*

In some circumstances, the velocity $\boldsymbol{v}$ *and so also the function* V *do not depend on the time* t*; the motion is then one of steady flow.*

(...)

We can now verify what was said earlier about the potential flow of an incompressible fluid: namely, that

$$\boldsymbol{v} = grad\, V,$$

$$\nabla^2 V = 0,$$

do actually satisfy the complete set of equations (Euler equations with mass density coefficient ϱ and external force (X, Y, Z), note mine)*, if the function* ϱ *is defined correspondingly, and if further*

$$X = \frac{\partial U}{\partial x},\ Y = \frac{\partial U}{\partial y},\ Z = \frac{\partial U}{\partial z},$$

i.e., if the external force have a potential.

It suffices to show that if we take

$$v_x = \frac{\partial V}{\partial x},\ \ v_y = \frac{\partial V}{\partial y},\ \ v_z = \frac{\partial V}{\partial z},$$

then the equations (1.22) (the Euler equations) *allow the function* p *to be constructed. When the expressions for* v_x, v_y, v_z *are substituted, these equations yield explicit expressions for*

$$\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}.$$

And it is known from the theory of partial differential equations of the first order that the equations will be compatible provided that the mixed second-order derivatives

$$\frac{\partial^2 p}{\partial x \partial y}, \frac{\partial^2 p}{\partial y \partial z}, \frac{\partial^2 p}{\partial z \partial x}$$

determined from the different equations have the same values. (...)

Then, following Sobolev, if the external force is gradient, if it have a potential, the solutions for velocity in the Euler's equations in case of incompressible flows are given by Laplace's equation, the velocity is a harmonic function in the three orthogonal directions, not only one possibility among others, but in fact they are the unique possible cases of solution, only harmonic functions, when the external force is gradient (for example also without external force, $X = Y = Z = 0$) and the fluid is incompressible.

The same argument used by Sobolev for solve Euler's equations can be used for solve the Navier-Stokes equations: *Thus to solve a problem of fluid motion it suffices to know to find the requisite solutions of equation (1.17),* $\nabla^2 V = 0$. This is like the conclusion of Lagrange, viewed in section § 1, for Euler's equations in potential flow case.

All solution of Euler equations is solution of Navier-Stokes equations for potential and incompressible flows, when $\nabla^2 \boldsymbol{u} = \boldsymbol{0}$. If $\boldsymbol{u} = \nabla\phi$ then $\nabla \times \boldsymbol{u} = \boldsymbol{0}$, because

$$\text{(3.1)} \qquad \nabla \times \boldsymbol{u} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial k \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} = \boldsymbol{0}$$

being $\frac{\partial^2\phi}{\partial x_i \partial x_j} = \frac{\partial^2\phi}{\partial x_j \partial x_i}$, $1 \le i, j \le 3$.

If $\nabla \times \boldsymbol{u} = \boldsymbol{0}$ (potential flow) and $\nabla \cdot \boldsymbol{u} = 0$ (incompressible flow) then

$$\text{(3.2)} \qquad \nabla^2 \boldsymbol{u} = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla \times (\nabla \times \boldsymbol{u}) = \boldsymbol{0},$$

i.e. the derivatives of second order in Navier-Stokes equations vanishes in case of potential and incompressible flows and the Navier-Stokes equations reduced to the Euler equations, whose respective solutions are harmonic functions. In this case, solve Euler equations implies solve Navier-Stokes equations, supposing the same initial and boundary conditions, and if the Navier-Stokes equations has unique solution at least in a small and not null time interval $[0, T]$, with the boundary condition $\lim_{r\to\infty} |\boldsymbol{u}| = \boldsymbol{0}$, $r = \sqrt{x^2 + y^2 + z^2}$, then this first solution in time is also the solution of Euler equations and the velocity satisfies the Laplace's equation.

§ 4

How the condition

$$\text{(4.1)} \qquad \frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i}, \; 1 \le i, j \le 3,$$

equivalent to $\nabla \times \boldsymbol{u} = \boldsymbol{0}$, solve the Euler equations with a null or gradient external force $\boldsymbol{f} = \nabla U$, so with this external force the condition of irrotational or potential flow is a necessary condition for solution of these equations, at least for non-steady flows. This has been rigorously proven by Lagrange[3], for incompressible fluids. Including the incompressibility condition, we have the Laplace's equation in vector form, $\nabla^2 \boldsymbol{u} = \boldsymbol{0}$ and $\nabla^2 \boldsymbol{u^0} = \boldsymbol{0}$, where $\boldsymbol{u^0}$ is the initial velocity, even without uniqueness solution, as viewed in section § 2.

Lagrange also proved, as well as Laplace (*Mécanique Céleste*), Poisson (*Traité de Méchanique*), Cauchy (*Mémoire sur la Théorie des Ondes*) and Stokes (*On the Friction of Fluids in Motion and the Equilibrium and Motion of Elastic Solids*), that if the differential of the fluid's velocity $u_1 dx + u_2 dy + u_3 dz$ is a differential exact in some instant of time (for example, in $t = 0$) then it is also for all time ($t \geq 0$) of this movement on the same conditions. This means that a potential flow is always potential flow, since $t = 0$. Then, from § 1, if the initial velocity have not an exact differential (i.e., if the initial velocity is not a gradient function, irrotational, with potential) and the external force have potential then the Euler's equations have no solution in this case of incompressible and potential flows, for non-steady flows.

For steady flows, where $\frac{\partial \boldsymbol{u}}{\partial t} \equiv \boldsymbol{0}$ and $\boldsymbol{u} = \boldsymbol{u}^{\boldsymbol{0}}$ for all $t \geq 0$, the condition for existence of solution is that

(4.2) $$\frac{\partial S_i}{\partial x_j} = \frac{\partial S_j}{\partial x_i}$$

for all pair (i, j), $1 \leq i, j \leq 3$, defining

(4.3) $$S_i = f_i - \sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j},$$

where $\boldsymbol{f} \equiv \boldsymbol{f}^{\boldsymbol{0}}$ is the stationary external force. This is a common condition for existence of solution for a system $\nabla p = \boldsymbol{S}$, representing the stationary Euler's equations, that is $\nabla \times \boldsymbol{S} = \boldsymbol{0}$.

Then now is possible go to the solution related to the breakdown of the Euler equations, corresponding to the cases (C) and (D) of [1]: without external force or with an external force which have a potential, $\boldsymbol{f} = \nabla U$, $\boldsymbol{f} \in S(\mathbb{R}^3 \times [0, \infty))$, S representing the Schwartz space, if the initial velocity $\boldsymbol{u}^{\boldsymbol{0}} \in \mathbb{R}^3$ with $\nabla \cdot \boldsymbol{u}^{\boldsymbol{0}} = 0$ is not a potential flow and (considering also the steady flows) $\frac{\partial S_i}{\partial x_j} \neq \frac{\partial S_j}{\partial x_i}$ for some pair (i, j) such that $1 \leq i, j \leq 3$, with

(4.4) $$S_i = f_i^0 - \sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j},$$

$x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$, $\boldsymbol{f}^{\boldsymbol{0}} = \boldsymbol{f}(x, y, z, 0)$, there is no solution $(\boldsymbol{u}, p)$ for the Euler equations, belonging to C^∞ or not, periodic solution or not. In special, when $\boldsymbol{u}^{\boldsymbol{0}} \in S(\mathbb{R}^3)$, $\nabla \cdot \boldsymbol{u}^{\boldsymbol{0}} = 0$ and $\boldsymbol{u}^{\boldsymbol{0}}$ is not a gradient function, with $\frac{\partial S_i}{\partial x_j} \neq \frac{\partial S_j}{\partial x_i}$ for some (i, j), there is no solution for Euler equations, in the mentioned conditions for $\boldsymbol{f}$. Besides that the unique initial velocity $\boldsymbol{u}^{\boldsymbol{0}} \in S(\mathbb{R}^3)$, harmonic and gradient function is $\boldsymbol{u}^{\boldsymbol{0}} = \boldsymbol{0}$, which provide only the trivial solution $\boldsymbol{u} = \boldsymbol{0}$ for velocity in Schwartz space and infinite solutions for pressure, $p = U + \theta(t)$, $p \in C^\infty(\mathbb{R}^3 \times [0, \infty))$.

§ 5

I finish this work with a qualitative discussion of the conclusion which we have obtained in the previous section. Any student of physics, Gravitation or

Electromagnetism, knows that the most well-known non trivial solution of the Laplace's equation is of the form $1/r$, which diverges in origin and goes to zero at infinity. According Liouville's Theorem[4], a harmonic function which is limited is constant, and equal to zero if it tends to zero at infinity. How the Millennium Problem requires a limited solution in all space for velocity and a limited initial velocity which goes to zero at infinity (in cases (A) and (C)), then we are forced to choose $\boldsymbol{u^0} = \boldsymbol{0}$.

Without these requisites we can obtain other solutions for velocity, for example, $\boldsymbol{u} = \boldsymbol{A}(t)$, as well as potential flows in general (say, harmonic functions for incompressible flows), including spatially periodic functions of unitary period without singularities in the cube $[0,1]^3$, which refers to case (B). Initial velocities spatially periodic but non potential flows lead to case (D) if the external force is null or gradient and $\frac{\partial S_i}{\partial x_j} \neq \frac{\partial S_j}{\partial x_i}$ in $t = 0$ for any (i, j), S_i defined by (4.4), such as occurs in the case (C).

Specifically, without preoccupations with unbounded velocity in some region, a solution of Euler's (and Navier-Stokes) equations for incompressible, non-steady and potential flows with gradient external force is

$$\boldsymbol{u} = A(t)\boldsymbol{u^0} + \boldsymbol{B}(t), \tag{5.1}$$

where $A(0) = 1$, $\boldsymbol{B}(0) = \boldsymbol{0}$, $\nabla \cdot \boldsymbol{u^0} = 0$, $\boldsymbol{u^0} = \nabla\phi(t = 0)$ and $\boldsymbol{u^0}$ is the initial velocity, without uniqueness solution due to possibility of $A(t) \not\equiv 1$ or $\boldsymbol{B}(t) \not\equiv \boldsymbol{0}$, and the pressure is given by Bernoulli's law,

$$p + \frac{\partial\phi}{\partial t} + \frac{1}{2}|\boldsymbol{u}|^2 = U + \theta(t), \tag{5.2}$$

$\boldsymbol{f} = \nabla U$, $\boldsymbol{u} = \nabla\phi$, also without uniqueness solution due to $\theta(t)$ and $\boldsymbol{u}$. We can consider $A(t)$ and $\boldsymbol{B}(t)$ belonging to C^∞ in their respective domains.

Other class of solutions for velocity is built through of the transformations $x_i \mapsto \alpha(t)x_i + ct$, $1 \leq i \leq 3$, $\alpha(t) \in C^\infty(\mathbb{R})$, $c \in \mathbb{R}$, $\alpha(t) \neq 0$, $\alpha(0) = 1$, in the parameters of the initial velocity, i.e.,

$$\boldsymbol{u}(x, y, z, t) = A(t)\boldsymbol{u^0}(\alpha x + ct, \alpha y + ct, \alpha z + ct) + \boldsymbol{B}(t), \tag{5.3}$$

because if

$$\nabla^2\boldsymbol{u^0}(x, y, z) = \nabla^2[A(t)\boldsymbol{u^0}(x, y, z) + \boldsymbol{B}(t)] = \boldsymbol{0} \tag{5.4.1}$$

$$\nabla \cdot \boldsymbol{u^0}(x, y, z) = \nabla \cdot [A(t)\boldsymbol{u^0}(x, y, z) + \boldsymbol{B}(t)] = 0 \tag{5.4.2}$$

$$\nabla \times \boldsymbol{u^0}(x, y, z) = \nabla \times [A(t)\boldsymbol{u^0}(x, y, z) + \boldsymbol{B}(t)] = \boldsymbol{0} \tag{5.4.3}$$

then also

(5.5.1) $$\nabla^2[A(t)\boldsymbol{u^0}(\alpha x + ct, \alpha y + ct, \alpha z + ct) + \boldsymbol{B}(t)] = \boldsymbol{0}$$

(5.5.2) $$\nabla \cdot [A(t)\boldsymbol{u^0}(\alpha x + ct, \alpha y + ct, \alpha z + ct) + \boldsymbol{B}(t)] = 0$$

(5.5.3) $$\nabla \times [A(t)\boldsymbol{u^0}(\alpha x + ct, \alpha y + ct, \alpha z + ct) + \boldsymbol{B}(t)] = \boldsymbol{0}$$

α a function of time, that is, the velocity (5.3) with $A(0) = 1$, $\boldsymbol{B}(0) = \boldsymbol{0}$, $\alpha(0) = 1$, $\alpha(t) \neq 0$, is a solution for Euler (and Navier-Stokes) equations with initial velocity $\boldsymbol{u^0}(x, y, z)$, a general solution for incompressible and irrotational (potential) flows, in the case of conservative external forces. The respective pressure is again given by (5.2), obviously both velocity and pressure without uniqueness solution.

As pointed by Lagrange and Sobolev, the solution of Laplace's equation is essential in the solution of Euler's equations. How is not difficult to see, the Laplace´s equation is especially important for obtain the initial velocity of a motion of incompressible, irrotational and non-steady fluid, $\boldsymbol{u^0} = \nabla\phi(t = 0)$, $\nabla^2\phi = 0$, as well as $\boldsymbol{u} = \nabla\phi$. Will be possible obtain others velocities for $t > 0$ and the pressure using (5.3) and (5.2), respectively. For a system where is prescribed a specific velocity $\boldsymbol{u^\partial}$ in a boundary $\partial\Omega$, except when $\Omega = \mathbb{R}^3$, the application of (5.3) may not be adequate and will be needed in general the use of other methods of solution. If $\Omega = \mathbb{R}^3$ and $\boldsymbol{u}|_{r\to\infty} = \boldsymbol{0}$ then $\boldsymbol{B}(t) \equiv \boldsymbol{0}$.

According Courant[5] (p.241), for $n = 2$ the "general solution" of the potential equation (or Laplace's equation) is the real part of any analytic function of the complex variable $x + iy$. For $n = 3$ one can also easily obtain solutions which depend on arbitrary functions. For example, let $f(w, t)$ be analytic in the complex variable w for fixed real t. Then, for arbitrary values of t, both the real and imaginary parts of the function

$$u = f(z + ix\cos t + iy\sin t\,, t)$$

of the real variables x, y, z are solutions of the equation $\nabla^2 u = 0$. Further solutions may be obtained by superposition:

$$u = \int_a^b f(z + ix\cos t + iy\sin t\,, t)dt.$$

For example, if we set

$$f(w, t) = w^n e^{iht},$$

where n and h are integers, and integrate from $-\pi$ to $+\pi$, we get homogeneous polynomials

$$u = \int_{-\pi}^{\pi}(z + ix\cos t + iy\sin t)^n\, e^{iht}dt$$

in x, y, z, following example given by Courant. Introducing polar coordinates $z = r\cos\theta, x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi$, we obtain

$$u = 2r^n e^{ih\phi}\int_0^{\pi}(\cos\theta + i\sin\theta\cos t)^n \cos ht \ dt$$
$$= r^n e^{ih\phi} P_{n,h}(\cos\theta),$$

where $P_{n,h}(\cos\theta)$ are the associated Legendre functions.

Possibly when or if it is analyzed that the incompressibility condition is not so important, with little relation with the physical reality, the solutions of the Euler (and Navier-Stokes) equations will no longer comply with the solutions of the Laplace's equation in irrotational movements, and you can find regular solutions in the whole space as well as more compatible with experiences.

A musician must make music,
an artist must paint,
a poet must write,
if he is to be ultimately happy.
What a man can be, he must be.

This need we may call self-actualization.
It refers to the desire for self-fulfillment,
namely, to the tendency for him
to become actualized in what he is potentially.

Abraham H. Maslow
(in "A Theory of Human Motivation")

Um músico deve compor,
um artista deve pintar,
um poeta deve escrever,
caso pretendam ser felizes.
O que um homem pode ser, ele deve ser.

A essa necessidade podemos
dar o nome de autorrealização.
Refere-se ao desejo de autopreenchimento,
isto é, à tendência que ele apresenta
de se tornar, em realidade,
o que já é em potencial.

Abraham H. Maslow
(em "Uma Teoria da Motivação Humana")

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17 – General Solution for Navier-Stokes Equations with Conservative External Force

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Abstract – We present two proofs of theorems on solutions of the Navier-Stokes equations for incompressible case with a conservative external force in $n = 3$ spatial dimensions. Without major difficulties, it can be adapted to any spatial dimension, $n \geq 1$.

Keywords – Navier-Stokes equations, velocity, pressure, Eulerian description, formulation, conservative external force, equivalent equations, exact solutions, existence, inexistence, Cauchy, irrotational, potential flow, Bernoulli's law, Lagrange's theorem.

We find previously[1] a general solution for Navier-Stokes Equations, supposing that there is a solution for initial instant $t = 0$ and applying an additional initial condition $\frac{\partial u_i(X,t)}{\partial t}|_{t=0} = \sum_{j=1}^{3} \frac{\partial u_i^0(X)}{\partial x_j}$, $1 \leq i \leq 3$, in the case on what the external force is zero. We will now generalize that solution to the case where there is a conservative external force, $f = \nabla U$, being applied in the fluid, for example, gravity. The problem is resolved dividing the original pressure in two parts, $p = p_f + p_u$, one of them (p_f) depending exclusively of the potential of f and another (p_u) as the obtained previously, depending exclusively of the velocity u (and therefore u^0). The influence of the conservative external force is only change the total pressure, without influence in the velocity, as happens in the Bernoulli´s law.

Firstly, we will prove theorems without external force, using $p = p_u, p_f = 0$, the identical proofs of [1].

Let $u^0(x, y, z)$ and $p^0(x, y, z)$ be respectively the initial velocity and initial pressure of the three-dimensional incompressible $(\nabla \cdot u = \nabla \cdot u^0 = 0)$ Navier-Stokes equations without external force and with mass density equal to 1,

$$(1) \qquad \frac{\partial p(X,t)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t} + \sum_{j=1}^{3} u_j(X,t) \frac{\partial u_i(X,t)}{\partial x_j} = \nu \nabla^2 u_i(X,t),$$

$1 \leq i \leq 3,\ X = (x_1, x_2, x_3) \in \mathbb{R}^3,\ x_1 \equiv x,\ x_2 \equiv y,\ x_3 \equiv z,\ x_i, t \in \mathbb{R},\ t \geq 0.$

Then in $t = 0$ is valid, for each integer i belongs to $1 \leq i \leq 3$,

(2) $$\frac{\partial p^0(X)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t}|_{t=0} + \sum_{j=1}^{3} u_j^0(X)\frac{\partial u_i^0(X)}{\partial x_j} = \nu\nabla^2 u_i^0(X).$$

Supposing that $u(x,y,z,t) = u^0(x+t, y+t, z+t)$ and $p(x,y,z,t) = p^0(x+t, y+t, z+t)$ is a solution (u,p) for (1), we have

(3) $$\frac{\partial p^0(\xi)}{\partial x_i} + \frac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^{3} u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial x_j} = \nu\nabla^2 u_i^0(\xi),$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ and $\xi_i = \xi_i(X,t) = x_i + t$, $1 \le i \le 3$.

For $t = 0$ the equations (2) and (3) are equals, because in $t = 0$ we have $\xi_i = x_i$ and therefore $\xi = (\xi_1, \xi_2, \xi_3) = (x_1, x_2, x_3) = X$.

For $t > 0$, if (2) is valid for any $X = (x,y,z) \in \mathbb{R}^3$ then (3) is valid for any $\xi \in \mathbb{R}^3$ substituting $x \mapsto \xi_1 = x + t$, $y \mapsto \xi_2 = y + t$, $z \mapsto \xi_3 = z + t$, $x, y, z \in \mathbb{R}, t \ge 0$, so $u(x,y,z,t) = u^0(x+t, y+t, z+t)$ and $p(x,y,z,t) = p^0(x+t, y+t, z+t)$, i.e., $u(X,t) = u^0(\xi)$ and $p(X,t) = p^0(\xi)$, solve equation (3) and therefore the Navier-Stokes equation (1).

The initial motivation to prove it is as follows. Let $A(x), B(x), C(x)$ and $D(x)$ functions such that is always valid, for any $x \in \mathbb{R}$, the relation

(4) $$A(x) + B(x) + C(x) = D(x).$$

Then, as $(x+t) \in \mathbb{R}, x, t \in \mathbb{R}, t \ge 0$, need be valid too the relation

(5) $$A(x+t) + B(x+t) + C(x+t) = D(x+t).$$

The same argument can be used for functions of two and three spatial dimensions (or better, for n spatial dimensions), for example, $\forall x, y, z, t \in \mathbb{R}, t \ge 0$,

(6) $$\begin{aligned} &A_i(x,y,z) + B_i(x,y,z) + C_i(x,y,z) = D_i(x,y,z) \\ &\Longrightarrow A_i(x+t,\ y+t,\ z+t) + B_i(x+t,\ y+t,\ z+t) + \\ &\quad + C_i(x+t,\ y+t,\ z+t) = D_i(x+t,\ y+t,\ z+t). \end{aligned}$$

Applying the previous relation (6) to the Navier-Stokes equations (2) for $t = 0$, if

(7.1) $$A_i(x,y,z) = \frac{\partial p^0(X)}{\partial x_i},$$

(7.2) $$B_i(x,y,z) = \frac{\partial u_i(X,t)}{\partial t}|_{t=0},$$

(7.3) $$C_i(x,y,z) = \sum_{j=1}^{3} u_j^0(X)\frac{\partial u_i^0(X)}{\partial x_j},$$

(7.4) $\qquad D_i(x,y,z) = \nu \nabla^2 u_i^0(X),$

(7.5) $\qquad A_i(x,y,z) + B_i(x,y,z) + C_i(x,y,z) = D_i(x,y,z),$

$X = (x,y,z)$, then, using $\xi = \xi(X,t) = (x+t, y+t, z+t)$, need be valid too the equalities

(8.1) $\qquad A_i(x+t,\ y+t,\ z+t) = \frac{\partial p^0(\xi)}{\partial x_i},$

(8.2) $\qquad B_i(x+t,\ y+t,\ z+t) = (\frac{\partial u_i(X,t)}{\partial t}|_{t=0})(\xi),$

(8.3) $\qquad C_i(x+t,\ y+t,\ z+t) = \sum_{j=1}^3 u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial x_j},$

(8.4) $\qquad D_i(x+t,\ y+t,\ z+t) = \nu \nabla^2 u_i^0(\xi),$

(8.5) $\qquad A_i(x+t,\ y+t,\ z+t) + B_i(x+t,\ y+t,\ z+t) +$
$\qquad + C_i(x+t,\ y+t,\ z+t) = D_i(x+t,\ y+t,\ z+t).$

The expression $(\frac{\partial u_i(X,t)}{\partial t}|_{t=0})(\xi)$ in (8.2) means that first is calculated the value of $\frac{\partial u_i(X,t)}{\partial t}$, next we assign the value $t = 0$ in this result and then we substitute $x \mapsto \xi_1 = x+t,\ y \mapsto \xi_2 = y+t,\ z \mapsto \xi_3 = z+t$, i.e., $X \mapsto \xi$.

Note that the right side of the relations (8.1) to (8.4) corresponds to each parcel of the Navier-Stokes equations (8.5) with the solution (u,p) such that

(9.1) $\qquad u(X,t) = u^0(\xi),$
(9.2) $\qquad p(X,t) = p^0(\xi),$

$X = (x,y,z)$, $\xi = \xi(X,t) = (x+t,\ y+t,\ z+t)$, then (9) is a solution for (1) if $u^0(X)$ and $p^0(X)$ are initial conditions.

We will now prove that if the variables (9.1) and (9.2) solve (1) for $t \geq 0$ then $u^0(x,y,z)$ and $p^0(x,y,z)$ solve (1) for $t = 0$, i.e., then both $u^0(x,y,z)$ and $p^0(x,y,z)$ solve (2). This is an important result of this paper. We'll use the chain rule[2].

Proof: Starting from (1), the three-dimensional incompressible Navier-Stokes equations, where $\nabla \cdot u = \nabla \cdot u^0 = 0$,

(10) $\qquad \frac{\partial p(X,t)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t} + \sum_{j=1}^3 u_j(X,t) \frac{\partial u_i(X,t)}{\partial x_j} = \nu \nabla^2 u_i(X,t),$

$1 \leq i \leq 3,\ X = (x,y,z)$, if a solution (u,p) for them is (9), i.e.,

(11.1) $u(X,t) = u^0(\xi)$,
(11.2) $p(X,t) = p^0(\xi)$,

$\xi = \xi(X,t) = (x+t,\ y+t,\ z+t)$, then we have, according (3),

$$\frac{\partial p^0(\xi)}{\partial x_i} + \frac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^{3} u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial x_j} = \nu\nabla^2 u_i^0(\xi). \tag{12}$$

How $\xi_i = x_i + t$ then $\frac{\partial \xi_i}{\partial x_i} = \frac{\partial \xi_i}{\partial t} = 1$ and $\frac{\partial \xi_i}{\partial x_j} = 0$ if $i \neq j$, so using the chain rule[1] we have, for each parcel in (10) and (12),

$$\frac{\partial p(X,t)}{\partial x_i} = \frac{\partial p^0(\xi)}{\partial x_i} = \sum_{j=1}^{3}\frac{\partial p^0(\xi)}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_i} = \frac{\partial p^0(\xi)}{\partial \xi_i} \tag{13.1}$$

$$\frac{\partial u_i(X,t)}{\partial t} = \frac{\partial u_i^0(\xi)}{\partial t} = \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j}\frac{\partial \xi_j}{\partial t} = \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j} \tag{13.2}$$

$$\begin{aligned} u_j(X,t)\frac{\partial u_i(X,t)}{\partial x_j} &= u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial x_j} = u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_j} = \\ &= u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial \xi_j} \end{aligned} \tag{13.3}$$

$$\begin{aligned} \nabla^2 u_i(X,t) &= \nabla^2 u_i^0(\xi) = \left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\frac{\partial}{\partial x_3}\right)u_i^0(\xi) = \\ &= \sum_{j=1}^{3}\left(\frac{\partial}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_j}\frac{\partial}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_j}\right)u_i^0(\xi) = \sum_{j=1}^{3}\left(\frac{\partial}{\partial \xi_j}\frac{\partial}{\partial \xi_j}\right)u_i^0(\xi) = \\ &= \nabla_\xi^2 u_i^0(\xi) \end{aligned} \tag{13.4}$$

Adding the parcels (13), with (13.3) for each integer $j = 1,2,3$ and the multiplication of (13.4) by viscosity coefficient ν, we come to

$$\frac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j} + \sum_{j=1}^{3} u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial \xi_j} = \nu\nabla_\xi^2 u_i^0(\xi), \tag{14}$$

which is equivalent to previous Navier-Stokes equations (10) and (12) with the solution (11), although it is not a conventional Navier-Stokes equation because the time derivative disappears, i.e.,

$$\frac{\partial u_i(X,t)}{\partial t} \mapsto \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j}. \tag{15}$$

Note that the right side of (15) is not $\frac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j}$, because here u_i^0 is, initially, a function only of $\xi = (\xi_1, \xi_2, \xi_3)$, not including t, but each ξ_i is a function of t and for this reason here is $\frac{\partial u_i(X,t)}{\partial t} = \frac{\partial u_i^0(\xi)}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} \frac{\partial \xi_j}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j}$, with $\xi_j = x_j + t$, $\frac{\partial \xi_j}{\partial t} = 1$.

In $t = 0$, when $\xi_i = x_i$, the equation (14) became

(16) $$\frac{\partial p^0(X)}{\partial x_i} + \sum_{j=1}^{3} \frac{\partial u_i^0(X)}{\partial x_j} + \sum_{j=1}^{3} u_j^0(X) \frac{\partial u_i^0(X)}{\partial x_j} = \nu \nabla^2 u_i^0(X).$$

If this equation is equivalent to (2) then

(17) $$\frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0} = \sum_{j=1}^{3} \frac{\partial u_i^0(X)}{\partial x_j},$$

which is thereby a good manner of define or choose the temporal derivative of velocity at $t = 0$ when the solution for velocity is $u(X,t) = u^0(\xi)$.

Similarly, for $t > 0$ we have

(18) $$\frac{\partial u_i(X,t)}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j},$$

$X = (x, y, z)$, $\xi = (\xi_1, \xi_2, \xi_3)$, $\xi_i = \xi_i(X,t) = x_i + t$, $1 \le i \le 3$.

Concluding, assuming that (9), identical to (11), is a solution for (1), identical to (10), we come to (16) for $t = 0$, which is equivalent to (2) with the additional initial condition (17) and it has a solution $(u^0(X), p^0(X))$. This is what we wanted to prove. □

Next, we will prove the opposite way of the previous demonstration: if $u^0(x,y,z)$ and $p^0(x,y,z)$ solve (1) for $t = 0$, i.e., if both $u^0(x,y,z)$ and $p^0(x,y,z)$ solve (2), then the variables (u,p) given in (9.1) and (9.2) solve (1) for $t \ge 0$. This is the fundamental result of this paper. The proof basically follows what we write from beginning of this paper until the equations (9), increasing the transformations (13) and the conditions (17) and (18). We'll use the chain rule[2] again.

Proof: If $u^0(x,y,z)$ and $p^0(x,y,z)$ solve the three-dimensional incompressible ($\nabla \cdot u = \nabla \cdot u^0 = 0$) Navier-Stokes equations

(19) $$\frac{\partial p(X,t)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t} + \sum_{j=1}^{3} u_j(X,t) \frac{\partial u_i(X,t)}{\partial x_j} = \nu \nabla^2 u_i(X,t)$$

for $t = 0$, with $1 \le i \le 3$, $X = (x_1, x_2, x_3) \in \mathbb{R}^3$, $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$, $x_i, t \in \mathbb{R}$, $t \ge 0$, then in $t = 0$ is valid, for each integer i belongs to $1 \le i \le 3$,

(20) $$\frac{\partial p^0(X)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t}|_{t=0} + \sum_{j=1}^{3} u_j^0(X) \frac{\partial u_i^0(X)}{\partial x_j} = \nu \nabla^2 u_i^0(X).$$

Supposing that $u(x,y,z,t) = u^0(x+t,y+t,z+t)$ and $p(x,y,z,t) = p^0(x+t,y+t,z+t)$ is a solution (u,p) for (19), we have

(21) $$\frac{\partial p^0(\xi)}{\partial x_i} + \frac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^{3} u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial x_j} = \nu \nabla^2 u_i^0(\xi),$$

using $\xi = (\xi_1, \xi_2, \xi_3)$ and $\xi_i = \xi_i(X,t) = x_i + t$, $1 \le i \le 3$.

For $t = 0$ the equations (20) and (21) are equals, because in $t = 0$ we have $\xi_i = x_i$ and therefore $\xi = (\xi_1,\xi_2,\xi_3) = (x_1,x_2,x_3) = X$.

For $t > 0$, if (20) is valid for any $X = (x,y,z) \in \mathbb{R}^3$ then (21) is valid for any $\xi \in \mathbb{R}^3$ substituting $x \mapsto \xi_1 = x+t,\ y \mapsto \xi_2 = y+t,\ z \mapsto \xi_3 = z+t,\ x,y,z \in \mathbb{R}, t \ge 0$, according transformations (22) below, so $u(x,y,z,t) = u^0(x+t,y+t,z+t)$ and $p(x,y,z,t) = p^0(x+t,y+t,z+t)$, i.e., $u(X,t) = u^0(\xi)$ and $p(X,t) = p^0(\xi)$, solve equation (21) and therefore the Navier-Stokes equation (19).

How $\xi_i = x_i + t$ then $\frac{\partial \xi_i}{\partial x_i} = \frac{\partial \xi_i}{\partial t} = 1$ and $\frac{\partial \xi_i}{\partial x_j} = 0$ if $i \ne j$, so using the chain rule[2] we have, for each parcel in (21),

(22.1) $$\frac{\partial p^0(\xi)}{\partial x_i} = \frac{\partial p(\xi(X,t))}{\partial x_i} = \sum_{j=1}^{3} \frac{\partial p^0(\xi)}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_i} = \frac{\partial p^0(\xi)}{\partial \xi_i}$$

(22.2) $$\frac{\partial u_i^0(\xi)}{\partial t} = \frac{\partial u_i(\xi(X,t))}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j}\frac{\partial \xi_j}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j}$$

(22.3) $$u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial x_j} = u_j(\xi(X,t))\frac{\partial u_i(\xi(X,t))}{\partial x_j} = u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_j} =$$
$$= u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial \xi_j}$$

(22.4) $$\nabla^2 u_i^0(\xi) = \nabla^2 u_i(\xi(X,t)) = \sum_{j=1}^{3}\left(\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}\right) u_i^0(\xi(X,t)) =$$
$$= \sum_{j=1}^{3}\left(\frac{\partial}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_j}\frac{\partial}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_j}\right) u_i^0(\xi) = \sum_{j=1}^{3}\left(\frac{\partial}{\partial \xi_j}\frac{\partial}{\partial \xi_j}\right) u_i^0(\xi) =$$
$$= \nabla_\xi^2 u_i^0(\xi)$$

The equation (21) transformed through by (22) gives

(23) $$\frac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j} + \sum_{j=1}^{3} u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial \xi_j} = \nu \nabla_\xi^2 u_i^0(\xi),$$

that is, we transform $X \longmapsto \xi$ and from $\xi_i = x_i + t$ we have $\frac{\partial \xi_i}{\partial x_i} = 1$ and therefore $\partial x_i = \partial \xi_i$.

The unexpected transformation is

(24) $$\frac{\partial u_i^0(\xi)}{\partial t} = \frac{\partial u_i(\xi(X,t))}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j},$$

making (23) not be in the form of a standard Navier-Stokes equation. In $t = 0$ the transformation (24) becomes

(25) $$\frac{\partial u_i^0(\xi)}{\partial t}|_{t=0} = \frac{\partial u_i(\xi(X,t))}{\partial t}|_{t=0} = \frac{\partial u_i(X,t)}{\partial t}|_{t=0} = \sum_{j=1}^{3} \frac{\partial u_i^0(X)}{\partial x_j},$$

$\xi_j = x_j,\ \xi = X$, for $t = 0$, thus we need to assume the additional initial condition

(26) $$\frac{\partial u_i(X,t)}{\partial t}|_{t=0} = \sum_{j=1}^{3} \frac{\partial u_i^0(X)}{\partial x_j}$$

when the solution for Navier-Stokes equation (1), identical to (19), is given by (9), i.e.,

(27.1) $u(X,t) = u^0(\xi)$,
(27.2) $p(X,t) = p^0(\xi)$,

$X = (x, y, z),\ \xi = \xi(X,t) = (x + t,\ y + t,\ z + t)$.

Concluding, if $(u^0(X), p^0(X))$ solve (2), identical to (20), substituting in (20) the transformation $X \mapsto \xi,\ X = (x, y, z),\ \xi = (\xi_1, \xi_2, \xi_3),\ \xi_i = x_i + t$, we come to (23),

(28) $$\frac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} + \sum_{j=1}^{3} u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial \xi_j} = \nu \nabla_\xi^2 u_i^0(\xi),$$

assuming the additional initial condition (26)

(29) $$\frac{\partial u_i(X,t)}{\partial t}|_{t=0} = \sum_{j=1}^{3} \frac{\partial u_i^0(X)}{\partial x_j}$$

due to transformation (24),

(30) $$\frac{\partial u_i^0(\xi)}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j}.$$

Using (30) in (28) we come to

(31) $$\frac{\partial p^0(\xi)}{\partial \xi_i} + \frac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^{3} u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial \xi_j} = \nu \nabla_\xi^2 u_i^0(\xi),$$

the Navier-Stokes equations with the solution $(u^0(\xi), p^0(\xi))$, i.e., $(u(X,t), p(X,t))$, according (27), identical to (9).

Using (27) and $\partial\xi_i = \partial x_i$ in (31) we come finally to

(32) $$\frac{\partial p(X,t)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t} + \sum_{j=1}^{3} u_j(X,t)\frac{\partial u_i(X,t)}{\partial x_j} = \nu \nabla_X^2 u_i(X,t),$$

the Navier-Stokes equations (1) with the solution $(u(X,t), p(X,t))$. This is what we wanted to prove. □

What we see in the two previous proofs can be applied, with the obvious adaptations, to solutions of the form

(33.1) $u(X,t) = u^0(\xi),$

(33.2) $p(X,t) = p^0(\xi),$

$X = (x,y,z),\ \xi = (\xi_1, \xi_2, \xi_3),\ \xi_i = x_i + T_i(t),\ T_i(0) = 0,\ 1 \le i \le 3,$

with the conditions

(34) $$\frac{\partial u_i(X,t)}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j}\frac{\partial \xi_j}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} T_j'(t),$$

and

(35) $$\frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} T_j'(0) = \sum_{j=1}^{3} \frac{\partial u_i^0(X)}{\partial x_j} T_j'(0),$$

being the functions $T_i(t)$ differentiable of class $C^1([0,\infty))$. In this case the equations (23) and (28) are

(36) $$\frac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} T_j'(t) + \sum_{j=1}^{3} u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial \xi_j} =$$

$$= \frac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j}\left[T_j'(t) + u_j^0(\xi)\right] = \nu \nabla_\xi^2 u_i^0(\xi).$$

Note that the equation (34) implies

(37) $$u_i(X,t) = u_i^0(X) + \int_0^t \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} T_j'(t)\,dt =$$

$$= u_i^0(\xi_1, \xi_2, \xi_3) = u_i^0\big(x_1 + T_1(t),\ x_2 + T_2(t),\ x_3 + T_3(t)\big),$$

that must be true for all differentiable function $u_i^0(\xi)$ with $\xi_i = x_i + T_i(t)$, $T_i(t)$ differentiable, $T_i(0) = 0,\ 1 \le i \le 3$.

Also it is not hard see that, without major difficulties, it can be adapted to any integer spatial dimension, $n >= 1$.

Including in the system a conservative external force $f = (f_1, f_2, f_3)$ whose potential is U, $f = \nabla U$, we can separate the total pressure p in two parts, p_f and p_u, such that $p = p_f + p_u$. In this case, the more complete equations for incompressible Navier-Stokes equations are, for $1 \leq i \leq 3$,

(38) $$\frac{\partial p(X,t)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t} + \sum_{j=1}^{3} u_j(X,t) \frac{\partial u_i(X,t)}{\partial x_j} = \nu \nabla^2 u_i(X,t) + f_i,$$

with

(39) $$\nabla \cdot u = \nabla \cdot u^0 = 0.$$

Defining

(40) $$p(X,t) = p_f(X,t) + p_u(X,t)$$

and the respective initial pressures

(41) $$p^0(X) = p_f^0(X) + p_u^0(X),$$

the obtained results in equations (1) and (2) for the pressure without external force will be attributed to p_u and p_u^0, respectively, while $p_f(X,t)$ is equal to force-potential U, i.e.,

(42.1) $$\nabla p_f = f = \nabla U$$

(42.2) $$p_f = U + \theta_f(t),$$

$\theta_f(t)$ a generic physically and mathematically reasonable function of time, as we already know.

Of this manner, the introduction of an external force do not change the velocity, but only the total pressure, such that

(43) $$p = p_f + p_u.$$

Then, the velocity can be calculated without the use of external force, in case of a conservative external force $f = \nabla U$.

It is clear that in the Eulerian description[3] the computational and analytical challenges will be, more than solving the Navier-Stokes equations for $t > 0$, solve these equations for $t = 0$, the initial instant. Unfortunately, it is not for all pair of values (u^0, p^0) that exists solution to the equation (28) and related equations, so or u^0 is a function of p^0, or p^0 is a function of u^0, or both u^0 and p^0 are functions of another functions, for example, a potential function ϕ such that $u^0 = \nabla\phi(t = 0)$, $u = \nabla\phi$, resulting in the known Bernoulli's law.

It is convenient say that Cauchy[4] in his memorable and admirable *Mémoire sur la Théorie des Ondes*, winner of the Mathematical Analysis award, year 1815, firstly does a study on the equations to be obeyed by three-dimensional molecules in a homogeneous fluid in the initial instant $t = 0$, coming to the conclusion which the initial velocity must be irrotational, i.e., a potential flow. Of this manner, after, he comes to conclusion that the velocity is always irrotational, potential flow, if the external force is conservative, which is the Lagrange's theorem (a possible exception occurs if one or two components of velocity are identically zero, when the reasonings on 3-D molecular volume are not valid). The solution obtained by Cauchy for Euler's equations is the Bernoulli's law, as almost always happens.

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18 – General Solution For Navier-Stokes Equations With Any Smooth Initial Data

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Abstract – We present a solution for the Navier-Stokes equations for incompressible case with any smooth (C^∞) initial velocity given a pressure and external force in $n = 3$ spatial dimensions, based on expansion in Taylor's series of time. Without major difficulties, it can be adapted to any spatial dimension, $n \geq 1$.

Keywords – Lagrange, Mécanique Analitique, exact differential, Euler's equations, Navier-Stokes equations, Taylor's series, Cauchy, Mémoire sur la Théorie des Ondes, Lagrange's theorem, Bernoulli's law.

Let p, q, r be the three components of velocity of an element of fluid in the 3-D orthogonal Euclidean system of spatial coordinates (x, y, z) and t the time in this system.

Lagrange in his *Mécanique Analitique*, firstly published in 1788, proved that if the quantity $(p\,dx + q\,dy + r\,dz)$ is an exact differential when $t = 0$ it will also be an exact differential when t has any other value. If the quantity $(p\,dx + q\,dy + r\,dz)$ is an exact differential at an arbitrary instant, it should be such for all other instants. Consequently, if there is one instant during the motion for which it is not an exact differential, it cannot be exact for the entire period of motion. If it were exact at another arbitrary instant, it should also be exact at the first instant.[1]

To prove it Lagrange used

$$(1) \qquad \begin{cases} p = p^I + p^{II}t + p^{III}t^2 + p^{IV}t^3 + \cdots \\ q = q^I + q^{II}t + q^{III}t^2 + q^{IV}t^3 + \cdots \\ r = r^I + r^{II}t + r^{III}t^2 + r^{IV}t^3 + \cdots \end{cases}$$

in which the quantities p^I, p^{II}, p^{III}, etc., q^I, q^{II}, q^{III}, etc., r^I, r^{II}, r^{III}, etc., are functions of x, y, z but without t.

Here we will finally solve the equations of Euler and Navier-Stokes using this representation of the velocity components in infinite series, as pointed by Lagrange. We assume satisfied the condition of incompressibility, for brevity. Without it the resulting equations are more complicated, as we know, but the method of solution is essentially the same in both cases.

To facilitate and abbreviate our writing, we represent the fluid velocity by its three components in indicial notation, i.e., $u = (u_1, u_2, u_3)$, as well as the external force will be $f = (f_1, f_2, f_3)$ and the spatial coordinates $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$. The pressure, a scalar function, will be represented as p.

The representation (1) is as the expansion of the velocity in a Taylor´s series in relation to time around $t = 0$, considering x, y, z as constant, i.e., for $1 \leq i \leq 3$,

(2) $$u_i = u_i|_{t=0} + \frac{\partial u_i}{\partial t}|_{t=0}\ t + \frac{\partial^2 u_i}{\partial t^2}|_{t=0}\ \frac{t^2}{2} + \frac{\partial^3 u_i}{\partial t^3}|_{t=0}\ \frac{t^3}{6} + \cdots + \frac{\partial^k u_i}{\partial t^k}|_{t=0}\ \frac{t^k}{k!} + \cdots$$

or

(3) $$u_i = u_i^0 + \sum_{k=1}^{\infty} \frac{\partial^k u_i}{\partial t^k}|_{t=0}\ \frac{t^k}{k!}.$$

For the calculation of $\frac{\partial u_i}{\partial t}, \frac{\partial^2 u_i}{\partial t^2}, \frac{\partial^3 u_i}{\partial t^3}, \ldots$ we use the values that are obtained directly from the Navier-Stokes equations and its derivatives in relation to time, i.e.,

(4) $$\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} + \nu\, \nabla^2 u_i + f_i,$$

and therefore

(5) $$\frac{\partial^2 u_i}{\partial t^2} = -\frac{\partial^2 p}{\partial t\, \partial x_i} - \sum_{j=1}^{3} \left(\frac{\partial u_j}{\partial t} \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} \right) + \nu\, \nabla^2 \frac{\partial u_i}{\partial t} + \frac{\partial f_i}{\partial t},$$

(6) $$\frac{\partial^3 u_i}{\partial t^3} = -\frac{\partial^3 p}{\partial t^2\, \partial x_i} - \sum_{j=1}^{3} \left(\frac{\partial^2 u_j}{\partial t^2} \frac{\partial u_i}{\partial x_j} + 2 \frac{\partial u_j}{\partial t} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} \frac{\partial^2 u_i}{\partial t^2} \right) + \nu\, \nabla^2 \frac{\partial^2 u_i}{\partial t^2} + \frac{\partial^2 f_i}{\partial t^2},$$

(7) $$\frac{\partial^4 u_i}{\partial t^4} = -\frac{\partial^4 p}{\partial t^3\, \partial x_i} - \sum_{j=1}^{3} N_j^3 + \nu\, \nabla^2 \frac{\partial^3 u_i}{\partial t^3} + \frac{\partial^3 f_i}{\partial t^3},$$
$$N_j^3 = \frac{\partial}{\partial t} N_j^2,\ N_j^2 = \frac{\partial^2 u_j}{\partial t^2} \frac{\partial u_i}{\partial x_j} + 2 \frac{\partial u_j}{\partial t} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} \frac{\partial^2 u_i}{\partial t^2},$$
$$N_j^3 = \frac{\partial^3 u_j}{\partial t^3} \frac{\partial u_i}{\partial x_j} + 3 \frac{\partial^2 u_j}{\partial t^2} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} + 3 \frac{\partial u_j}{\partial t} \frac{\partial}{\partial x_j} \frac{\partial^2 u_i}{\partial t^2} + u_j \frac{\partial}{\partial x_j} \frac{\partial^3 u_i}{\partial t^3},$$

(8) $$\frac{\partial^5 u_i}{\partial t^5} = -\frac{\partial^5 p}{\partial t^4\, \partial x_i} - \sum_{j=1}^{3} N_j^4 + \nu\, \nabla^2 \frac{\partial^4 u_i}{\partial t^4} + \frac{\partial^4 f_i}{\partial t^4},$$
$$N_j^4 = \frac{\partial}{\partial t} N_j^3 = \frac{\partial^4 u_j}{\partial t^4} \frac{\partial u_i}{\partial x_j} + 4 \frac{\partial^3 u_j}{\partial t^3} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} + 6 \frac{\partial^2 u_j}{\partial t^2} \frac{\partial}{\partial x_j} \frac{\partial^2 u_i}{\partial t^2} + 4 \frac{\partial u_j}{\partial t} \frac{\partial}{\partial x_j} \frac{\partial^3 u_i}{\partial t^3} + u_j \frac{\partial}{\partial x_j} \frac{\partial^4 u_i}{\partial t^4},$$

and using induction we come to

(9) $$\frac{\partial^k u_i}{\partial t^k} = -\frac{\partial^k p}{\partial t^{k-1}\,\partial x_i} - \sum_{j=1}^{3} N_j^{k-1} + \nu\,\nabla^2 \frac{\partial^{k-1} u_i}{\partial t^{k-1}} + \frac{\partial^{k-1} f_i}{\partial t^{k-1}},$$
$$N_j^{k-1} = \frac{\partial}{\partial t} N_j^{k-2} = \sum_{l=0}^{k-1} \binom{k-1}{l}\, \partial_t^{k-1-l} u_j\, \frac{\partial}{\partial x_j} \partial_t^l u_i,$$
$$\partial_t^0 u_n = u_n,\ \partial_t^m u_n = \frac{\partial^m u_n}{\partial t^m},\ \binom{k-1}{l} = \frac{(k-1)!}{(k-1-l)!\,l!}.$$

In (2) and (3) it is necessary to know the values of the derivatives $\frac{\partial u_i}{\partial t}, \frac{\partial^2 u_i}{\partial t^2}, \dots, \frac{\partial^k u_i}{\partial t^k}$ in $t = 0$ then we must to calculate, from (4) to (9),

(10) $$\frac{\partial u_i}{\partial t}\Big|_{t=0} = -\frac{\partial p^0}{\partial x_i} - \sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j} + \nu\,\nabla^2 u_i^0 + f_i^0,$$

the superior index 0 meaning the value of the respective function at $t = 0$, and

(11) $$\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0} = -\frac{\partial^2 p}{\partial t\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^{3} N_j^1\Big|_{t=0} +$$
$$+\nu\,\nabla^2 \frac{\partial u_i}{\partial t}\Big|_{t=0} + \frac{\partial f_i}{\partial t}\Big|_{t=0},$$
$$N_j^1\Big|_{t=0} = \sum_{j=1}^{3}\left(\frac{\partial u_j}{\partial t}\Big|_{t=0} \frac{\partial u_i^0}{\partial x_j} + u_j^0 \frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}\Big|_{t=0}\right),$$

(12) $$\frac{\partial^3 u_i}{\partial t^3}\Big|_{t=0} = -\frac{\partial^3 p}{\partial t^2\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^{3} N_j^2\Big|_{t=0} +$$
$$+\nu\,\nabla^2 \frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0} + \frac{\partial^2 f_i}{\partial t^2}\Big|_{t=0},$$
$$N_j^2\Big|_{t=0} = \frac{\partial^2 u_j}{\partial t^2}\Big|_{t=0} \frac{\partial u_i^0}{\partial x_j} + 2\frac{\partial u_j}{\partial t}\Big|_{t=0} \frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}\Big|_{t=0} +$$
$$+ u_j^0 \frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0},$$

(13) $$\frac{\partial^4 u_i}{\partial t^4}\Big|_{t=0} = -\frac{\partial^4 p}{\partial t^3\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^{3} N_j^3\Big|_{t=0} +$$
$$+\nu\,\nabla^2 \frac{\partial^3 u_i}{\partial t^3}\Big|_{t=0} + \frac{\partial^3 f_i}{\partial t^3}\Big|_{t=0},$$
$$N_j^3\Big|_{t=0} = \frac{\partial^3 u_j}{\partial t^3}\Big|_{t=0} \frac{\partial u_i^0}{\partial x_j} + 3\frac{\partial^2 u_j}{\partial t^2}\Big|_{t=0} \frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}\Big|_{t=0} +$$
$$+ 3\frac{\partial u_j}{\partial t}\Big|_{t=0} \frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0} + u_j^0 \frac{\partial}{\partial x_j}\frac{\partial^3 u_i}{\partial t^3}\Big|_{t=0},$$

(14) $$\frac{\partial^5 u_i}{\partial t^5}\Big|_{t=0} = -\frac{\partial^5 p}{\partial t^4\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^{3} N_j^4\Big|_{t=0} +$$
$$+\nu\,\nabla^2 \frac{\partial^4 u_i}{\partial t^4}\Big|_{t=0} + \frac{\partial^4 f_i}{\partial t^4}\Big|_{t=0},$$

$$N_j^4|_{t=0} = \frac{\partial^4 u_j}{\partial t^4}|_{t=0} \frac{\partial u_i^0}{\partial x_j} + 4\frac{\partial^3 u_j}{\partial t^3}|_{t=0} \frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t} +$$
$$+ 6\frac{\partial^2 u_j}{\partial t^2}|_{t=0} \frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2}|_{t=0} + 4\frac{\partial u_j}{\partial t}|_{t=0} \frac{\partial}{\partial x_j}\frac{\partial^3 u_i}{\partial t^3}|_{t=0} +$$
$$+ u_j^0 \frac{\partial}{\partial x_j}\frac{\partial^4 u_i}{\partial t^4}|_{t=0},$$

and of generic form,

(15) $$\frac{\partial^k u_i}{\partial t^k}|_{t=0} = -\frac{\partial^k p}{\partial t^{k-1}\,\partial x_i}|_{t=0} - \sum_{j=1}^3 N_j^{k-1}|_{t=0} +$$
$$+ \nu\,\nabla^2 \frac{\partial^{k-1} u_i}{\partial t^{k-1}}|_{t=0} + \frac{\partial^{k-1} f_i}{\partial t^{k-1}}|_{t=0},$$
$$N_j^{k-1}|_{t=0} = \sum_{l=0}^{k-1} \binom{k-1}{l} \partial_t^{k-1-l} u_j|_{t=0} \frac{\partial}{\partial x_j} \partial_t^l u_i|_{t=0},$$
$$\partial_t^0 u_n|_{t=0} = u_n^0,\ \partial_t^m u_n|_{t=0} = \frac{\partial^m u_n}{\partial t^m}|_{t=0}.$$

If the external force is conservative there is a scalar potential U such as $f = \nabla U$ and the pressure can be calculated from this potential U, i.e.,

(16) $$\frac{\partial p}{\partial x_i} = f_i = \frac{\partial U}{\partial x_i},$$

and then

(17) $$p = U + \theta(t),$$

$\theta(t)$ a generic function of time of class C^∞, so it is not necessary the use of the pressure p and external force f, and respective derivatives, in (4) to (15) if the external force is conservative. In this case, the velocity can be independent of the both pressure and external force, otherwise it will be necessary to use both the pressure and external force derivatives to calculate the velocity in powers of time.

The result that we obtain here in this development in Taylor's series seems to me a great advance in the search of the solutions of the Euler's and Navier-Stokes equations. It is possible now to know on the possibility of non-uniqueness solutions as well as breakdown solution respect to unbounded energy of another manner.

We now can choose previously an infinity of different pressures such that the calculation of $\frac{\partial u}{\partial t}$ and derivatives can be done, for a given initial velocity and external force, although such calculation can be very hard.

It is convenient say that Cauchy[2] in his memorable and admirable *Mémoire sur la Théorie des Ondes*, winner of the Mathematical Analysis award, year 1815, firstly does a study on the equations to be obeyed by three-dimensional molecules in a homogeneous fluid in the initial instant $t = 0$, coming to the conclusion which the initial velocity must be irrotational, i.e., a potential flow. Of this manner, after, he comes to conclusion that the velocity is always irrotational, potential flow, if the external force is conservative, which is essentially the Lagrange's theorem described in the begin of this article, but it is shown without the use of series expansion (a possible exception occurs if one or two components of velocity are identically zero, when the reasonings on 3-D molecular volume are not valid). The solution obtained by Cauchy for Euler's equations is the Bernoulli's law, as almost always happens. Perhaps a solution in Eulerian description not always corresponds to some solution in Lagrangian description, and vice-versa, I yet don't know for sure. There can be no contradiction in science, particularly in mathematics.

I began my study of the Navier-Stokes equations verifying the lack (inexistence, I called *breakdown*) of solutions, but realizing that given the pressure and initial velocity there would be no problem about not being possible to integrate the equations of Navier-Stokes and find the velocity, in general case. Now with more clarity and conviction I realize that, given only the velocity may not be possible to find the corresponding pressure, but given the pressure we can find the velocity, in special using the expansion in Taylor's series, as we see here.

If the mentioned series is divergent may be an indicative of that the correspondent velocity and its square diverge, again going to the case of breakdown solution due to unbounded energy. Without pressure and with initial velocity and external force both belonging to Schwartz Space is expected that the solution for velocity also belonging to Schwartz Space, obtaining physically reasonable and well-behaved solutions throughout the space.

The method presented here can also be applied in other equations, of course, for example in the heat equation. Always will be necessary that the remainder in the Taylor's series goes to zero when the order k of the derivative tends to infinity.[3] Applying this concept in (3) and (9), substituting t by τ, the remainder $R_{i,k}$ of order k for velocity component i is

(18) $$R_{i,k} = \frac{1}{k!}\int_0^t (t-\tau)^k \frac{\partial^{k+1} u_i}{\partial t^{k+1}} d\tau,$$

which can be estimated by Lagrange's remainder,

(19) $$R_{i,k} = \frac{t^{k+1}}{(k+1)!} \frac{\partial^{k+1} u_i}{\partial t^{k+1}}(\xi),$$

or by Cauchy's remainder,

(20) $$R_{i,k} = \frac{t^{k+1}}{k!}(1-\theta)^k \frac{\partial^{k+1} u_i}{\partial t^{k+1}}(\xi),$$

with $0 \leq \xi \leq t$ and $0 \leq \theta \leq 1$.

To Jean-Christophe Yoccoz, *in memorian*. I have just know of his premature death, great friend of mathematicians of IMPA. I'm not one successful, I do not have fame, I did not win any awards. In common we have only a great love for mathematics. He was a genius man who now leaves the Earth, but I know that even in heaven there are math and science to be done. He did an excellent job.

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19 – Solutions for Euler and Navier-Stokes Equations in Powers of Time

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Abstract – We present a solution for the Euler and Navier-Stokes equations for incompressible case given any smooth (C^∞) initial velocity, pressure and external force in $N = 3$ spatial dimensions, based on expansion in Taylor's series of time. Without major difficulties, it can be adapted to any spatial dimension, $N \geq 1$.

Keywords – Lagrange, Mécanique Analitique, exact differential, Euler's equations, Navier-Stokes equations, Taylor's series, Cauchy, Mémoire sur la Théorie des Ondes, Lagrange's theorem, Bernoulli's law, non-uniqueness solutions.

§ 1

Let p, q, r be the three components of velocity of an element of fluid in the 3-D orthogonal Euclidean system of spatial coordinates (x, y, z) and t the time in this system.

Lagrange in his *Mécanique Analitique*, firstly published in 1788, proved that if the quantity $(p\,dx + q\,dy + r\,dz)$ is an exact differential when $t = 0$ it will also be an exact differential when t has any other value. If the quantity $(p\,dx + q\,dy + r\,dz)$ is an exact differential at an arbitrary instant, it should be such for all other instants. Consequently, if there is one instant during the motion for which it is not an exact differential, it cannot be exact for the entire period of motion. If it were exact at another arbitrary instant, it should also be exact at the first instant.[1]

To prove it Lagrange used

$$(1.1) \qquad \begin{cases} p = p^I + p^{II}t + p^{III}t^2 + p^{IV}t^3 + \cdots \\ q = q^I + q^{II}t + q^{III}t^2 + q^{IV}t^3 + \cdots \\ r = r^I + r^{II}t + r^{III}t^2 + r^{IV}t^3 + \cdots \end{cases}$$

in which the quantities p^I, p^{II}, p^{III}, etc., q^I, q^{II}, q^{III}, etc., r^I, r^{II}, r^{III}, etc., are functions of x, y, z but without t.

Here we will finally solve the equations of Euler and Navier-Stokes using this representation of the velocity components in infinite series, as pointed by Lagrange. We assume satisfied the condition of incompressibility, for brevity. Without it the resulting equations are more complicated, as we know, but the method of solution is essentially the same in both cases. We focus our attention in the general case of the Navier-Stokes equations, and for the Euler equations simply set the viscosity coefficient as $\nu = 0$.

To facilitate and abbreviate our writing, we represent the fluid velocity by its three components in indicial notation, i.e., $u = (u_1, u_2, u_3)$, as well as the external force will be $f = (f_1, f_2, f_3)$ and the spatial coordinates $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$. The pressure, a scalar function, will be represented as p. As frequently used in mathematics approach, the density mass will be $\rho = 1$ (otherwise substitute p by p/ρ and ν by ν/ρ). We consider all functions belonging to C^∞, being valid the use of $\frac{\partial^2 u_i}{\partial x_j \partial x_k} = \frac{\partial^2 u_i}{\partial x_k \partial x_j}$ and other inversions in order of derivatives, so much in relation to space as to time.

The representation (1.1) is as the expansion of the velocity in a Taylor´s series in relation to time around $t = 0$, considering x, y, z as constant, i.e., for $1 \le i \le 3$,

(1.2) $$u_i = u_i|_{t=0} + \frac{\partial u_i}{\partial t}|_{t=0}\, t + \frac{\partial^2 u_i}{\partial t^2}|_{t=0}\, \frac{t^2}{2} + \frac{\partial^3 u_i}{\partial t^3}|_{t=0}\, \frac{t^3}{6} + \cdots + \frac{\partial^k u_i}{\partial t^k}|_{t=0}\, \frac{t^k}{k!} + \cdots$$

or

(1.3) $$u_i = u_i^0 + \sum_{k=1}^{\infty} \frac{\partial^k u_i}{\partial t^k}|_{t=0}\, \frac{t^k}{k!}.$$

For the calculation of $\frac{\partial u_i}{\partial t}, \frac{\partial^2 u_i}{\partial t^2}, \frac{\partial^3 u_i}{\partial t^3}, \ldots$ we use the values that are obtained directly from the Navier-Stokes equations and its derivatives in relation to time, i.e.,

(1.4) $$\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} + \nu\, \nabla^2 u_i + f_i,$$

and therefore

(1.5) $$\frac{\partial^2 u_i}{\partial t^2} = -\frac{\partial^2 p}{\partial t\, \partial x_i} - \sum_{j=1}^{3} \left(\frac{\partial u_j}{\partial t} \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} \right) + \nu\, \nabla^2 \frac{\partial u_i}{\partial t} + \frac{\partial f_i}{\partial t},$$

(1.6) $$\frac{\partial^3 u_i}{\partial t^3} = -\frac{\partial^3 p}{\partial t^2\, \partial x_i} - \sum_{j=1}^{3} \left(\frac{\partial^2 u_j}{\partial t^2} \frac{\partial u_i}{\partial x_j} + 2 \frac{\partial u_j}{\partial t} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} \frac{\partial^2 u_i}{\partial t^2} \right) + \nu\, \nabla^2 \frac{\partial^2 u_i}{\partial t^2} + \frac{\partial^2 f_i}{\partial t^2},$$

(1.7) $$\frac{\partial^4 u_i}{\partial t^4} = -\frac{\partial^4 p}{\partial t^3\, \partial x_i} - \sum_{j=1}^{3} N_j^3 + \nu\, \nabla^2 \frac{\partial^3 u_i}{\partial t^3} + \frac{\partial^3 f_i}{\partial t^3},$$
$$N_j^3 = \frac{\partial}{\partial t} N_j^2,\ N_j^2 = \frac{\partial^2 u_j}{\partial t^2} \frac{\partial u_i}{\partial x_j} + 2 \frac{\partial u_j}{\partial t} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} \frac{\partial^2 u_i}{\partial t^2},$$
$$N_j^3 = \frac{\partial^3 u_j}{\partial t^3} \frac{\partial u_i}{\partial x_j} + 3 \frac{\partial^2 u_j}{\partial t^2} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} + 3 \frac{\partial u_j}{\partial t} \frac{\partial}{\partial x_j} \frac{\partial^2 u_i}{\partial t^2} + u_j \frac{\partial}{\partial x_j} \frac{\partial^3 u_i}{\partial t^3},$$

(1.8) $$\frac{\partial^5 u_i}{\partial t^5} = -\frac{\partial^5 p}{\partial t^4\,\partial x_i} - \sum_{j=1}^{3} N_j^4 + \nu\,\nabla^2 \frac{\partial^4 u_i}{\partial t^4} + \frac{\partial^4 f_i}{\partial t^4},$$
$$N_j^4 = \frac{\partial}{\partial t} N_j^3 = \frac{\partial^4 u_j}{\partial t^4}\frac{\partial u_i}{\partial x_j} + 4\frac{\partial^3 u_j}{\partial t^3}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t} + 6\frac{\partial^2 u_j}{\partial t^2}\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2} +$$
$$+4\frac{\partial u_j}{\partial t}\frac{\partial}{\partial x_j}\frac{\partial^3 u_i}{\partial t^3} + u_j\frac{\partial}{\partial x_j}\frac{\partial^4 u_i}{\partial t^4},$$

and using induction we come to

(1.9) $$\frac{\partial^k u_i}{\partial t^k} = -\frac{\partial^k p}{\partial t^{k-1}\,\partial x_i} - \sum_{j=1}^{3} N_j^{k-1} + \nu\,\nabla^2 \frac{\partial^{k-1} u_i}{\partial t^{k-1}} + \frac{\partial^{k-1} f_i}{\partial t^{k-1}},$$
$$N_j^{k-1} = \frac{\partial}{\partial t} N_j^{k-2} = \sum_{l=0}^{k-1} \binom{k-1}{l} \partial_t^{k-1-l} u_j \,\frac{\partial}{\partial x_j}\partial_t^l u_i,$$
$$\partial_t^0 u_n = u_n,\ \partial_t^m u_n = \frac{\partial^m u_n}{\partial t^m},\ \binom{k-1}{l} = \frac{(k-1)!}{(k-1-l)!\,l!}.$$

In (1.2) and (1.3) it is necessary to know the values of the derivatives $\frac{\partial u_i}{\partial t}, \frac{\partial^2 u_i}{\partial t^2}, \ldots, \frac{\partial^k u_i}{\partial t^k}$ in $t = 0$ then we must to calculate, from (1.4) to (1.9),

(1.10) $$\frac{\partial u_i}{\partial t}\Big|_{t=0} = -\frac{\partial p^0}{\partial x_i} - \sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j} + \nu\,\nabla^2 u_i^0 + f_i^0,$$

the superior index 0 meaning the value of the respective function at $t = 0$, and

(1.11) $$\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0} = -\frac{\partial^2 p}{\partial t\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^{3} N_j^1\Big|_{t=0} +$$
$$+\nu\,\nabla^2\frac{\partial u_i}{\partial t}\Big|_{t=0} + \frac{\partial f_i}{\partial t}\Big|_{t=0},$$
$$N_j^1\Big|_{t=0} = \sum_{j=1}^{3}\left(\frac{\partial u_j}{\partial t}\Big|_{t=0}\frac{\partial u_i^0}{\partial x_j} + u_j^0\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}\Big|_{t=0}\right),$$

(1.12) $$\frac{\partial^3 u_i}{\partial t^3}\Big|_{t=0} = -\frac{\partial^3 p}{\partial t^2\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^{3} N_j^2\Big|_{t=0} +$$
$$+\nu\,\nabla^2\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0} + \frac{\partial^2 f_i}{\partial t^2}\Big|_{t=0},$$
$$N_j^2\Big|_{t=0} = \frac{\partial^2 u_j}{\partial t^2}\Big|_{t=0}\frac{\partial u_i^0}{\partial x_j} + 2\frac{\partial u_j}{\partial t}\Big|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}\Big|_{t=0} +$$
$$+u_j^0\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0},$$

(1.13) $$\frac{\partial^4 u_i}{\partial t^4}\Big|_{t=0} = -\frac{\partial^4 p}{\partial t^3\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^{3} N_j^3\Big|_{t=0} +$$
$$+\nu\,\nabla^2\frac{\partial^3 u_i}{\partial t^3}\Big|_{t=0} + \frac{\partial^3 f_i}{\partial t^3}\Big|_{t=0},$$
$$N_j^3\Big|_{t=0} = \frac{\partial^3 u_j}{\partial t^3}\Big|_{t=0}\frac{\partial u_i^0}{\partial x_j} + 3\frac{\partial^2 u_j}{\partial t^2}\Big|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}\Big|_{t=0} +$$

$$+ 3\frac{\partial u_j}{\partial t}|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2}|_{t=0} + u_j^0\frac{\partial}{\partial x_j}\frac{\partial^3 u_i}{\partial t^3}|_{t=0},$$

(1.14)
$$\frac{\partial^5 u_i}{\partial t^5}|_{t=0} = -\frac{\partial^5 p}{\partial t^4\,\partial x_i}|_{t=0} - \sum_{j=1}^{3} N_j^4\,|_{t=0} +$$
$$+ \nu\,\nabla^2\frac{\partial^4 u_i}{\partial t^4}|_{t=0} + \frac{\partial^4 f_i}{\partial t^4}|_{t=0},$$
$$N_j^4|_{t=0} = \frac{\partial^4 u_j}{\partial t^4}|_{t=0}\frac{\partial u_i^0}{\partial x_j} + 4\frac{\partial^3 u_j}{\partial t^3}|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t} +$$
$$+ 6\frac{\partial^2 u_j}{\partial t^2}|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2}|_{t=0} + 4\frac{\partial u_j}{\partial t}|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial^3 u_i}{\partial t^3}|_{t=0} +$$
$$+ u_j^0\frac{\partial}{\partial x_j}\frac{\partial^4 u_i}{\partial t^4}|_{t=0},$$

and of generic form,

(1.15)
$$\frac{\partial^k u_i}{\partial t^k}|_{t=0} = -\frac{\partial^k p}{\partial t^{k-1}\,\partial x_i}|_{t=0} - \sum_{j=1}^{3} N_j^{k-1}\,|_{t=0} +$$
$$+ \nu\,\nabla^2\frac{\partial^{k-1} u_i}{\partial t^{k-1}}|_{t=0} + \frac{\partial^{k-1} f_i}{\partial t^{k-1}}|_{t=0},$$
$$N_j^{k-1}|_{t=0} = \sum_{l=0}^{k-1}\binom{k-1}{l}\,\partial_t^{k-1-l}u_j|_{t=0}\,\frac{\partial}{\partial x_j}\partial_t^l u_i|_{t=0},$$
$$\partial_t^0 u_n|_{t=0} = u_n^0,\;\; \partial_t^m u_n|_{t=0} = \frac{\partial^m u_n}{\partial t^m}|_{t=0}.$$

If the external force is conservative there is a scalar potential U such as $f = \nabla U$ and the pressure can be calculated from this potential U, i.e.,

(1.16)
$$\frac{\partial p}{\partial x_i} = f_i = \frac{\partial U}{\partial x_i},$$

and then

(1.17)
$$p = U + \theta(t),$$

$\theta(t)$ a generic function of time of class C^∞, so it is not necessary the use of the pressure p and external force f, and respective derivatives, in (1.4) to (1.15) if the external force is conservative. In this case, the velocity can be independent of the both pressure and external force, otherwise it will be necessary to use both the pressure and external force derivatives to calculate the velocity in powers of time.

The result that we obtain here in this development in Taylor's series seems to me a great advance in the search of the solutions of the Euler's and Navier-Stokes equations. It is possible now to know on the possibility of non-uniqueness solutions as well as breakdown solution respect to unbounded energy of another manner. We now can choose previously an infinity of different pressures such that the calculation of $\frac{\partial u}{\partial t}$ and derivatives can be done, for a given initial velocity and external force, although such calculation can be very hard.

It is convenient say that Cauchy[2] in his memorable and admirable *Mémoire sur la Théorie des Ondes*, winner of the Mathematical Analysis award, year 1815, firstly does a study on the equations to be obeyed by three-dimensional molecules in a homogeneous fluid in the initial instant $t = 0$, coming to the conclusion which the initial velocity must be irrotational, i.e., a potential flow. Of this manner, after, he comes to conclusion that the velocity is always irrotational, potential flow, if the external force is conservative, which is essentially the Lagrange's theorem described in the begin of this article, but it is shown without the use of series expansion (a possible exception to the theorem occurs if one or two components of velocity are identically zero, when the reasonings on 3-D molecular volume are not valid). The solution obtained by Cauchy for Euler's equations is the Bernoulli's law, as almost always happens. Now a more generic solution is obtained, in special when it is possible a solution be expanded in polynomial series of time. Though not always a function can be expanded in Taylor's series, there is certainly an infinity of possible cases of solution where this is possible.

If the mentioned series is divergent in some point or region may be an indicative of that the correspondent velocity and its square diverge, again going to the case of breakdown solution due to unbounded energy. With the three functions initial velocity, pressure and external force belonging to Schwartz Space is expected that the solution for velocity also belongs to Schwartz Space, obtaining physically reasonable and well-behaved solution throughout the space.

The method presented here in this first section can also be applied in other equations, of course, for example in the heat equation, Schrödinger equation, wave equation and many others. Always will be necessary that the remainder in the Taylor's series goes to zero when the order k of the derivative tends to infinity (Courant[3], chap. VI). Applying this concept in (1.3) and (1.9), substituting t by τ, the remainder $R_{i,k}$ of order k for velocity component i is

(1.18) $$R_{i,k} = \frac{1}{k!}\int_0^t (t-\tau)^k \frac{\partial^{k+1} u_i}{\partial t^{k+1}}\, d\tau,$$

which can be estimated by Lagrange's remainder,

(1.19) $$R_{i,k} = \frac{t^{k+1}}{(k+1)!}\frac{\partial^{k+1} u_i}{\partial t^{k+1}}(\xi),$$

or by Cauchy's remainder,

(1.20) $$R_{i,k} = \frac{t^{k+1}}{k!}(1-\theta)^k \frac{\partial^{k+1} u_i}{\partial t^{k+1}}(\xi),$$

with $0 \le \xi \le t$ and $0 \le \theta \le 1$.

Note that if it is not possible to make a series around $t = 0$ (for example, to the functions $\log t$, $\sqrt[3]{t}$, e^{-1/t^2}, according Courant[3], chap. VI) an other instant t_0 of convergence and remainder $R_{i,k\to\infty}$ zero must be found, and then replacing t^k by $(t - t_0)^k$ and the calculations in $t = 0$ by $t = t_0$ in previous equations.

§ 2

In this section we will build a series of powers of time solving the Navier-Stokes equations, differently than that used in the previous section. From theorem of uniqueness of series of powers (*A function $f(x)$ can be represented by a power series in x in only one way, if it all, i.e., the representation of a function by a power series is "unique"; Every power series which converges for points other than $x = 0$ is the Taylor series of the function which it represents* (Courant[3], chap. VIII)), both solutions need be the same, for a same initial velocity, pressure, external force, compressibility condition and all boundary conditions.

Defining

(2.1) $$u_i = u_i^0 + X_{i,1} t + X_{i,2} t^2 + \cdots + X_{i,n} t^n + \cdots = \sum_{n=0}^{\infty} X_{i,n} t^n,$$
$$X_{i,0} = u_i^0 = u_i(x_1, x_2, x_3, 0),$$

where each $X_{i,n}$ is a function of position (x_1, x_2, x_3), without t, and

(2.2) $$\frac{\partial p}{\partial x_i} = q_i^0 + q_{i,1} t + q_{i,2} t^2 + \cdots + q_{i,n} t^n + \cdots = \sum_{n=0}^{\infty} q_{i,n} t^n,$$
$$q_{i,0} = q_i^0 = \frac{\partial p^0}{\partial x_i},\ p^0 = p(x_1, x_2, x_3, 0),$$

(2.3) $$f_i = f_i^0 + f_{i,1} t + f_{i,2} t^2 + \cdots + f_{i,n} t^n + \cdots = \sum_{n=0}^{\infty} f_{i,n} t^n,$$
$$f_{i,0} = f_i^0 = f_i(x_1, x_2, x_3, 0),$$

we can put these series in the Navier-Stokes equation,

(2.4) $$\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} + \nu\, \nabla^2 u_i + f_i.$$

The velocity derivative in relation to time is

(2.5) $$\frac{\partial u_i}{\partial t} = X_{i,1} + 2X_{i,2} t + 3X_{i,3} t^2 + \cdots + nX_{i,n} t^{n-1} + \cdots =$$
$$= \sum_{n=0}^{\infty} (n+1) X_{i,n+1} t^n,$$

the nonlinear terms are, of order zero (constant in time)

(2.6) $$\sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j},$$

of order 1,

(2.7) $$\sum_{j=1}^{3} \left(u_j^0 \frac{\partial X_{i,1}}{\partial x_j} + X_{j,1} \frac{\partial u_i^0}{\partial x_j} \right) t,$$

of order 2,

(2.8) $$\sum_{j=1}^{3} \left(u_j^0 \frac{\partial X_{i,2}}{\partial x_j} + X_{j,1} \frac{\partial X_{i,1}}{\partial x_j} + X_{j,2} \frac{\partial u_i^0}{\partial x_j} \right) t^2,$$

of order 3,

(2.9) $$\sum_{j=1}^{3} \left(u_j^0 \frac{\partial X_{i,3}}{\partial x_j} + X_{j,1} \frac{\partial X_{i,2}}{\partial x_j} + X_{j,2} \frac{\partial X_{i,1}}{\partial x_j} + X_{j,3} \frac{\partial u_i^0}{\partial x_j} \right) t^3,$$

and of order n, of generic form, equal to

(2.10) $$\sum_{j=1}^{3} \sum_{k=0}^{n} X_{j,k} \frac{\partial X_{i,n-k}}{\partial x_j} t^n,$$

with $X_{j,0} = u_j^0$, $\frac{\partial X_{i,0}}{\partial x_j} = \frac{\partial u_i^0}{\partial x_j}$.

Applying these sums in (2.4) we have

(2.11) $$\sum_{n=0}^{\infty} (n+1) X_{i,n+1} t^n = -\sum_{n=0}^{\infty} q_{i,n} t^n - \\ - \sum_{n=0}^{\infty} \sum_{j=1}^{3} \sum_{k=0}^{n} X_{j,k} \frac{\partial X_{i,n-k}}{\partial x_j} t^n + \nu \sum_{n=0}^{\infty} \nabla^2 X_{i,n}\, t^n + \\ + \sum_{n=0}^{\infty} f_{i,n} t^n,$$

and then

(2.12) $$(n+1) X_{i,n+1} = -q_{i,n} - \sum_{j=1}^{3} \sum_{k=0}^{n} X_{j,k} \frac{\partial X_{i,n-k}}{\partial x_j} + \\ + \nu \nabla^2 X_{i,n} + f_{i,n},$$

which allows us to obtain, by recurrence, $X_{i,1}$, $X_{i,2}$, $X_{i,3}$, etc., that is, for $1 \le i \le 3$ and $n \ge 0$,

(2.13) $$X_{i,n+1} = \frac{1}{n+1} S_n, \\ S_n = -q_{i,n} - \sum_{j=1}^{3} \sum_{k=0}^{n} X_{j,k} \frac{\partial X_{i,n-k}}{\partial x_j} + \nu \nabla^2 X_{i,n} + f_{i,n}.$$

You can see how much will become increasingly difficult calculate the terms $X_{i,n}$ with increasing the values of n, for example, will appear terms in

$\nu^n, \nabla^2\nabla^2 \ldots \nabla^2 u_i^0$, etc. If $\nu > 1$ (say, $1/\rho > 1$ or $\nu/\rho > 1$) certainly there is a specific problem to be studied with relation to convergence of the series, which of course also occurs in the representation given in section § 1. The same can be said for $t \to \infty$. In fact, I do not understand why a particle fluid initially in motion, without any collision with another particle and submitted to a permanent impulsive force need always be with finite velocity as $t \to \infty$. For example, a constant resulting force f, not equal to zero, applied all time on a body will produce an infinite velocity u to this body when $t \to \infty$, supposing possible such force and a way no obstacles, etc.

§ 3

The previous solutions show us that we need to have, for all integers $1 \leq i \leq 3$ and $n \geq 0$,

$$\frac{1}{n!}\frac{\partial^n u_i}{\partial t^n}\Big|_{t=0} = X_{i,n}, \tag{3.1}$$

and both members of this relation are very difficult to be calculated, either equation (1.15) as well as (2.13). Add to this difficulty the fact that besides the main Navier-Stokes equations (1.4)-(2.4) must be included the condition of incompressibility,

$$\nabla \cdot u = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} u_i = 0. \tag{3.2}$$

Using (2.1) in (3.2) we have

$$\nabla \cdot u = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \sum_{n=0}^{\infty} X_{i,n} t^n = \sum_{n=0}^{\infty} \left(\sum_{i=1}^{3} \frac{\partial}{\partial x_i} X_{i,n} \right) t^n = 0. \tag{3.3}$$

As this equation need be valid for all $t \geq 0$ we have

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i} X_{i,n} = \nabla \cdot X_n = 0, \tag{3.4}$$

defining $X_n = \left(X_{1,n},\ X_{2,n},\ X_{3,n} \right)$, i.e., all coefficients X_n must obey the condition of incompressibility in the vector representation of velocity,

$$u = \sum_{n=0}^{\infty} X_n\, t^n. \tag{3.5}$$

Following Lagrange[1], getting two differentiable and continuous functions α and β of class C^2 and defining

$$u_1 = \frac{\partial \alpha}{\partial z}, \qquad u_2 = \frac{\partial \beta}{\partial z}, \qquad u_3 = -\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} \right), \tag{3.6.1}$$

(3.6.2) $$u_1^0 = \frac{\partial \alpha^0}{\partial z}, \qquad u_2^0 = \frac{\partial \beta^0}{\partial z}, \qquad u_3^0 = -\left(\frac{\partial \alpha^0}{\partial x} + \frac{\partial \beta^0}{\partial y}\right),$$

with $\alpha^0 = \alpha(t=0)$ and $\beta^0 = \beta(t=0)$, we have satisfied the condition (3.2), which it is easy to see. Other manner is when u is derived from a vector potential A, i.e.,

(3.7.1) $$u = \nabla \times A,$$

(3.7.2) $$u^0 = \nabla \times A^0,$$

with $A^0 = A(t=0)$.

The relations (3.6) are very useful and easy to be implemented and we will use them to solve the Euler and Navier-Stokes equations when the incompressibility condition is required. Given any continuous, differentiable and integrable vector components u_1 and u_2 then

(3.8.1) $$\alpha = \int u_1 \, dz,$$

(3.8.2) $$\beta = \int u_2 \, dz,$$

and thus u_3 and u_3^0 need to be according

(3.9.1) $$u_3 = -\int \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right) dz = -\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y}\right),$$

(3.9.2) $$u_3^0 = -\int \left(\frac{\partial u_1^0}{\partial x} + \frac{\partial u_2^0}{\partial y}\right) dz = -\left(\frac{\partial \alpha^0}{\partial x} + \frac{\partial \beta^0}{\partial y}\right),$$

which reminds us that the components of the velocity vector maintains conditions to be complied to each other, i.e., it is not any initial velocity which can be used for solution of Euler and Navier-Stokes equations in incompressible flows case.

In the equations of the sections § 1 and § 2, instead u_1 we will use $\frac{\partial \alpha}{\partial z}$, instead u_2 will be $\frac{\partial \beta}{\partial z}$, and $-\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y}\right)$ instead u_3, as well as the correspondents initial values, replacing u_1^0 by $\frac{\partial \alpha^0}{\partial z}$, u_2^0 by $\frac{\partial \beta^0}{\partial z}$, and u_3^0 by $-\left(\frac{\partial \alpha^0}{\partial x} + \frac{\partial \beta^0}{\partial y}\right)$. Of this manner, we will be developing series for $\frac{\partial \alpha}{\partial z}$, $\frac{\partial \beta}{\partial z}$ and $-\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y}\right)$, so that $\nabla \cdot u = 0$. Then this is a preliminary problem to be solved, the calculation of α^0 and β^0 giving u_1^0, u_2^0 and u_3^0 when $\nabla \cdot u^0 = 0$ and it is necessary that $\nabla \cdot u = 0$, i.e.,

(3.10.1) $$\alpha^0 = \int u_1^0 \, dz,$$

(3.10.2) $$\beta^0 = \int u_2^0 \, dz,$$

with the validity of (3.9.2). Done this, the exact solution for the principal problem can be calculated from reasoning exposed here, if there is not an equivalent solution described in a most simplified formulation, for example, according Bernoulli's law and Laplace's equation.

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What good would living on a planet without destruction, greed and envy,
where the nations were dedicated to building a beautiful world
and to the salvation of those in need.
That there were no enemies and everyone could be happy where they live,
in their own way.

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20 – Solutions for Euler and Navier-Stokes Equations in Finite and Infinite Series of Time

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Abstract – We present solutions for the Euler and Navier-Stokes equations in finite and infinite series of time, in spatial dimension $N = 3$, firstly based on expansion in Taylor's series of time and then, in special case, solutions for velocity given by irrotational vectors, for incompressible flows and conservative external force, the Bernoulli's law. A little description of the Lamb's solution for Euler equations is done.

Keywords – Lagrange, Mécanique Analitique, exact differential, Euler's equations, Navier-Stokes equations, Taylor's series, series of time, Cauchy, Mémoire sur la Théorie des Ondes, Lagrange's theorem, Bernoulli's law, non-uniqueness solutions, Millenium Problem, velocity potential, Liouville's theorem, Helmholtz decomposition, Hodge decomposition.

§ 1

Let p, q, r be the three components of velocity of an element of fluid in the 3-D orthogonal Euclidean system of spatial coordinates (x, y, z) and t the time in this system.

Lagrange in his *Mécanique Analitique*, firstly published in 1788, proved that if the quantity $(p\,dx + q\,dy + r\,dz)$ is an exact differential when $t = 0$ it will also be an exact differential when t has any other value. If the quantity $(p\,dx + q\,dy + r\,dz)$ is an exact differential at an arbitrary instant, it should be such for all other instants. Consequently, if there is one instant during the motion for which it is not an exact differential, it cannot be exact for the entire period of motion. If it were exact at another arbitrary instant, it should also be exact at the first instant.[1]

To prove it Lagrange used

$$(1.1)\qquad \begin{cases} p = p^I + p^{II}t + p^{III}t^2 + p^{IV}t^3 + \cdots \\ q = q^I + q^{II}t + q^{III}t^2 + q^{IV}t^3 + \cdots \\ r = r^I + r^{II}t + r^{III}t^2 + r^{IV}t^3 + \cdots \end{cases}$$

in which the quantities p^I, p^{II}, p^{III}, etc., q^I, q^{II}, q^{III}, etc., r^I, r^{II}, r^{III}, etc., are functions of x, y, z but without t.

Here we will finally solve the equations of Euler and Navier-Stokes using this representation of the velocity components in infinite series, as pointed by Lagrange. We assume satisfied the condition of incompressibility, for brevity.

Without it the resulting equations are more complicated, as we know, but the method of solution is essentially the same in both cases. We focus our attention in the general case of the Navier-Stokes equations, with $\nu \geq 0$ constant, and for the Euler equations simply set the viscosity coefficient as $\nu = 0$.

To facilitate and abbreviate our writing, we represent the fluid velocity by its three components in indicial notation, i.e., $u = (u_1, u_2, u_3)$, as well as the external force will be $f = (f_1, f_2, f_3)$ and the spatial coordinates $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$. The pressure, a scalar function, will be represented as p. As frequently used in mathematics approach, the density mass will be $\rho = 1$. We consider all functions belonging to C^∞, being valid the use of $\frac{\partial^2 u_i}{\partial x_j \partial x_k} = \frac{\partial^2 u_i}{\partial x_k \partial x_j}$ and other inversions in order of derivatives, so much in relation to space as to time.

The representation (1.1) is as the expansion of the velocity in a Taylor´s series in relation to time around $t = 0$, considering x, y, z as constant, i.e., for $1 \leq i \leq 3$,

(1.2) $$u_i = u_i|_{t=0} + \frac{\partial u_i}{\partial t}|_{t=0}\, t + \frac{\partial^2 u_i}{\partial t^2}|_{t=0}\, \frac{t^2}{2} + \frac{\partial^3 u_i}{\partial t^3}|_{t=0}\, \frac{t^3}{6} + \cdots + \frac{\partial^k u_i}{\partial t^k}|_{t=0}\, \frac{t^k}{k!} + \cdots$$

or

(1.3) $$u_i = u_i^0 + \sum_{k=1}^{\infty} \frac{\partial^k u_i}{\partial t^k}|_{t=0}\, \frac{t^k}{k!}.$$

For the calculation of $\frac{\partial u_i}{\partial t}, \frac{\partial^2 u_i}{\partial t^2}, \frac{\partial^3 u_i}{\partial t^3}, \ldots$ we use the values that are obtained directly from the Navier-Stokes equations and its derivatives in relation to time, i.e.,

(1.4) $$\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} + \nu\, \nabla^2 u_i + f_i,$$

and therefore

(1.5) $$\frac{\partial^2 u_i}{\partial t^2} = -\frac{\partial^2 p}{\partial t\, \partial x_i} - \sum_{j=1}^{3} \left(\frac{\partial u_j}{\partial t} \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} \right) + \nu\, \nabla^2 \frac{\partial u_i}{\partial t} + \frac{\partial f_i}{\partial t},$$

(1.6) $$\frac{\partial^3 u_i}{\partial t^3} = -\frac{\partial^3 p}{\partial t^2\, \partial x_i} - \sum_{j=1}^{3} \left(\frac{\partial^2 u_j}{\partial t^2} \frac{\partial u_i}{\partial x_j} + 2 \frac{\partial u_j}{\partial t} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} \frac{\partial^2 u_i}{\partial t^2} \right) + \nu\, \nabla^2 \frac{\partial^2 u_i}{\partial t^2} + \frac{\partial^2 f_i}{\partial t^2},$$

(1.7) $$\frac{\partial^4 u_i}{\partial t^4} = -\frac{\partial^4 p}{\partial t^3\, \partial x_i} - \sum_{j=1}^{3} N_j^3 + \nu\, \nabla^2 \frac{\partial^3 u_i}{\partial t^3} + \frac{\partial^3 f_i}{\partial t^3},$$

$$N_j^3 = \frac{\partial}{\partial t} N_j^2,\ N_j^2 = \frac{\partial^2 u_j}{\partial t^2}\frac{\partial u_i}{\partial x_j} + 2\frac{\partial u_j}{\partial t}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t} + u_j\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2},$$

$$N_j^3 = \frac{\partial^3 u_j}{\partial t^3}\frac{\partial u_i}{\partial x_j} + 3\frac{\partial^2 u_j}{\partial t^2}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t} + 3\frac{\partial u_j}{\partial t}\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2} + u_j\frac{\partial}{\partial x_j}\frac{\partial^3 u_i}{\partial t^3},$$

(1.8) $$\frac{\partial^5 u_i}{\partial t^5} = -\frac{\partial^5 p}{\partial t^4\,\partial x_i} - \sum_{j=1}^3 N_j^4 + \nu\,\nabla^2\frac{\partial^4 u_i}{\partial t^4} + \frac{\partial^4 f_i}{\partial t^4},$$

$$N_j^4 = \frac{\partial}{\partial t}N_j^3 = \frac{\partial^4 u_j}{\partial t^4}\frac{\partial u_i}{\partial x_j} + 4\frac{\partial^3 u_j}{\partial t^3}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t} + 6\frac{\partial^2 u_j}{\partial t^2}\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2} +$$
$$+4\frac{\partial u_j}{\partial t}\frac{\partial}{\partial x_j}\frac{\partial^3 u_i}{\partial t^3} + u_j\frac{\partial}{\partial x_j}\frac{\partial^4 u_i}{\partial t^4},$$

and using induction we come to

(1.9) $$\frac{\partial^k u_i}{\partial t^k} = -\frac{\partial^k p}{\partial t^{k-1}\,\partial x_i} - \sum_{j=1}^3 N_j^{k-1} + \nu\,\nabla^2\frac{\partial^{k-1} u_i}{\partial t^{k-1}} + \frac{\partial^{k-1} f_i}{\partial t^{k-1}},$$

$$N_j^{k-1} = \frac{\partial}{\partial t}N_j^{k-2} = \sum_{l=0}^{k-1}\binom{k-1}{l}\partial_t^{k-1-l}u_j\,\frac{\partial}{\partial x_j}\partial_t^l u_i,$$

$$\partial_t^0 u_n = u_n,\ \partial_t^m u_n = \frac{\partial^m u_n}{\partial t^m},\ \binom{k-1}{l} = \frac{(k-1)!}{(k-1-l)!\,l!}.$$

In (1.2) and (1.3) it is necessary to know the values of the derivatives $\frac{\partial u_i}{\partial t}, \frac{\partial^2 u_i}{\partial t^2}, \dots, \frac{\partial^k u_i}{\partial t^k}$ in $t = 0$ then we must to calculate, from (1.4) to (1.9),

(1.10) $$\frac{\partial u_i}{\partial t}\Big|_{t=0} = -\frac{\partial p^0}{\partial x_i} - \sum_{j=1}^3 u_j^0\frac{\partial u_i^0}{\partial x_j} + \nu\,\nabla^2 u_i^0 + f_i^0,$$

the superior index 0 meaning the value of the respective function at $t = 0$, and

(1.11) $$\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0} = -\frac{\partial^2 p}{\partial t\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^3 N_j^1\Big|_{t=0} +$$
$$+\nu\,\nabla^2\frac{\partial u_i}{\partial t}\Big|_{t=0} + \frac{\partial f_i}{\partial t}\Big|_{t=0},$$

$$N_j^1\Big|_{t=0} = \sum_{j=1}^3\left(\frac{\partial u_j}{\partial t}\Big|_{t=0}\frac{\partial u_i^0}{\partial x_j} + u_j^0\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}\Big|_{t=0}\right),$$

(1.12) $$\frac{\partial^3 u_i}{\partial t^3}\Big|_{t=0} = -\frac{\partial^3 p}{\partial t^2\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^3 N_j^2\Big|_{t=0} +$$
$$+\nu\,\nabla^2\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0} + \frac{\partial^2 f_i}{\partial t^2}\Big|_{t=0},$$

$$N_j^2\Big|_{t=0} = \frac{\partial^2 u_j}{\partial t^2}\Big|_{t=0}\frac{\partial u_i^0}{\partial x_j} + 2\frac{\partial u_j}{\partial t}\Big|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}\Big|_{t=0} +$$
$$+u_j^0\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0},$$

(1.13) $$\frac{\partial^4 u_i}{\partial t^4}\Big|_{t=0} = -\frac{\partial^4 p}{\partial t^3\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^3 N_j^3\Big|_{t=0} +$$

$$+\nu\,\nabla^2\frac{\partial^3 u_i}{\partial t^3}\big|_{t=0}+\frac{\partial^3 f_i}{\partial t^3}\big|_{t=0},$$

$$N_j^3\big|_{t=0}=\frac{\partial^3 u_j}{\partial t^3}\big|_{t=0}\frac{\partial u_i^0}{\partial x_j}+3\frac{\partial^2 u_j}{\partial t^2}\big|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}\big|_{t=0}+$$

$$+3\frac{\partial u_j}{\partial t}\big|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2}\big|_{t=0}+u_j^0\frac{\partial}{\partial x_j}\frac{\partial^3 u_i}{\partial t^3}\big|_{t=0},$$

(1.14)
$$\frac{\partial^5 u_i}{\partial t^5}\big|_{t=0}=-\frac{\partial^5 p}{\partial t^4\,\partial x_i}\big|_{t=0}-\sum_{j=1}^3 N_j^4\big|_{t=0}+$$

$$+\nu\,\nabla^2\frac{\partial^4 u_i}{\partial t^4}\big|_{t=0}+\frac{\partial^4 f_i}{\partial t^4}\big|_{t=0},$$

$$N_j^4\big|_{t=0}=\frac{\partial^4 u_j}{\partial t^4}\big|_{t=0}\frac{\partial u_i^0}{\partial x_j}+4\frac{\partial^3 u_j}{\partial t^3}\big|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}+$$

$$+6\frac{\partial^2 u_j}{\partial t^2}\big|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2}\big|_{t=0}+4\frac{\partial u_j}{\partial t}\big|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial^3 u_i}{\partial t^3}\big|_{t=0}+$$

$$+u_j^0\frac{\partial}{\partial x_j}\frac{\partial^4 u_i}{\partial t^4}\big|_{t=0},$$

and of generic form,

(1.15)
$$\frac{\partial^k u_i}{\partial t^k}\big|_{t=0}=-\frac{\partial^k p}{\partial t^{k-1}\,\partial x_i}\big|_{t=0}-\sum_{j=1}^3 N_j^{k-1}\big|_{t=0}+$$

$$+\nu\,\nabla^2\frac{\partial^{k-1} u_i}{\partial t^{k-1}}\big|_{t=0}+\frac{\partial^{k-1} f_i}{\partial t^{k-1}}\big|_{t=0},$$

$$N_j^{k-1}\big|_{t=0}=\sum_{l=0}^{k-1}\binom{k-1}{l}\,\partial_t^{k-1-l}u_j\big|_{t=0}\,\frac{\partial}{\partial x_j}\partial_t^l u_i\big|_{t=0},$$

$$\partial_t^0 u_n\big|_{t=0}=u_n^0,\ \ \partial_t^m u_n\big|_{t=0}=\frac{\partial^m u_n}{\partial t^m}\big|_{t=0}.$$

If the external force is conservative there is a scalar potential U such as $f=\nabla U$ and the pressure can be calculated from this potential U, i.e.,

(1.16)
$$\frac{\partial p}{\partial x_i}=f_i=\frac{\partial U}{\partial x_i},$$

and then

(1.17)
$$p=U+\theta(t),$$

$\theta(t)$ a generic function of time of class C^∞, so it is not necessary the use of the pressure p and external force f, and respective derivatives, in (1.4) to (1.15) if the external force is conservative. In this case, the velocity can be independent of the both pressure and external force, otherwise it will be necessary to use both the pressure and external force derivatives to calculate the velocity in powers of time.

The result that we obtain here in this development in Taylor's series seems to me a great advance in the search of the solutions of the Euler's and Navier-Stokes equations. It is possible now to know on the possibility of non-uniqueness solutions as well as breakdown solution respect to unbounded energy of another manner. We now can choose previously an infinity of different pressures such that the calculation of $\frac{\partial u}{\partial t}$ and derivatives can be done, for a given initial velocity and external force, although such calculation can be very hard.

It is convenient say that Cauchy[2] in his memorable and admirable *Mémoire sur la Théorie des Ondes*, winner of the Mathematical Analysis award, year 1815, firstly does a study on the equations to be obeyed by three-dimensional molecules in a homogeneous fluid in the initial instant $t = 0$, coming to the conclusion which the initial velocity must be irrotational, i.e., a potential flow. Of this manner, after, he comes to conclusion that the velocity is always irrotational, potential flow, if the external force is conservative, which is essentially the Lagrange's theorem described in the begin of this article, but it is shown without the use of series expansion (a possible exception to the theorem occurs if one or two components of velocity are identically zero, when the reasonings on 3-D molecular volume are not valid). The solution obtained by Cauchy for Euler's equations is the Bernoulli's law, as almost always happens. Now at first a more generic solution is obtained, in special when it is possible a solution be expanded in polynomial series of time. Though not always a function can be expanded in Taylor's series, there is certainly an infinity of possible cases of solutions where this is possible.

If the mentioned series is divergent in some point or region may be an indicative of that the correspondent velocity and its square diverge, again going to the case of breakdown solution due to unbounded energy. With the three functions initial velocity, pressure and external force belonging to Schwartz Space is expected that the solution for velocity also belongs to Schwartz Space, obtaining physically reasonable and well-behaved solution throughout the space.

The method presented here in this first section can also be applied in other equations, of course, for example in the heat equation, Schrödinger equation, wave equation and many others. Always will be necessary that the remainder in the Taylor's series goes to zero when the order k of the derivative tends to infinity (Courant[3], chap. VI). Applying this concept in (1.3) and (1.9), substituting t by τ, the remainder $R_{i,k}$ of order k for velocity component i is

$$(1.18) \qquad R_{i,k} = \frac{1}{k!}\int_0^t (t-\tau)^k \frac{\partial^{k+1} u_i}{\partial t^{k+1}} d\tau,$$

which can be estimated by Lagrange's remainder,

$$(1.19) \qquad R_{i,k} = \frac{t^{k+1}}{(k+1)!}\frac{\partial^{k+1} u_i}{\partial t^{k+1}}(\xi),$$

or by Cauchy's remainder,

(1.20) $$R_{i,k} = \frac{t^{k+1}}{k!}(1-\theta)^k \frac{\partial^{k+1} u_i}{\partial t^{k+1}}(\xi),$$

with $0 \le \xi \le t$ and $0 \le \theta \le 1$.

§ 2

In this section we will build a series of powers of time solving the Navier-Stokes equations, differently than that used in the previous section. From theorem of uniqueness of series of powers (*A function $f(x)$ can be represented by a power series in x in only one way, if it all, i.e., the representation of a function by a power series is "unique"; Every power series which converges for points other than $x = 0$ is the Taylor series of the function which it represents* (Courant[3], chap. VIII)), both solutions need be the same, for a same initial velocity, pressure, external force, compressibility condition and all boundary conditions.

Defining

(2.1) $$u_i = u_i^0 + X_{i,1}t + X_{i,2}t^2 + \cdots + X_{i,n}t^n + \cdots = \sum_{n=0}^{\infty} X_{i,n}t^n,$$
$$X_{i,0} = u_i^0 = u_i(x_1, x_2, x_3, 0),$$

where each $X_{i,n}$ is a function of position (x_1, x_2, x_3), without t, and

(2.2) $$\frac{\partial p}{\partial x_i} = q_i^0 + q_{i,1}t + q_{i,2}t^2 + \cdots + q_{i,n}t^n + \cdots = \sum_{n=0}^{\infty} q_{i,n}t^n,$$
$$q_{i,0} = q_i^0 = \frac{\partial p^0}{\partial x_i}, \ p^0 = p(x_1, x_2, x_3, 0),$$

(2.3) $$f_i = f_i^0 + f_{i,1}t + f_{i,2}t^2 + \cdots + f_{i,n}t^n + \cdots = \sum_{n=0}^{\infty} f_{i,n}t^n,$$
$$f_{i,0} = f_i^0 = f_i(x_1, x_2, x_3, 0),$$

we can put these series in the Navier-Stokes equation

(2.4) $$\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} + \nu \nabla^2 u_i + f_i.$$

The velocity derivative in relation to time is

(2.5) $$\frac{\partial u_i}{\partial t} = X_{i,1} + 2X_{i,2}t + 3X_{i,3}t^2 + \cdots + nX_{i,n}t^{n-1} + \cdots =$$
$$= \sum_{n=0}^{\infty}(n+1)X_{i,n+1}t^n,$$

the nonlinear terms are, of order zero (constant in time),

(2.6) $$\sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j},$$

of order 1,

(2.7) $$\sum_{j=1}^{3}\left(u_j^0\frac{\partial X_{i,1}}{\partial x_j}+X_{j,1}\frac{\partial u_i^0}{\partial x_j}\right)t,$$

of order 2,

(2.8) $$\sum_{j=1}^{3}\left(u_j^0\frac{\partial X_{i,2}}{\partial x_j}+X_{j,1}\frac{\partial X_{i,1}}{\partial x_j}+X_{j,2}\frac{\partial u_i^0}{\partial x_j}\right)t^2,$$

of order 3,

(2.9) $$\sum_{j=1}^{3}\left(u_j^0\frac{\partial X_{i,3}}{\partial x_j}+X_{j,1}\frac{\partial X_{i,2}}{\partial x_j}+X_{j,2}\frac{\partial X_{i,1}}{\partial x_j}+X_{j,3}\frac{\partial u_i^0}{\partial x_j}\right)t^3,$$

and of order n, of generic form, equal to

(2.10) $$\sum_{j=1}^{3}\sum_{k=0}^{n}X_{j,k}\frac{\partial X_{i,n-k}}{\partial x_j}t^n,$$

with $X_{j,0}=u_j^0$, $\frac{\partial X_{i,0}}{\partial x_j}=\frac{\partial u_i^0}{\partial x_j}$.

Applying these sums in (2.4) we have

(2.11) $$\sum_{n=0}^{\infty}(n+1)X_{i,n+1}t^n=-\sum_{n=0}^{\infty}q_{i,n}t^n-$$
$$-\sum_{n=0}^{\infty}\sum_{j=1}^{3}\sum_{k=0}^{n}X_{j,k}\frac{\partial X_{i,n-k}}{\partial x_j}t^n+\nu\sum_{n=0}^{\infty}\nabla^2X_{i,n}\,t^n+$$
$$+\sum_{n=0}^{\infty}f_{i,n}t^n,$$

and then

(2.12) $$(n+1)X_{i,n+1}=-q_{i,n}-\sum_{j=1}^{3}\sum_{k=0}^{n}X_{j,k}\frac{\partial X_{i,n-k}}{\partial x_j}+$$
$$+\nu\nabla^2X_{i,n}+f_{i,n},$$

which allows us to obtain, by recurrence, $X_{i,1}$, $X_{i,2}$, $X_{i,3}$, etc., that is, for $1\le i\le 3$ and $n\ge 0$,

(2.13) $$X_{i,n+1}=\frac{1}{n+1}S_n,$$
$$S_n=-q_{i,n}-\sum_{j=1}^{3}\sum_{k=0}^{n}X_{j,k}\frac{\partial X_{i,n-k}}{\partial x_j}+\nu\nabla^2X_{i,n}+f_{i,n}.$$

You can see how much will become increasingly difficult calculate the terms $X_{i,n}$ with increasing the values of n, for example, will appear terms in $\nu^n, \nabla^2\nabla^2 \dots \nabla^2 u_i^0$, etc. If $\nu>1$ certainly there is a specific problem to be studied with relation to convergence of the series, which of course also occurs in the representation given in section § 1. The same can be said for $t\to\infty$.

§ 3

The previous solutions show us that we need to have, for all integers $1 \le i \le 3$ and $n \ge 0$,

(3.1) $$\frac{1}{n!}\frac{\partial^n u_i}{\partial t^n}|_{t=0} = X_{i,n},$$

and both members of this relation are very difficult to be calculated, either equation (1.15) as well as (2.13). Add to this difficulty the fact that besides the main Navier-Stokes equations (1.4)-(2.4) must be included the condition of incompressibility,

(3.2) $$\nabla \cdot u = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} u_i = 0.$$

Using (2.1) in (3.2) we have

(3.3) $$\nabla \cdot u = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \sum_{n=0}^{\infty} X_{i,n} t^n = \sum_{n=0}^{\infty} \left(\sum_{i=1}^{3} \frac{\partial}{\partial x_i} X_{i,n} \right) t^n = 0.$$

As this equation need be valid for all $t \ge 0$ we have

(3.4) $$\sum_{i=1}^{3} \frac{\partial}{\partial x_i} X_{i,n} = \nabla \cdot X_n = 0,$$

defining $X_n = (X_{1,n},\ X_{2,n},\ X_{3,n})$, i.e., all coefficients X_n must obey the condition of incompressibility in the vector representation of velocity,

(3.5) $$u = \sum_{n=0}^{\infty} X_n\, t^n.$$

As we realized that it is possible infinite solutions for a same initial condition for velocity then we can try choose a more easier solution, whose maximum value of n is finite, in special $n = 1$, i.e.,

(3.6) $$u_i = u_i^0 + X_{i,1} t,$$

where

(3.7) $$X_{i,1} = \frac{\partial u_i}{\partial t}|_{t=0}.$$

Then of (3.4), for $n = 0$, it is necessary that

(3.8) $$\nabla \cdot u^0 = 0,$$

which is also an initial condition, for $n = 1$ it is necessary that

(3.9) $$\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(-q_{i,0} - \sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j} + \nu \nabla^2 u_i^0 + f_i^0 \right) = 0,$$

$q_{i,0} = \frac{\partial p}{\partial x_i}(x_1, x_2, x_3, 0)$, and for $n \geq 2$ do not have no term for velocity, by definition in (3.6), but we need that the nonlinear terms of second order in time vanishes in the sum $\sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}$, i.e.,

(3.10) $$\sum_{j=1}^{3} X_{j,1} \frac{\partial X_{i,1}}{\partial x_j} = 0.$$

The two last conditions are obviously satisfied when $X_{i,1}$ is zero, any other numerical constant or a generic smooth function of time $\tau_i(t)$, i.e.,

(3.11) $$X_{i,1} = -q_{i,0} - \sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j} + \nu\nabla^2 u_i^0 + f_i^0 = \tau_i(t),$$

and then

(3.12) $$u_i = u_i^0 + T_i(t)$$

may be our solution obtained for velocity, with $T_i(0) = 0$ and $T = \tau\, t$, still lacking the calculation of pressure.

As our initial hypothesis was that $X_{i,1}$ was not time dependent we need review the solution or simply choose $T_i(t) = c_i t$, c_i a numerical constant. Note that for compliance with the Millenium Problem[4], case (A), only if $T_i(t)$ is identically zero we can have the condition of bounded energy satisfied, i.e.,

(3.13) $$\int_{\mathbb{R}^3} |u|^2 dx < C \text{ finite}, x \in \mathbb{R}^3,$$

what force us to choose $T_i(t) \equiv 0$ and thus the final solution for velocity will be $u = u^0$ for any $t \geq 0$, since that u^0 obey to the necessary conditions of case (A). For case (B), related with periodical spatially solutions, where the condition of bounded energy in whole space is not necessary, it is possible $T_i(t) \neq 0$, and in fact this is a promising and well behaved solution for Euler and Navier-Stokes equations in periodic spatially solutions cases, choosing $T(t) \in C^\infty$ limited, as well as u^0 and consequently u.

Note that the solution (3.12) also is compatible with the condition (3.4) of divergence free for series of form

(3.14) $$u_i = u_i^0 + \sum_{n=1}^{\infty} c_{i,n}\, t^n,$$

i.e.,

(3.15) $$T_i(t) = \sum_{n=1}^{\infty} c_{i,n}\, t^n, \qquad X_{i,n} = c_{i,n},$$

supposing that (3.8) is valid. The same is said if the maximum value of n is finite, obviously. All $c_{i,n}$ are numerical constants for $n \geq 1$.

§ 4

How calculate the pressure value, if it is not given nor previously chosen?

From Navier-Stokes equations, it is equal to the line integral

(4.1) $$p = \int_L \left(-\frac{\partial u}{\partial t} - (u \cdot \nabla)u + \nu \nabla^2 u + f\right) \cdot dl,$$

where L is any sectionally smooth curve going from a point (x^0, y^0, z^0) to (x, y, z), for a fixed time t, and for this calculation it is necessary that the vector

(4.2) $$S = -\frac{\partial u}{\partial t} - (u \cdot \nabla)u + \nu \nabla^2 u + f$$

is gradient, i.e.,

(4.3) $$S = \nabla p, \qquad p = \int_L S \cdot dl,$$

so the condition

(4.4) $$\frac{\partial S_i}{\partial x_j} = \frac{\partial S_j}{\partial x_i},$$

need be satisfied for all integer $1 \leq i, j \leq 3$ in order that (4.1) can be calculated, where $S = (S_1,\ S_2,\ S_3)$ and

(4.5) $$S_i = -\frac{\partial u_i}{\partial t} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} + \nu\, \nabla^2 u_i + f_i.$$

Obviously, even if the pressure is given or chosen previously, as indicated in sections § 1 and § 2, the equations (4.1) and (4.4) need to be fulfilled.

The condition (4.4) is a very hard condition to be satisfied, instead the incompressibility condition

(4.6) $$\nabla \cdot u = \nabla \cdot u^0 = 0.$$

Following Lagrange[1], getting two differentiable and continuous functions α and β of class C^2 and defining

(4.7.1) $$u_1 = \frac{\partial \alpha}{\partial z}, \qquad u_2 = \frac{\partial \beta}{\partial z}, \qquad u_3 = -\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y}\right),$$

(4.7.2) $$u_1^0 = \frac{\partial \alpha^0}{\partial z}, \qquad u_2^0 = \frac{\partial \beta^0}{\partial z}, \qquad u_3^0 = -\left(\frac{\partial \alpha^0}{\partial x} + \frac{\partial \beta^0}{\partial y}\right),$$

with $\alpha^0 = \alpha(t = 0)$ and $\beta^0 = \beta(t = 0)$, we have satisfied the condition (4.6), which it is easy to see. Other manner is when u is derived from a vector potential A, i.e.,

(4.8.1) $$u = \nabla \times A,$$
(4.8.2) $$u^0 = \nabla \times A^0,$$

with $A^0 = A(t = 0)$.

The relations (4.7) are very useful and easy to be implemented. Given any continuous, differentiable and integrable vector components u_1 and u_2 then

(4.9.1) $$\alpha = \int u_1 \, dz,$$
(4.9.2) $$\beta = \int u_2 \, dz,$$

and thus u_3 and u_3^0 need to be according

(4.10.1) $$u_3 = -\int \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right) dz = -\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y}\right),$$
(4.10.2) $$u_3^0 = -\int \left(\frac{\partial u_1^0}{\partial x} + \frac{\partial u_2^0}{\partial y}\right) dz = -\left(\frac{\partial \alpha^0}{\partial x} + \frac{\partial \beta^0}{\partial y}\right),$$

which reminds us that the components of the velocity vector maintains conditions to be complied to each other, i.e., it is not any initial velocity which can be used for solution of Euler and Navier-Stokes equations in incompressible flows case.

Following these transformations, in the equations of the sections § 1 and § 2, instead u_1 we will use $\frac{\partial \alpha}{\partial z}$, instead u_2 will be $\frac{\partial \beta}{\partial z}$, and $-\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y}\right)$ instead u_3, as well as the correspondents initial values, replacing u_1^0 by $\frac{\partial \alpha^0}{\partial z}$, u_2^0 by $\frac{\partial \beta^0}{\partial z}$, and u_3^0 by $-\left(\frac{\partial \alpha^0}{\partial x} + \frac{\partial \beta^0}{\partial y}\right)$. Of this manner, we will be developing series for $\frac{\partial \alpha}{\partial z}$, $\frac{\partial \beta}{\partial z}$ and $-\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y}\right)$, so that $\nabla \cdot u = 0$. Then this is a preliminary problem to be solved, the calculation of α^0 and β^0 giving u_1^0, u_2^0 and u_3^0 when $\nabla \cdot u^0 = 0$ and it is necessary that $\nabla \cdot u = 0$, i.e.,

(4.11.1) $$\alpha^0 = \int u_1^0 \, dz,$$
(4.11.2) $$\beta^0 = \int u_2^0 \, dz,$$

with the validity of (4.10.2). Done this, the exact solution for the principal problem can be calculated from reasoning exposed here, if there is not an equivalent solution described in a most simplified formulation, for example, according Bernoulli's law and Laplace's equation.

Another relation need be obeyed for obtaining the mentioned solution, both in compressible as incompressible flows, described below.

For obtaining a solution for the system

$$(4.12)\qquad \begin{cases} \dfrac{\partial p}{\partial x} = S_1 \\ \dfrac{\partial p}{\partial y} = S_2 \\ \dfrac{\partial p}{\partial z} = S_3 \end{cases}$$

representing the Euler ($\nu = 0$) and Navier-Stokes equations, with S_i given by (4.5) using $x \equiv x_1,\ y \equiv x_2,\ z \equiv x_3$, it is necessary that $\nabla \times S = 0,\ S = (S_1, S_2, S_3)$. This condition is equivalent to follow system

$$(4.13)\qquad \begin{cases} \dfrac{\partial S_1}{\partial y} = \dfrac{\partial S_2}{\partial x} \\ \dfrac{\partial S_1}{\partial z} = \dfrac{\partial S_3}{\partial x} \\ \dfrac{\partial S_2}{\partial z} = \dfrac{\partial S_3}{\partial y} \end{cases}$$

which is the mentioned condition (4.4).

The first of these equations leads to

$$(4.14)\qquad \frac{\partial}{\partial y}\frac{\partial u_1}{\partial t} = \frac{\partial}{\partial x}\frac{\partial u_2}{\partial t}$$

respect to equality of temporal derivatives of velocity components 1 and 2, or

$$(4.15)\qquad \frac{\partial}{\partial t}\frac{\partial u_1}{\partial y} = \frac{\partial}{\partial t}\frac{\partial u_2}{\partial x}.$$

Repeating this reasoning for the second and third equations of (4.13) we come to

$$(4.16.1)\qquad \frac{\partial}{\partial t}\frac{\partial u_1}{\partial z} = \frac{\partial}{\partial t}\frac{\partial u_3}{\partial x},$$

$$(4.16.2)\qquad \frac{\partial}{\partial t}\frac{\partial u_2}{\partial z} = \frac{\partial}{\partial t}\frac{\partial u_3}{\partial y},$$

or

$$(4.17)\qquad \frac{\partial}{\partial t}\frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial t}\frac{\partial u_j}{\partial x_i}, \qquad 1 \le i, j \le 3,$$

i.e., $\nabla \times \frac{\partial}{\partial t} u = \frac{\partial}{\partial t} \nabla \times u = 0$ for all $t \ge 0$, which contains $\nabla \times u = 0$ and $\frac{\partial u}{\partial t} = 0$ as solutions.

Continuing the reasoning for the Laplacian terms, we have for the first equation of (4.13)

(4.18) $$\frac{\partial}{\partial y}\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u_1=\frac{\partial}{\partial x}\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u_2,$$

where we can have

(4.19) $$\frac{\partial}{\partial y}\frac{\partial^2 u_1}{\partial x^2}=\frac{\partial^3 u_2}{\partial x^3},\qquad \frac{\partial}{\partial y}\frac{\partial^2 u_1}{\partial y^2}=\frac{\partial^3 u_2}{\partial x\partial y^2},\qquad \frac{\partial}{\partial y}\frac{\partial^2 u_1}{\partial z^2}=\frac{\partial^3 u_2}{\partial x\partial z^2},$$

for the second equation of (4.13)

(4.20) $$\frac{\partial}{\partial z}\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u_1=\frac{\partial}{\partial x}\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u_3,$$

with the possibility

(4.21) $$\frac{\partial}{\partial z}\frac{\partial^2 u_1}{\partial x^2}=\frac{\partial^3 u_3}{\partial x^3},\qquad \frac{\partial}{\partial z}\frac{\partial^2 u_1}{\partial y^2}=\frac{\partial^3 u_3}{\partial x\partial y^2},\qquad \frac{\partial}{\partial z}\frac{\partial^2 u_1}{\partial z^2}=\frac{\partial^3 u_3}{\partial x\partial z^2},$$

and for the third equation of (4.13)

(4.22) $$\frac{\partial}{\partial z}\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u_2=\frac{\partial}{\partial y}\nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u_3,$$

again as the respective previous equalities, equaling each parcel of the left side to the respective parcel of the right side, we can have

(4.23) $$\frac{\partial}{\partial z}\frac{\partial^2 u_2}{\partial x^2}=\frac{\partial^3 u_3}{\partial y\partial x^2},\qquad \frac{\partial}{\partial z}\frac{\partial^2 u_2}{\partial y^2}=\frac{\partial^3 u_3}{\partial y^3},\qquad \frac{\partial}{\partial z}\frac{\partial^2 u_2}{\partial z^2}=\frac{\partial^3 u_3}{\partial y\partial z^2}.$$

For the nonlinear terms of (4.13) we have, for the first equation

(4.24) $$\frac{\partial}{\partial y}\left(u_1\frac{\partial u_1}{\partial x}+u_2\frac{\partial u_1}{\partial y}+u_3\frac{\partial u_1}{\partial z}\right)=\frac{\partial}{\partial x}\left(u_1\frac{\partial u_2}{\partial x}+u_2\frac{\partial u_2}{\partial y}+u_3\frac{\partial u_2}{\partial z}\right)$$

(4.25) $$\frac{\partial u_1}{\partial x}\frac{\partial u_1}{\partial y}+u_1\frac{\partial^2 u_1}{\partial x\partial y}+\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial y}+u_2\frac{\partial^2 u_1}{\partial y^2}+\frac{\partial u_1}{\partial z}\frac{\partial u_3}{\partial y}+u_3\frac{\partial^2 u_1}{\partial y\partial z}=$$
$$=\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x}+u_1\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial u_2}{\partial x}\frac{\partial u_2}{\partial y}+u_2\frac{\partial^2 u_2}{\partial x\partial y}+\frac{\partial u_3}{\partial x}\frac{\partial u_2}{\partial z}+u_3\frac{\partial^2 u_2}{\partial x\partial z}$$

for the second equation

(4.26) $$\frac{\partial}{\partial z}\left(u_1\frac{\partial u_1}{\partial x}+u_2\frac{\partial u_1}{\partial y}+u_3\frac{\partial u_1}{\partial z}\right)=\frac{\partial}{\partial x}\left(u_1\frac{\partial u_3}{\partial x}+u_2\frac{\partial u_3}{\partial y}+u_3\frac{\partial u_3}{\partial z}\right)$$

(4.27) $$\frac{\partial u_1}{\partial x}\frac{\partial u_1}{\partial z}+u_1\frac{\partial^2 u_1}{\partial x\partial z}+\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial z}+u_2\frac{\partial^2 u_1}{\partial y\partial z}+\frac{\partial u_1}{\partial z}\frac{\partial u_3}{\partial z}+u_3\frac{\partial^2 u_1}{\partial z^2}=$$
$$=\frac{\partial u_1}{\partial x}\frac{\partial u_3}{\partial x}+u_1\frac{\partial^2 u_3}{\partial x^2}+\frac{\partial u_2}{\partial x}\frac{\partial u_3}{\partial y}+u_2\frac{\partial^2 u_3}{\partial x\partial y}+\frac{\partial u_3}{\partial x}\frac{\partial u_3}{\partial z}+u_3\frac{\partial^2 u_3}{\partial x\partial z}$$

and for the last equation

$$\frac{\partial}{\partial z}\left(u_1\frac{\partial u_2}{\partial x}+u_2\frac{\partial u_2}{\partial y}+u_3\frac{\partial u_2}{\partial z}\right)=\frac{\partial}{\partial y}\left(u_1\frac{\partial u_3}{\partial x}+u_2\frac{\partial u_3}{\partial y}+u_3\frac{\partial u_3}{\partial z}\right) \tag{4.28}$$

$$\begin{aligned}\frac{\partial u_1}{\partial z}\frac{\partial u_2}{\partial x}+u_1\frac{\partial^2 u_2}{\partial x\partial z}+\frac{\partial u_2}{\partial y}\frac{\partial u_2}{\partial z}+u_2\frac{\partial^2 u_2}{\partial y\partial z}+\frac{\partial u_2}{\partial z}\frac{\partial u_3}{\partial z}+u_3\frac{\partial^2 u_2}{\partial z^2}=\\=\frac{\partial u_1}{\partial y}\frac{\partial u_3}{\partial x}+u_1\frac{\partial^2 u_3}{\partial x\partial y}+\frac{\partial u_2}{\partial y}\frac{\partial u_3}{\partial y}+u_2\frac{\partial^2 u_3}{\partial y^2}+\frac{\partial u_3}{\partial y}\frac{\partial u_3}{\partial z}+u_3\frac{\partial^2 u_3}{\partial y\partial z}.\end{aligned} \tag{4.29}$$

All these equations, from (4.17) to (4.29), admit for solution the condition

$$\frac{\partial u_i}{\partial x_j}=\frac{\partial u_j}{\partial x_i},\quad 1\le i,j\le 3, \tag{4.30}$$

in which case u is an irrotational vector, $\nabla\times u=0$, and so there is a velocity potential ϕ such that $u=\nabla\phi$. We will use this condition that u is irrotational and that also it is incompressible for the calculation of pressure in this special situation, coming to the known Bernoulli's law. For this also it is necessary to consider that the external force is conservative, i.e., it has a potential U such that $f=\nabla U$ and $\nabla\times f=0$, because then we will have satisfied the system (4.13) completely, when

$$\frac{\partial f_i}{\partial x_j}=\frac{\partial f_j}{\partial x_i},\quad 1\le i,j\le 3. \tag{4.31}$$

If $\nabla\times u=0$ and $\nabla\cdot u=0$ then

$$\nabla^2 u=\nabla(\nabla\cdot u)-\nabla\times(\nabla\times u)=0, \tag{4.32}$$

i.e., the Laplacian in the Navier-Stokes equations vanishes for any viscosity coefficient and the Navier-Stokes reduced to Euler equations.

If $\nabla\times u=0$ then the nonlinear term in vector form is simplified, according

$$(u\cdot\nabla)u=(\nabla\times u)\times u+\tfrac{1}{2}\nabla|u|^2=\tfrac{1}{2}\nabla|u|^2, \tag{4.33}$$

thus, using (4.32) and (4.33) and more the potentials of the velocity and external force, the Navier-Stokes (and Euler) equations reduced to

$$\nabla p+\frac{\partial}{\partial t}\nabla\phi+\tfrac{1}{2}\nabla|u|^2=\nabla U, \tag{4.34}$$

therefore the solution for pressure is

$$p=-\frac{\partial\phi}{\partial t}-\tfrac{1}{2}|u|^2+U+\theta(t), \tag{4.35}$$

the Bernoulli's law, where $\theta(t)$ is a generic time function, let's suppose $\theta(t) \in C^{\infty}$ a limited time function, a numeric constant or even zero.

If $u = \nabla\phi$ and $\nabla \cdot u = 0$, according we are admitting, then from incompressibility condition

(4.36) $$\nabla \cdot u = \frac{\partial}{\partial x}\frac{\partial\phi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial\phi}{\partial y} + \frac{\partial}{\partial z}\frac{\partial\phi}{\partial z} = 0$$

we come to the Laplace's equation

(4.37) $$\nabla^2\phi = 0,$$

where each possible solution gives the respective values of velocity components, such that

(4.38) $$u_1 = \frac{\partial\phi}{\partial x}, \qquad u_2 = \frac{\partial\phi}{\partial y}, \qquad u_3 = \frac{\partial\phi}{\partial z},$$

and the pressure is given by (4.35), with

(4.39) $$|u|^2 = u_1^2 + u_2^2 + u_3^2 = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2.$$

According Courant[5] (p.241), for $n = 2$ the "general solution" of the potential equation (or Laplace's equation) is the real part of any analytic function of the complex variable $x + iy$. For $n = 3$ one can also easily obtain solutions which depend on arbitrary functions. For example, let $f(w,t)$ be analytic in the complex variable w for fixed real t. Then, for arbitrary values of t, both the real and imaginary parts of the function

(4.40) $$u = f(z + ix\cos t + iy\sin t\,, t)$$

of the real variables x, y, z are solutions of the equation $\nabla^2 u = 0$. Further solutions may be obtained by superposition:

(4.41) $$u = \int_a^b f(z + ix\cos t + iy\sin t\,, t)dt.$$

For example, if we set

(4.42) $$f(w,t) = w^n e^{iht},$$

where n and h are integers, and integrate from $-\pi$ to $+\pi$, we get homogeneous polynomials

(4.43) $$u = \int_{-\pi}^{\pi}(z + ix\cos t + iy\sin t)^n\, e^{iht}dt$$

in x, y, z, following example given by Courant. Introducing polar coordinates $z = r\cos\theta, x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi$, we obtain

(4.44) $$u = 2r^n e^{ih\phi} \int_0^{\pi} (\cos\theta + i\sin\theta\cos t)^n \cos ht \ dt$$
$$= r^n e^{ih\phi} P_{n,h}(\cos\theta),$$

where $P_{n,h}(\cos\theta)$ are the associated Legendre functions.

§ 5

The series obtained in two first sections admitted that the incompressibility condition is satisfied for any $t \geq 0$, but we saw how difficult are the expressions (1.15) and (2.13) for that this can really occur for $t > 0$. In $t = 0$ this can be satisfied without great problems because the terms in t, t^2, t^3, etc. vanish. We can construct a solution using the indicated in equations (4.7), more general than (4.8), but the easier solution is to consider all coefficients, since the order zero, the free time power coefficient, as components of an irrotational and incompressible vector, this when the initial velocity is compatible with these conditions, i.e., our solutions for velocity in series of time (finite and also infinite) are, in this case, of a generic form

(5.1) $$u(x, y, z, t) = \sum_{k=0}^{m} X_k(x, y, z)\, T_k(t),$$

where all $X_k(x, y, z)$ are irrotational and incompressible vectors, i.e., solutions of Laplace's equation in vector form, they are harmonic functions, according superposition principle, as well as the respective velocity potentials are the scalar functions $\phi_k(x, y, z)$ such that

(5.2) $$\phi(x, y, z, t) = \sum_{k=0}^{m} \phi_k(x, y, z)\, T_k(t),$$

solutions of

(5.3) $$\nabla^2 \phi_k = 0,$$

where

(5.4) $$X_{i,k} = \frac{\partial}{\partial x_i} \phi_k,$$

with $X_k = (X_{1,k},\ X_{2,k},\ X_{3,k})$, $X_0 = u^0$ the initial velocity, $T_0(0) = 1$ and $T_k(0) = 0$ if $k \geq 1$. The functions $T_k(t) \in C^\infty(\mathbb{R})$ are limited for t finite, by our convention.

Resorting again to the mentioned theorem of uniqueness of series of powers in § 2 and using the Taylor's theorem (Courant[3], chap. VI), we can choose $T_k(t) = t^k$ and $m \to \infty$ in (5.1), i.e.,

(5.5) $$u(x, y, z, t) = \sum_{k=0}^{\infty} X_k(x, y, z)\, t^k,$$

and conclude that the coefficients of series of time given by (1.15) and (2.13) are, when u^0 is irrotational, solutions of Laplace's equation, at least in cases of conservative external forces and incompressible flows, for a same initial velocity, pressure, external force, compressibility condition and all boundary conditions, without contradictions with Lagrange[1] and Cauchy[2], and for this reason in these cases the Bernoulli's law is the correct solution for pressure in Euler and Navier-Stokes equations. We are assuming, but it is possible to prove in more detail, that always there some solution for Euler and Navier-Stokes equations in series of power (even, for example, $u = 0$), that it is analytical in a non-empty region for all $t \geq 0$ finite, and even not existing uniqueness of solutions, for each possible solution u it can be put in the form (5.5) using (1.15) or (2.13) or, for irrotational and incompressible flows, (5.3) and (5.4), existing the relation of equivalence (3.1), i.e.,

(5.6) $$\frac{1}{k!}\frac{\partial^k u_i}{\partial t^k}\Big|_{t=0} = X_{i,k},$$

for $i = 1, 2, 3$. Note that if it is not possible to make a series around $t = 0$ (for example, to the functions $\log t$, $\sqrt[3]{t}$, e^{-1/t^2}, according Courant[3], chap. VI) an other instant t_0 of convergence and remainder $R_{i,k\to\infty}$ zero must be found, and then replacing t^k by $(t - t_0)^k$ and the calculations in $t = 0$ by $t = t_0$ in previous equations.

It is not necessary the use of viscosity coefficient for smooth and incompressible fluids with conservative external force (or without any force). For non-stationary flows it is knows that the Lagrange's theorem[6],[7], as well as the Kelvin's circulation theorem[7],[8], is not valid for Navier-Stokes equations, but here it is implied that $\nu\nabla^2 u \neq 0$, the general case. The necessity of smooth velocity in whole space leads us to exclude all obstacles and regions without velocity of the fluid in study, which naturally occur using boundaries. The vorticity $\omega = \nabla \times u \neq 0$ is generated at solid boundaries[9], thus without boundaries (spatial domain $\Omega = \mathbb{R}^3$) no generation of vorticity, and without vorticity there is potential flow and vanishes the Laplacian of velocity if $\nabla \cdot u = 0$, then it is possible again the validity of Lagrange's theorem in an unlimited domain without boundaries and with both smooth and irrotational initial velocities and external forces, for incompressible fluids, because thus $\nabla^2 u = 0$, independently of viscosity coefficient.

Note that according Liouville's theorem[10], a harmonic function which is limited is constant, and equal to zero if it tends to zero at infinity. How the Millennium Problem requires a limited solution in all space for velocity and a limited initial velocity which goes to zero at infinity (in cases (A) and (C)), then we are forced to choose only $u^0 = 0$ for case (A) if $\nabla^2 u = 0$, what automatically implies the occurrence of case (C) due to infinite examples of prohibited u^0 and using any conservative external force f. If $\nabla^2 u = 0$ and $u^0 = 0$ then the unique possible solution for case (A), where it is necessary that $f = 0$, is $u = 0$ otherwise u would not be limited or u would be equal to constant greater than zero or any not null function of time, that violated the condition of bounded energy, equation (3.13).

If $\nabla^2 u \neq 0$ then the suitable general solution for Navier-Stokes equations is as indicated in sections § 1 and § 2 using (4.7), for an infinity of possible pressures of C^∞ class.

§ 6

I finish this article mentioning that Lamb[7] (chap. VII) gives a complete solution for velocity in Euler equations when the velocity vanishes at infinity.

He said that "no irrotational motion is possible in an incompressible fluid filling infinite space, and subject to the condition that the velocity vanishes at infinity." This is equivalent to the unique possible solution $u = 0$.

From this result he proved the following theorem: "The motion of a fluid which fills infinite space, and is at rest at infinity, is determinate when we know the values of the expansion (θ, say) and of the component angular velocities ξ, η, ζ, at all points of the region.", where

(6.1) $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \theta$$

is the named expansion and

(6.2.1) $$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 2\xi$$

(6.2.2) $$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 2\eta$$

(6.2.3) $$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2\zeta$$

are the equations for angular velocities. The components of the velocity are u, v, w, and vanish at infinity as well as $\theta,\ \xi,\ \eta,\ \zeta$.

Lamb obtain the solution for velocity

(6.3.1) $$u = -\frac{\partial \Phi}{\partial x} + \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}$$

(6.3.2) $$v = -\frac{\partial \Phi}{\partial y} + \frac{\partial F}{\partial z} - \frac{\partial H}{\partial x}$$

(6.3.3) $$w = -\frac{\partial \Phi}{\partial z} + \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}$$

where

(6.4.1) $$\Phi = \frac{1}{4\pi}\iiint \frac{\theta\prime}{r}\,dx'dy'dz'$$

(6.4.2) $$F = \frac{1}{2\pi}\iiint \frac{\xi\prime}{r}\,dx'dy'dz'$$

(6.4.3) $$G = \frac{1}{2\pi}\iiint \frac{\eta\prime}{r}\,dx'dy'dz'$$

(6.4.4) $$H = \frac{1}{2\pi}\iiint \frac{\zeta\prime}{r}\,dx'dy'dz'$$

the accents attached to θ, ξ, η, ζ are used to distinguish the values of these quantities at the point (x', y', z'), r denoting the distance

(6.5) $$r = \{(x - x')^2 + (y - y')^2 + (z - z')^2\}^{1/2}$$

and the integrations including all places which θ', ξ', η', ζ' differ from zero, respectively.

The following relations are valid:

(6.6) $$u_1 = -\frac{\partial\Phi}{\partial x}, \qquad v_1 = -\frac{\partial\Phi}{\partial y}, \qquad w_1 = -\frac{\partial\Phi}{\partial z},$$

(6.7) $$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = -\nabla^2\Phi = \theta$$

for solution of (6.1), and

(6.8) $$u_2 = \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}, \qquad v_2 = \frac{\partial F}{\partial z} - \frac{\partial H}{\partial x}, \qquad w_2 = \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y},$$

(6.9) $$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial w_2}{\partial z} = 0$$

(6.10) $$2\xi = \frac{\partial w_2}{\partial y} - \frac{\partial v_2}{\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z}\right) - \nabla^2 F$$

(6.11) $$2\eta = \frac{\partial u_2}{\partial z} - \frac{\partial w_2}{\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z}\right) - \nabla^2 G$$

(6.12) $$2\zeta = \frac{\partial v_2}{\partial x} - \frac{\partial u_2}{\partial y} = \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z}\right) - \nabla^2 H$$

(6.13) $$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = 0$$

(6.14) $$\nabla^2 F = -2\xi, \qquad \nabla^2 G = -2\eta, \qquad \nabla^2 H = -2\zeta,$$

for solution of (6.2).

The solution (u, v, w) given by (6.3) is obtained by superposition

(6.15.1) $$u = u_1 + u_2$$

(6.15.2) $$v = v_1 + v_2$$

(6.15.3) $$w = w_1 + w_2$$

From the reasoning of Lamb, derived from von Helmholtz, and following your calculations, we cannot understand *a priori* that the equations (6.3) are the correct solutions of Euler equations because the equations (6.2) are not the Euler equations and the pressure is not mentioned, i.e., the relation (4.4) is not verified.

The equations (6.3) are a form of representation of any vector $\boldsymbol{u} = (u, v, w)$, a fluid flow or not, satisfying appropriate smoothness and decay conditions, in a sum of one gradient vector ($\boldsymbol{u_\Phi} = -\nabla\Phi$), the velocity potential, and one rotational vector ($\boldsymbol{u_\omega} = \nabla \times (F, G, H)$, with $\nabla \cdot (F, G, H) = 0$), which is the know Helmholtz or Hodge decomposition[11]. Adopting the minus sign of Lamb in $\nabla\Phi$,

(6.16) $$\boldsymbol{u} = \boldsymbol{u_\Phi} + \boldsymbol{u_\omega} = -\nabla\Phi + \nabla \times \boldsymbol{\psi},$$

where Φ is the scalar potential and $\boldsymbol{\psi} = (F, G, H)$ is the vector potential, with

(6.17) $$\nabla^2\boldsymbol{\psi} = -\boldsymbol{u_\omega}.$$

The solution given by Lamb in a sum derived of one scalar potential and one vector potential can be expressed as a single vector, gradient of scalar potential, in case of incompressible flow.

If $u = (u_1, u_2, u_3)$ and $A = (A_1, A_2, A_3)$ are vectors, ϕ is a scalar function, u, A, ϕ are smooth functions and we define

(6.18) $$u = \nabla \times A = \nabla\phi$$

then we have

(6.19) $$\nabla \cdot u = 0, \qquad \nabla \times u = 0, \qquad \nabla^2 u = 0,$$

and

(6.20) $$\phi = \int_L (\nabla \times A) \cdot dl = \int_L u \cdot dl,$$

since that $\nabla \times A$ is a gradient vector function, as well as the velocity u.

For that $\nabla \times A$ is gradient it is necessary that, for $1 \le i, j \le 3$,

(6.21) $$\frac{\partial}{\partial x_i}(\nabla \times A)_j = \frac{\partial}{\partial x_j}(\nabla \times A)_i.$$

Developing we have, with $x_1 \equiv x, x_2 \equiv y, x_3 \equiv z$,

(6.22.1) $$\frac{\partial^2 A_3}{\partial x^2}+\frac{\partial^2 A_3}{\partial y^2}=\frac{\partial}{\partial x}\frac{\partial A_1}{\partial z}+\frac{\partial}{\partial y}\frac{\partial A_2}{\partial z}=\frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}\right)$$

(6.22.2) $$\frac{\partial^2 A_2}{\partial x^2}+\frac{\partial^2 A_2}{\partial z^2}=\frac{\partial}{\partial x}\frac{\partial A_1}{\partial y}+\frac{\partial}{\partial z}\frac{\partial A_3}{\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_3}{\partial z}\right)$$

(6.22.3) $$\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}=\frac{\partial}{\partial y}\frac{\partial A_2}{\partial x}+\frac{\partial}{\partial z}\frac{\partial A_3}{\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)$$

When $\nabla \cdot A = 0$ then comes

(6.23.1) $$\frac{\partial^2 A_1}{\partial x^2}+\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}=0$$

(6.23.2) $$\frac{\partial^2 A_2}{\partial x^2}+\frac{\partial^2 A_2}{\partial y^2}+\frac{\partial^2 A_2}{\partial z^2}=0$$

(6.23.3) $$\frac{\partial^2 A_3}{\partial x^2}+\frac{\partial^2 A_3}{\partial y^2}+\frac{\partial^2 A_3}{\partial z^2}=0$$

i.e., each component of the vector A is a harmonic function and so

(6.24) $$\nabla^2 A = 0.$$

We see then that it is possible to have simultaneously a potential flow ($u = \nabla\phi$) and a vortex motion ($u = \nabla \times A$), since that $\nabla \cdot A = 0$, without be necessary that neither $\nabla \times A = 0$ nor $u = 0$. In this case the equation (6.16) can be rewritten as

(6.25) $$u = u_\phi = u_\omega = \nabla\phi = \nabla \times \mathrm{A},$$

where we use $\phi = -\Phi$ and $A = \psi$, without use of bold characters for indicate vectors. As we saw in section § 4 for incompressible and potential flow, the pressure is given by Bernoulli's law, equation (4.34),

(6.26) $$p = -\frac{\partial\phi}{\partial t} - \frac{1}{2}|u|^2 + U + \theta(t),$$

because here too $\nabla \cdot u = 0$ and $u = \nabla\phi$, even though $u = \nabla \times A$ (due to lack of a better name I also called *vortex motion* the not null velocity generated by a curl of a not null vector).

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Lagrange, grande matemático.

A Matemática é um desafio quando se começa,
uma alegria quando pensamos estar certos pela 1ª vez,
uma vergonha quando se erra,
tortura quando o problema é difícil,
esporte gostoso quando o problema é fácil,
um alívio quando se termina,
um luxo quando se prova tudo.
Acima de tudo é grande beleza.

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21 – Draft on a Problem in Euler and Navier-Stokes Equations

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Abstract – A brief draft respect to a problem found in the equations of Euler and Navier-Stokes, whose adequate treatment solves a centennial problem about the solution of these equations and a most correct modeling of fluid movement.

Keywords – Euler equations, Navier-Stokes equations, Eulerian description, Lagrangian description.

§ 1

Motived by my work on Lagrangian and Eulerian descriptions in Euler[1] and Navier-Stokes[2] equations, where I used for velocity's components the relation

$$(1) \qquad \begin{cases} \dfrac{\partial u_i}{\partial x_j} = 0, \ i \neq j, \\ \partial x_i = u_i \partial t \end{cases}$$

because the construction of the non-linear terms $u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y} + u_3 \frac{\partial u_i}{\partial z}$ in these equations was based on the 2nd law of Newton, $F = ma$, making

$$(2) \qquad a = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt},$$

with

$$(3) \qquad \begin{cases} \dfrac{dx}{dt} = u_1 \\ \dfrac{dy}{dt} = u_2 \\ \dfrac{dz}{dt} = u_3 \end{cases}$$

I now realize that it is possible, or better said, it is necessary for a more appropriate modeling of fluids in motion, the simultaneous use of both velocities, in the Lagrangian and Eulerian descriptions, in the same equation (Euler equations or Navier-Stokes equations), what we will see in § 3. For while, we think in each description or formulation separate of the other, i.e., used exclusively, in an equation.

The equations (3), writing synthetically as $\frac{dx_i}{dt} = u_i$, with $x_1 \equiv x,\ x_2 \equiv y$, $x_3 \equiv z$, show us that the velocity's component u_i is dependent only of coordinate x_i, regardless of the values of others $x_j,\ j \neq i$, justifying the use of (1).

Following this idea, the original system

$$(4)\quad \begin{cases} \frac{\partial p}{\partial x} + \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + u_3 \frac{\partial u_1}{\partial z} = \nu \nabla^2 u_1 + \frac{1}{3} \nu \nabla_1 (\nabla \cdot u) + f_1 \\ \frac{\partial p}{\partial y} + \frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + u_3 \frac{\partial u_2}{\partial z} = \nu \nabla^2 u_2 + \frac{1}{3} \nu \nabla_2 (\nabla \cdot u) + f_2 \\ \frac{\partial p}{\partial z} + \frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x} + u_2 \frac{\partial u_3}{\partial y} + u_3 \frac{\partial u_3}{\partial z} = \nu \nabla^2 u_3 + \frac{1}{3} \nu \nabla_3 (\nabla \cdot u) + f_3 \end{cases}$$

can be transformed in

$$(5)\quad \begin{cases} \frac{1}{u_1} \frac{\partial p}{\partial t} + \frac{Du_1}{Dt} = \nu (\nabla^2 u_1)|_t + \frac{1}{3} \nu (\nabla_1 (\nabla \cdot u))|_t + f_1|_t \\ \frac{1}{u_2} \frac{\partial p}{\partial t} + \frac{Du_2}{Dt} = \nu (\nabla^2 u_2)|_t + \frac{1}{3} \nu (\nabla_2 (\nabla \cdot u))|_t + f_2|_t \\ \frac{1}{u_3} \frac{\partial p}{\partial t} + \frac{Du_3}{Dt} = \nu (\nabla^2 u_3)|_t + \frac{1}{3} \nu (\nabla_3 (\nabla \cdot u))|_t + f_3|_t \end{cases}$$

thus (4) and (5) are equivalent systems, according validity of (2) and (3), since that the partial derivatives of the pressure and velocities were correctly transformed to the variable time, using $\partial x = u_1 \partial t,\ \partial y = u_2 \partial t,\ \partial z = u_3 \partial t$. The nabla and Laplacian operators are considered calculated in Lagrangian formulation, i.e., in the variable time. Likewise for the calculation of $\frac{Du}{Dt}$, following (2), and external force f, using $x = x(t), y = y(t), z = z(t)$. The integration of the system (5) shows that anyone of its equations can be used for solve it, and the results must be equals each other. Then this is a condition to the occurrence of solutions.

We use the following transformations (omitting the use of $|_t$, the calculation at time t of the position (x, y, z) of the moving particle):

$$(6.1)\quad \frac{\partial u_i}{\partial x_j} = \begin{cases} \frac{\partial u_i/\partial t}{\partial x_i/\partial t} = \frac{1}{u_i} \frac{\partial u_i}{\partial t},\ i = j \\ 0,\ i \neq j \end{cases}$$

$$(6.2)\quad \nabla \cdot u = \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j} = \sum_{j=1}^{3} \frac{1}{u_j} \frac{\partial u_j}{\partial t}$$

$$(6.3)\quad \nabla_i (\nabla \cdot u) = \frac{\partial}{\partial x_i} \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} \right) = \frac{\partial}{\partial x_i} \frac{\partial u_i}{\partial x_i} = \frac{\partial/\partial t}{\partial x_i/\partial t} \frac{1}{u_i} \frac{\partial u_i}{\partial t} = \frac{1}{u_i^2} \left[-\frac{1}{u_i} \left(\frac{\partial u_i}{\partial t} \right)^2 + \frac{\partial^2 u_i}{\partial t^2} \right]$$

and

(7.1) $$\frac{\partial^2 u_i}{\partial x_j^2} = \begin{cases} \frac{1}{u_i^2}\left[-\frac{1}{u_i}\left(\frac{\partial u_i}{\partial t}\right)^2 + \frac{\partial^2 u_i}{\partial t^2}\right], \ i = j \\ 0, \ i \neq j \end{cases}$$

(7.2) $$\nabla^2 u_i = \frac{\partial^2 u_i}{\partial x_i^2} = \frac{1}{u_i^2}\left[-\frac{1}{u_i}\left(\frac{\partial u_i}{\partial t}\right)^2 + \frac{\partial^2 u_i}{\partial t^2}\right]$$

and thus the system (5) can be integrated, finding the pressure p.

§ 2

Without passing through the Lagrangian formulation, given a velocity $u(x, y, z, t)$ at least two times differentiable with respect to spatial coordinates and one respect to time and an integrable external force $f(x, y, z, t)$, perhaps the better expression for the solution of the equation (4) is

(8) $$p(x, y, z, t) = \int_L S \cdot dl + \theta(t) = \sum_{i=1}^{3} \int_{P_i^0}^{P_i} S_i dx_i + \theta(t),$$
$$S = (S_1, \ S_2, \ S_3),$$
$$S_i = -\left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}\right) + \nu(\nabla^2 u_i) + \frac{1}{3}\nu\big(\nabla_i(\nabla \cdot u)\big) + f_i,$$

supposing possible the integrations and that the vector $S = -\left[\frac{\partial u}{\partial t} + (u \cdot \nabla)u\right] + \nu\nabla^2 u + \frac{1}{3}\nu\nabla(\nabla \cdot u) + f$ is a gradient function, where it is necessary that

(9) $$\frac{\partial S_i}{\partial x_j} = \frac{\partial S_j}{\partial x_i}.$$

This is the development of the solution of (4) for the specific path L going parallely (or perpendicularly) to axes X, Y and Z from $(x_1^0, x_2^0, x_3^0) \equiv (x_0, y_0, z_0)$ to $(x_1, x_2, x_3) \equiv (x, y, z)$, since that the solution (8) is valid for any piecewise smooth path L. We can choose $P_1^0 = (x_0, y_0, z_0)$, $P_2^0 = (x, y_0, z_0)$, $P_3^0 = (x, y, z_0)$ for the origin points and $P_1 = (x, y_0, z_0)$, $P_2 = (x, y, z_0)$, $P_3 = (x, y, z)$ for the destination points. $\theta(t)$ is a generic time function, physically and mathematically reasonable, for example with $\theta(0) = 0$ or adjustable for some given condition. Again we have seen that the system of Navier-Stokes equations has no unique solution, only given initial conditions, supposing that there is some solution. We can choose different velocities that have the same initial velocity and also result, in general, in different pressures.

The remark given for the system (5), when used in (4), leads us to the following conclusion: the integration of the system (4), confronting with (5), shows that anyone of its equations can be used for solve it, and the results must be equals each other. Then again this is a condition to the occurrence of solutions, which shows to us the possibility of existence of "breakdown" solutions, as defined in [3].

By other side, using the first condition (1), $\frac{\partial u_i}{\partial x_j} = 0$ if $i \neq j$, due to Lagrangian formulation, where $u_i = \frac{dx_i}{dt}$, the original system (4) is simplified as

$$(10) \qquad \begin{cases} \frac{\partial p}{\partial x} + \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} = \frac{4}{3}\nu \frac{\partial^2 u_1}{\partial x^2} + f_1 \\ \frac{\partial p}{\partial y} + \frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial y} = \frac{4}{3}\nu \frac{\partial^2 u_2}{\partial y^2} + f_2 \\ \frac{\partial p}{\partial z} + \frac{\partial u_3}{\partial t} + u_3 \frac{\partial u_3}{\partial z} = \frac{4}{3}\nu \frac{\partial^2 u_3}{\partial z^2} + f_3 \end{cases}$$

where u_i is a function only of the respective x_i and t, but not x_j if $j \neq i$. When it is required the incompressibility condition, $\nabla \cdot u = \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right) = 0$, then the constant $\frac{4}{3}$ in (10) should be replaced by 1.

If the external force has potential, $f = \nabla V$, then the system (10) has solution

$$(11) \qquad \begin{aligned} p &= \sum_{i=1}^{3} \int_{P_i^0}^{P_i} \left[-\left(\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x_i}\right) + \frac{4}{3}\nu \frac{\partial^2 u_i}{\partial x_i^2} + f_i \right] dx_i + \theta(t) \\ &= V + \sum_{i=1}^{3} \int_{x_i^0}^{x_i} \left[-\left(\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x_i}\right) + \frac{4}{3}\nu \frac{\partial^2 u_i}{\partial x_i^2} \right] dx_i + \theta(t), \end{aligned}$$

$V = \int_L f \cdot dl$, which although similar to (8) has the solubility guaranteed by the special functional dependence of the components of the vector u, i.e., $u_i = u_i(x_i, t)$, with $\frac{\partial u_i}{\partial x_j} = 0$ if $i \neq j$, supposing u, its derivatives and f integrable vectors. In this case the vector S described in (8) is always a gradient function, i.e., the relation (9) is satisfied. Note that if f is not an irrotational or gradient vector, i.e., if it does not have a potential, then the system (10), with $u_i = u_i(x_i, t)$, it has no solution, the case of "breakdown" solution in [3].

When the incompressibility condition is imposed ($\nabla \cdot u = 0$) we have, using (1), a small variety of possible solutions for velocity, of the form

$$(12) \qquad u_i(x_i, t) = A_i(t) x_i + B_i(t),$$

$A_i, B_i \in C^\infty([0, \infty[)$, with

$$(13) \qquad A_1(t) + A_2(t) + A_3(t) = 0,$$

if the coordinates x_1, x_2, x_3 are independent of each other. In this case it is valid $\nabla^2 u = 0$, i.e., the system of equations has a solution for velocity independent of viscosity coefficient, equal to Euler equations, and except when $u = 0$ (for some or all $t \geq 0$) we have always $\int_{\mathbb{R}^3} |u|^2 \, dxdydz \to \infty$, the occurrence of unbounded or unlimited energy, what is not difficult to see.

Another class of solutions S for velocity gives more possibility for the construction of the components of velocity u_i, but maintains a bond between $x_1,\ x_2,\ x_3$ and t such that

(14) $\quad S = \{(u_1, u_2, u_3);\ u_i \in C^1(\mathbb{R} \times \mathbb{R}_0^+),\ (x_1, x_2, x_3, t) \in \mathbb{R}^3 \times \mathbb{R}_0^+,\ \nabla \cdot u = 0\},$

where $\mathbb{R}_0^+ = [0, \infty[$, and there is a scalar function φ_3 with $x_3 = \varphi_3(x_1, x_2, t)$ or similarly $x_1 = \varphi_1(x_2, x_3, t)$ or $x_2 = \varphi_2(x_1, x_3, t)$. The dependence between $x_1,\ x_2,\ x_3$ and t is necessary for that $\nabla \cdot u = 0$ in these points (x_1, x_2, x_3) at each time t, forming a surface or manifold which is the domain of the solutions and which varies in time.

§ 3

The system (3), for the sake of mathematical rigor, needs to be replaced by

$$(15) \qquad \begin{cases} \frac{dx}{dt} = u_1(t) \\ \frac{dy}{dt} = u_2(t) \\ \frac{dz}{dt} = u_3(t) \end{cases}$$

emphasizing that the velocity components that appear as the time derivative of the coordinate (x, y, z) are legitimate functions of time, i.e., can be considered as representative of the Lagrangian description, $u_i(t)$, unlike the derivatives of u_i in $\frac{\partial u_i}{\partial t}, \frac{\partial u_i}{\partial x_j}$, $\nabla \cdot u$ and $\nabla^2 u_i$, that are in the Eulerian description, function of (x, y, z, t).

Representing the Eulerian velocity and respective components with the letter E indicated as upper index, and the corresponding Lagrangian components with the letter L, the system (4) is rewritten as

$$(16) \quad \begin{cases} \frac{\partial p}{\partial x} + \frac{\partial u_1^E}{\partial t} + u_1^L \frac{\partial u_1^E}{\partial x} + u_2^L \frac{\partial u_1^E}{\partial y} + u_3^L \frac{\partial u_1^E}{\partial z} = \nu \nabla^2 u_1^E + \frac{1}{3} \nu \nabla_1 (\nabla \cdot u^E) + f_1 \\ \frac{\partial p}{\partial y} + \frac{\partial u_2^E}{\partial t} + u_1^L \frac{\partial u_2^E}{\partial x} + u_2^L \frac{\partial u_2^E}{\partial y} + u_3^L \frac{\partial u_2^E}{\partial z} = \nu \nabla^2 u_2^E + \frac{1}{3} \nu \nabla_2 (\nabla \cdot u^E) + f_2 \\ \frac{\partial p}{\partial z} + \frac{\partial u_3^E}{\partial t} + u_1^L \frac{\partial u_3^E}{\partial x} + u_2^L \frac{\partial u_3^E}{\partial y} + u_3^L \frac{\partial u_3^E}{\partial z} = \nu \nabla^2 u_3^E + \frac{1}{3} \nu \nabla_3 (\nabla \cdot u^E) + f_3 \end{cases}$$

being the pressure p and external force f implicitly defined in the Eulerian description. A more concise notation for (16) is simply, for $i = 1, 2, 3$,

$$(17)\quad \frac{\partial p}{\partial x_i}+\frac{\partial u_i}{\partial t}+\alpha_1\frac{\partial u_i}{\partial x}+\alpha_2\frac{\partial u_i}{\partial y}+\alpha_3\frac{\partial u_i}{\partial z}=\nu\nabla^2 u_i+\frac{1}{3}\nu\nabla_i(\nabla\cdot u)+f_i,$$

where p, f_i, u and u_i are in Eulerian description and $\alpha_i=\alpha_i(t)$ in Lagrangian description, i.e., $\alpha_i=\frac{dx_i}{dt}$, with the radius vector $r=(x_1,x_2,x_3)=(x,y,z)$ function of time and indicating a motion of a specific particle of fluid.

The equations (16) and (17) shows us that the nonlinear form disappear, facilitating the obtaining of its solutions, transforming when $\nabla\cdot u=0$ into a linear and second-order partial differential equation of the elliptic type, already well-studied[4]. If $\nu=0$ (Euler equations) we have equations of first order, obviously, which is also widely studied[5]. We realize that for each possible value of α_i it is possible to obtain different values of u_i, and reciprocally, i.e., there is not an one-one correspondence between α_i and u_i, thus it is convenient choose more easy time functions for the $\alpha_i(t)$, provided that compatible with the physical problem to be studied.

Nevertheless, even though it is very interesting to study other mathematical solutions for the original system (4) or the new system (16), I understand that the final conclusion made in [1] and [2] remains valid: it is possible to exist velocities in the Eulerian formulation that do not correspond to a real movement of particles of a fluid, according to the Lagrangian formulation. When I wrote this the first time I did not have the equations (16) and (17) deduced here, but if it is true (as it is) that we should have (3) and (15) for a motion of fluid particle, then x_i and its respective velocity u_i are closely related, and the initial use of (1) in section § 1 is valid. This is an excellent question to be examined computationally. Being correct, the solution (11) for pressure must therefore be replaced by

$$(18)\qquad \begin{aligned} p &= \sum_{i=1}^{3}\int_{P_i^0}^{P_i}\left[-\left(\frac{\partial u_i}{\partial t}+\alpha_i\frac{\partial u_i}{\partial x_i}\right)+\frac{4}{3}\nu\frac{\partial^2 u_i}{\partial x_i^2}+f_i\right]dx_i+\theta(t)\\ &= V+\sum_{i=1}^{3}\int_{x_i^0}^{x_i}\left[-\left(\frac{\partial u_i}{\partial t}+\alpha_i\frac{\partial u_i}{\partial x_i}\right)+\frac{4}{3}\nu\frac{\partial^2 u_i}{\partial x_i^2}\right]dx_i+\theta(t),\end{aligned}$$

where $\alpha_i=\alpha_i(t)$ is the component i of the velocity in Lagrangian description of a particle of fluid in motion, $u_i=u_i(x_i,\ t)$ is the component i of the velocity in Eulerian description, and the other meanings already given in the previous section. As we have already seen, when it is required the incompressibility condition then the constant $\frac{4}{3}$ in (18) should be replaced by 1 and the general solution (12) for velocity with the condition (13) remains valid, if the coordinates $x_1,\ x_2,\ x_3$ are independent of each other, as well as (14) with possible dependence between $x_1,\ x_2,\ x_3$ and t.

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22 – On a Problem in Euler and Navier-Stokes Equations

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Abstract – A study respect to a problem found in the equations of Euler and Navier-Stokes, whose adequate treatment solves a centennial problem about the solution of these equations and a most correct modeling of fluid in movement.

Keywords – Euler equations, Navier-Stokes equations, Eulerian description, Lagrangian description, breakdown solutions, non-uniqueness, vector pressure.

1 – Introduction

This article is a better version of [1], which in turn was motived by my works on Lagrangian and Eulerian descriptions in Euler[2] and Navier-Stokes[3] equations, where I used for velocity's components the relation

(1.1) $$\begin{cases} \frac{\partial u_i}{\partial x_j} = 0, \ i \neq j, \\ \partial x_i = u_i \partial t \end{cases}$$

because the construction of the non-linear terms $u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y} + u_3 \frac{\partial u_i}{\partial z}$ in these equations was based on the 2nd law of Newton, $F = ma$, making

(1.2) $$a = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt},$$

with

(1.3) $$\begin{cases} \frac{dx}{dt} = u_1 \\ \frac{dy}{dt} = u_2 \\ \frac{dz}{dt} = u_3 \end{cases}$$

I now realize that it is possible, or better said, it is necessary for a more appropriate modeling of fluids in motion, the simultaneous use of both velocities, in the Lagrangian and Eulerian descriptions, in the same equation (Euler equations or Navier-Stokes equations), what we will see in section 4. For while, we think in each description or formulation separate of the other, i.e., used exclusively, in an equation.

The equations (1.3), writing synthetically as $\frac{dx_i}{dt} = u_i$, with $x_1 \equiv x,\ x_2 \equiv y$, $x_3 \equiv z$, show us that the velocity's component u_i is dependent only of coordinate x_i, regardless of the values of others $x_j,\ j \neq i$, justifying the use of (1.1).

Following this idea, the original system for $n = 3$ spatial dimension and volumetric mass density $\rho = 1$,

$$(1.4)\qquad \begin{cases} \frac{\partial p}{\partial x} + \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + u_3 \frac{\partial u_1}{\partial z} = \nu \nabla^2 u_1 + \frac{1}{3} \nu \nabla_1 (\nabla \cdot u) + f_1 \\ \frac{\partial p}{\partial y} + \frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + u_3 \frac{\partial u_2}{\partial z} = \nu \nabla^2 u_2 + \frac{1}{3} \nu \nabla_2 (\nabla \cdot u) + f_2 \\ \frac{\partial p}{\partial z} + \frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x} + u_2 \frac{\partial u_3}{\partial y} + u_3 \frac{\partial u_3}{\partial z} = \nu \nabla^2 u_3 + \frac{1}{3} \nu \nabla_3 (\nabla \cdot u) + f_3 \end{cases}$$

can be transformed in

$$(1.5)\qquad \begin{cases} \frac{1}{u_1} \frac{\partial p}{\partial t} + \frac{Du_1}{Dt} = \nu (\nabla^2 u_1)|_t + \frac{1}{3} \nu (\nabla_1 (\nabla \cdot u))|_t + f_1|_t \\ \frac{1}{u_2} \frac{\partial p}{\partial t} + \frac{Du_2}{Dt} = \nu (\nabla^2 u_2)|_t \ + \frac{1}{3} \nu (\nabla_2 (\nabla \cdot u))|_t + f_2|_t \\ \frac{1}{u_3} \frac{\partial p}{\partial t} + \frac{Du_3}{Dt} = \nu (\nabla^2 u_3)|_t + \frac{1}{3} \nu (\nabla_3 (\nabla \cdot u))|_t + f_3|_t \end{cases}$$

thus (1.4) and (1.5) are equivalent systems, according validity of (1.2) and (1.3), since that the partial derivatives of the pressure and velocities were correctly transformed to the variable time, using $\partial x = u_1 \partial t,\ \partial y = u_2 \partial t,\ \partial z = u_3 \partial t$. The nabla and Laplacian operators are considered calculated in Lagrangian formulation, i.e., in the variable time. Likewise for the calculation of $\frac{Du}{Dt}$, following (1.2), and external force f, using $x = x(t), y = y(t), z = z(t)$. The integration of the system (1.5) shows that anyone of its equations can be used for solve it, and the results must be equals each other, except for a constant of integration. Then this is a condition to the occurrence of solutions, if the velocity u and external force f are given and the pressure p must be calculated.

We use the following transformations (omitting the use of $|_t$, the calculation at time t of the position (x, y, z) of the moving particle):

$$(1.6.1)\qquad \frac{\partial u_i}{\partial x_j} = \begin{cases} \frac{\partial u_i / \partial t}{\partial x_i / \partial t} = \frac{1}{u_i} \frac{\partial u_i}{\partial t},\ i = j \\ 0,\ i \neq j \end{cases}$$

$$(1.6.2)\qquad \nabla \cdot u = \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j} = \sum_{j=1}^{3} \frac{1}{u_j} \frac{\partial u_j}{\partial t}$$

$$(1.6.3)\qquad \nabla_i (\nabla \cdot u) = \frac{\partial}{\partial x_i} \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} \right) = \frac{\partial}{\partial x_i} \frac{\partial u_i}{\partial x_i} = \frac{\partial / \partial t}{\partial x_i / \partial t} \frac{1}{u_i} \frac{\partial u_i}{\partial t}$$

$$= \frac{1}{u_i^2}\left[-\frac{1}{u_i}\left(\frac{\partial u_i}{\partial t}\right)^2 + \frac{\partial^2 u_i}{\partial t^2}\right]$$

and

$$(1.7.1) \qquad \frac{\partial^2 u_i}{\partial x_j^2} = \begin{cases} \frac{1}{u_i^2}\left[-\frac{1}{u_i}\left(\frac{\partial u_i}{\partial t}\right)^2 + \frac{\partial^2 u_i}{\partial t^2}\right], \ i = j \\ 0, \ i \neq j \end{cases}$$

$$(1.7.2) \qquad \nabla^2 u_i = \frac{\partial^2 u_i}{\partial x_i^2} = \frac{1}{u_i^2}\left[-\frac{1}{u_i}\left(\frac{\partial u_i}{\partial t}\right)^2 + \frac{\partial^2 u_i}{\partial t^2}\right]$$

and thus the system (1.5) can be integrated, finding the pressure p on the particle in motion.

From equations (1.5) to (1.7) it is possible to construct the Euler and Navier-Stokes equations in a new Lagrangian description from the respective Eulerian description. Although in the Eulerian description a position (x, y, z) refers to any position, generally adopted as fixed in time, when we want it to refer to a particle motion we arrive at this new Lagrangian description aforementioned. While in this Introduction the equations (1.5) to (1.7) lead to a new Lagrangian formulation of the Euler and Navier-Stokes equations, in section 4 and Conclusion we will see the respective modification of the Eulerian formulation, or a kind of mixed description.

Next, in section 2 we will deduce the equations of Euler, in section 3 we will deduce the equations of Navier-Stokes, the section 4 will show a new expression for the equations of Euler and Navier-Stokes, with the simultaneous use of the Eulerian and Lagrangian formulations (or a correction of the Eulerian formulation), and in the section 5 we will give examples of the need to use the new equations here deduced, rather than the traditional equations known.

The section 6 deals with the issue of breakdown solutions, section 7 on non-uniqueness of solutions, and section 8, finally, will be our conclusion.

Except for sections 2 and 3 we use mass density $\rho = 1$, otherwise if it is necessary replace the pressure p by p/ρ and the viscosity coefficient ν by ν/ρ. I believe that the new equations presented here really need to be accepted, and we will have exact solutions found faster for the various applications.

2 – Deduction of Euler equations

Many deductions of the Euler (and Navier-Stokes) equations start from the assumption that the pressure is a scalar magnitude, equal in all directions at the same point. Particularly I do not think this needs to be this way, or rather, I believe that the pressure can be a vector entity, rather than a scalar, so there is a vector pressure such that $p = (p_1,\ p_2,\ p_3)$, which would make it extraordinarily simple to solve the Euler and Navier-Stokes equations. Instead of using the gradient of p, the vector $\nabla p \equiv \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right)$, we should use the vector $\left(\frac{\partial p_1}{\partial x}, \frac{\partial p_2}{\partial y}, \frac{\partial p_3}{\partial z}\right)$, and then

(2.1) $$p_i = \int_{x_i^0}^{x_i} \left[-\left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}\right) + f_i\right] dx_i + \theta_i(t),$$

for $i = 1, 2, 3$, solves the Euler equations, i.e., calculate the components of pressure given the velocity and an external force, conservative or not, and an "arbitrary" (well behaved, smooth, physically reasonable) function of time $\theta(t)$. This will be a pressure with independence of path, depending only of the initial and final points, $(x_1^0,\ x_2^0,\ x_3^0)$ and $(x_1,\ x_2,\ x_3)$ respectively. Without wanting to deepen this subject now, we will continue using scalar pressure, at least in general.

We will follow the deduction of Landau & Lifshitz[4] and as they we will use $\boldsymbol{v}$ to indicate velocity and bold characters for vectors. They *emphasize that* $\boldsymbol{v}(x, y, z, t)$ *is the velocity of the fluid at a given point* (x, y, z) *in space and at a given time* t*, i.e., it refers to fixed points in space and not to specific particles of the fluid; in the course of time, the latter move about in space. The same remarks apply to* ρ *and* p.

Let us considerer some volume in the fluid. The total force acting on this volume is equal to the integral (the minus signal indicates a compressive force)

$$-\oint p\, d\boldsymbol{f}$$

of the pressure, taken over the surface bounding the volume. Transforming it to a volume integral, we have

(2.2) $$-\oint p\, d\boldsymbol{f} = -\int \boldsymbol{grad}\, p\, dV.$$

Hence we see that the fluid surrounding any volume element dV *exerts on that element a force* $-dV\, \boldsymbol{grad}\, p$*. In other words, we can say that a force* $-\boldsymbol{grad}\, p$ *acts on unit volume of the fluid.*

See that an equality similar to Gauss's law was used with the previous acceptance of scalar pressure. The same equality, with equal reason, could be rewritten, using a vector pressure $\boldsymbol{p} = (p_1,\ p_2,\ p_3)$, as

(2.3) $$-\oint \boldsymbol{p}\, df = -\int \left(\frac{\partial p_1}{\partial x}, \frac{\partial p_2}{\partial y}, \frac{\partial p_3}{\partial z}\right) dV,$$

i.e., without assuming that $p_1 = p_2 = p_3 = p$ and with the convention that $\boldsymbol{p}$ is a resultant vector of pressures applied on a volume element $dV = dx\, dy\, dz$ centered at point (x, y, z) and time t.

Continuing Landau & Lifshitz, *we can now write the equation of motion of a volume element in the fluid by equating the force* $-\boldsymbol{grad}\, p$ *to the produt of the mass per unit volume* (ρ) *and the acceleration* $d\boldsymbol{v}/dt$*:*

(2.4) $$\rho\, d\boldsymbol{v}/dt = -\boldsymbol{grad}\, p.$$

The derivative $d\boldsymbol{v}/dt$ *which appears here denotes not the rate of change of the fluid velocity at a fixed point in space, but the rate of change of the velocity of a given fluid particle as it moves about in space. This derivative has to be expressed in terms of quantities referring to points fixed in space. To do so, we notice that the change* $d\boldsymbol{v}$ *in the velocity of the given fluid particle during the time* dt *is composed of two parts, namely the change during* dt *in the velocity at a point fixed in space, and the difference between the velocities (at the same instant) at two points* $d\boldsymbol{r}$ *apart, where* $d\boldsymbol{r}$ *is the distance moved by the given fluid particle during the time* dt*. The first part is* $(\partial\boldsymbol{v}/\partial t)dt$*, where the derivative* $\partial\boldsymbol{v}/\partial t$ *is taken for constant* x, y, z*, i.e., at the given point in space. The second part is*

(2.5) $$dx\frac{\partial\boldsymbol{v}}{\partial x} + dy\frac{\partial\boldsymbol{v}}{\partial y} + dz\frac{\partial\boldsymbol{v}}{\partial z} = (d\boldsymbol{r}\cdot\boldsymbol{grad})\boldsymbol{v}.$$

Thus

(2.6) $$d\boldsymbol{v} = (\partial\boldsymbol{v}/\partial t)dt + (d\boldsymbol{r}\cdot\boldsymbol{grad})\boldsymbol{v},$$

or, dividing both sides by dt*,*

(2.7) $$\frac{d\boldsymbol{v}}{dt} = \frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{grad})\boldsymbol{v}.$$

Substituting this in (2.4)*, we find*

(2.8) $$\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{grad})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{grad}\, p;$$

it was first obtained by L. Euler in 1755.

If the fluid is in a gravitational field, an additional force $\rho\boldsymbol{g}$*, where* $\boldsymbol{g}$ *is the acceleration due to gravity, acts on any unit volume. This force must be added to the right-side of equation* (2.4)*, so the equation* (2.8) *takes the form*

(2.9) $$\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{grad})\boldsymbol{v} = -\frac{\boldsymbol{grad}\, p}{\rho} + \boldsymbol{g}.$$

Using the vector pressure, the correspondent to equation (2.9), with a generic density of external force $\boldsymbol{f}$ (not only gravitational), is

$$(2.10) \qquad \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{grad})\boldsymbol{v} = -\frac{1}{\rho}\left(\frac{\partial p_1}{\partial x}, \frac{\partial p_2}{\partial y}, \frac{\partial p_3}{\partial z}\right) + \boldsymbol{f},$$

therefore a new kind of Euler's equation, and whose integration does not involve major difficulties.

It is interesting observe that Batchelor[5] wrote (chap. 3.3) "*The simple notion of a pressure acting equally in all directions is lost in most cases of a fluid in motion"*, thus shown that the imposition or acceptation of a pressure equal in the three rectangular coordinates is, in fact, something fragile, possibly not true in the nature, for fluids in motion.

3 – Deduction of Navier-Stokes equations

Among several deductions of the equations of Navier-Stokes, we will choose the one described in Richardson[6](1950), for its brevity, simplicity and understanding.

Richardson firstly makes his deduction of the Euler equations (*Acad. Berlin*, 1755),

$$(3.1) \qquad \begin{cases} \dfrac{\partial U}{\partial t} + U\dfrac{\partial U}{\partial x} + V\dfrac{\partial U}{\partial y} + W\dfrac{\partial U}{\partial z} = X - \dfrac{1}{\rho}\dfrac{\partial p}{\partial x} \\ \dfrac{\partial V}{\partial t} + U\dfrac{\partial V}{\partial x} + V\dfrac{\partial V}{\partial y} + W\dfrac{\partial V}{\partial z} = Y - \dfrac{1}{\rho}\dfrac{\partial p}{\partial y} \\ \dfrac{\partial W}{\partial t} + U\dfrac{\partial W}{\partial x} + V\dfrac{\partial W}{\partial y} + W\dfrac{\partial W}{\partial z} = Z - \dfrac{1}{\rho}\dfrac{\partial p}{\partial z} \end{cases}$$

where the velocity of fluid is (U, V, W), the external force (on unit mass) is (X, Y, Z), the pressure is p and the volumetric density of mass is ρ.

The equations are constructed from the statement of Newton's Second Law of Motion, i.e., that the total force acting on a particle is the product of its mass and acceleration.

If x, y, z *are the rectilinear co-ordinates of a small cube of the material (density* ρ*) of volume* δv*,* $\ddot{x}, \ddot{y}, \ddot{z}$ *the components of its acceleration and* X, Y, Z *of forces on unit mass, let* X_p*,* Y_p*,* Z_p *be the components of the external forces acting normally on the three surfaces of area* δS *due to the differences of pressure (Fig. 1).*

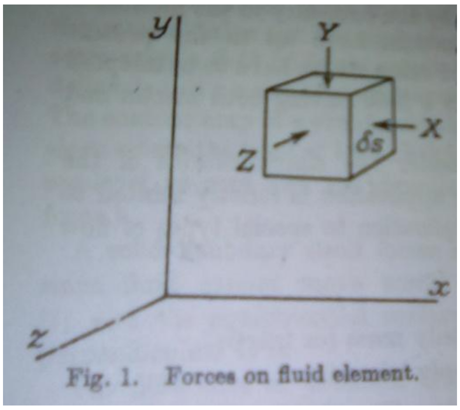

Fig. 1 – Forces on fluid element.

Setting aside the frictional forces for the moment (which resulting in Navier-Stokes equations)*, we have these conditions of equilibrium:*

$$(3.2)\qquad \begin{cases} \rho\ddot{x}\delta v = X\rho\delta v + X_p\delta S \\ \rho\ddot{y}\delta v = Y\rho\delta v + Y_p\delta S \\ \rho\ddot{z}\delta v = Z\rho\delta v + Z_p\delta S \end{cases}$$

In place of X_p, Y_p, Z_p *we shall insert the pressure gradients in the corresponding directions, i.e.*

$$(3.3)\qquad \begin{cases} X_p \cdot \delta S = \frac{\partial p}{\partial x} \cdot \delta v \\ Y_p \cdot \delta S = \frac{\partial p}{\partial y} \cdot \delta v \\ Z_p \cdot \delta S = \frac{\partial p}{\partial z} \cdot \delta v \end{cases}$$

For (3.3), in an ideal fluid, the pressure acts equally in all directions in the interior and at right angles to any surface presented to it. Then X_p, Y_p, Z_p *are each derived from* p*, the mean hydrostatic pressure at the point in the fluid circumscribed by the cube.*

Substituting in (3.2) we get

$$(3.4)\qquad \begin{cases} \rho\ddot{x} = \rho X - \frac{\partial p}{\partial x} \\ \rho\ddot{y} = \rho Y - \frac{\partial p}{\partial y} \\ \rho\ddot{z} = \rho Z - \frac{\partial p}{\partial z} \end{cases}$$

These equations are not suited to direct application since the quantities x, y, z *appear in them at once as dependent and independent variables. There are two ways of adapting them to suit experimental observation. We can ask ourselves, "At a given point, what fluid occupies the element of space subsequently?" or, "Where does a given particle find itself as times goes on?" The first attitude* *corresponds to that of a fixed observer, the second to that of an observer who moves with the general velocity of the medium.*

Mathematically, the first question can be put thus: "What function of x, y, z *and* t *are the velocity components* $U(=\dot{x})$, $V(=\dot{y})$, $W(=\dot{z})$*?" We proceed to retain* x, y, z *as independent variables but eliminate their dependent aspects to obtain*

$$\frac{d^2x}{dt^2} = \frac{dU}{dt} = \frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial U}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial U}{\partial z} \cdot \frac{dz}{dt}, \text{ etc.} \tag{3.5}$$

which with (3.4) resolve into the Eulerian equations (3.1).

Answering to the first question, Richardson says that *the second form of our question* ("Where does a given particle find itself as times goes on?") *can be translated thus: "What functions of time and place are those co-ordinates – let them be* a, b, c *– which characterize a given particle?" To answer this, we get rid of* x, y, z *as independent variables but retain them where dependent and arrive at the Lagrangian (Mem. Acad. (Berlin), 1781) form of the equations of motion:*

$$\begin{cases} \left(\frac{\partial^2 x}{\partial t^2} - X\right)\frac{\partial x}{\partial a} + \left(\frac{\partial^2 y}{\partial t^2} - Y\right)\frac{\partial y}{\partial a} + \left(\frac{\partial^2 z}{\partial t^2} - Z\right)\frac{\partial z}{\partial a} + \frac{1}{\rho}\frac{\partial p}{\partial a} = 0 \\ \left(\frac{\partial^2 x}{\partial t^2} - X\right)\frac{\partial x}{\partial b} + \left(\frac{\partial^2 y}{\partial t^2} - Y\right)\frac{\partial y}{\partial b} + \left(\frac{\partial^2 z}{\partial t^2} - Z\right)\frac{\partial z}{\partial b} + \frac{1}{\rho}\frac{\partial p}{\partial b} = 0 \\ \left(\frac{\partial^2 x}{\partial t^2} - X\right)\frac{\partial x}{\partial c} + \left(\frac{\partial^2 y}{\partial t^2} - Y\right)\frac{\partial y}{\partial c} + \left(\frac{\partial^2 z}{\partial t^2} - Z\right)\frac{\partial z}{\partial c} + \frac{1}{\rho}\frac{\partial p}{\partial c} = 0 \end{cases} \tag{3.6}$$

As we known, *the form due to Euler is, however, more generally used.*

Now let us introduce the frictional forces. We define the coefficient of viscosity, η*, as the force per unit area of two parallel laminae of fluid unit distance apart, measured across the direction of flow. Thus, if* U *and* $U + \delta U$ *(Fig. 2) are the velocities (in the direction of* x*) at two planes* δy *apart, the force per unit area on the fluid in either plane is* $\eta \cdot \partial U/\partial y$*, i.e., the product of the coefficient of viscosity and the velocity gradient perpendicular to the direction of flow. If* A, B *and* C *are such laminae, each of area* S*,* A *exerts a force on* B *equal to* $-\eta \cdot \partial U/\partial y \cdot S$*;* C *exerts a force on* B *equal to* $\eta \cdot (\partial U/\partial y + \partial^2 U/\partial y^2 \cdot \delta y) \cdot S$*, so that the net force on* B *is*

$$\eta \cdot \frac{\partial^2 U}{\partial y^2} \cdot \delta y \cdot S = \frac{\eta}{\rho} \cdot \delta m \cdot \frac{\partial^2 U}{\partial y^2} = \eta \cdot \delta v \cdot \frac{\partial^2 U}{\partial y^2} \tag{3.7}$$

where δm *is the mass of fluid between* A *and* B and δv is the respective volume. *The factor* η/ρ*, written* ν*, which we shall often require, is called the kinematic (coefficient of) viscosity. (It should be noted that it is here assumed that* η *is constant for a given fluid, invariable with* $\partial U/\partial y$*,* but a more general proof also is made posteriorly in [6], here omitted.)

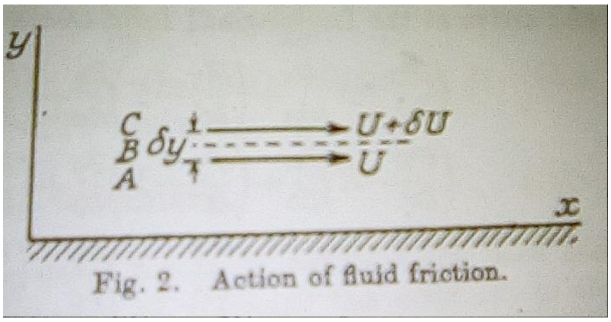

Fig. 2 – Action of fluid friction.

In the general case, the total viscous force on an element of mass m *due to the component* U *will be*

$$\eta \cdot \delta v \cdot \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right)$$

written shortly $\nu m \nabla^2 U$. *This force must be added to those on the right-hand side of the equations we have already derived* (Euler equations), resulting in the *equations ascribed to Navier (Mem. Acad. Sci. (Paris), 1822) and Stokes (Camb. Trans., 1845),*

$$(3.8) \qquad \begin{cases} \dfrac{\partial U}{\partial t} + U\dfrac{\partial U}{\partial x} + V\dfrac{\partial U}{\partial y} + W\dfrac{\partial U}{\partial z} = X - \dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu\,\nabla^2 U \\ \dfrac{\partial V}{\partial t} + U\dfrac{\partial V}{\partial x} + V\dfrac{\partial V}{\partial y} + W\dfrac{\partial V}{\partial z} = Y - \dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + \nu\,\nabla^2 V \\ \dfrac{\partial W}{\partial t} + U\dfrac{\partial W}{\partial x} + V\dfrac{\partial W}{\partial y} + W\dfrac{\partial W}{\partial z} = Z - \dfrac{1}{\rho}\dfrac{\partial p}{\partial z} + \nu\,\nabla^2 W \end{cases}$$

with $\nu = \eta/\rho$ the (kinematic) viscosity coefficient.

Confirming the difficulty of the Lagrangian description of the Euler and Navier-stokes equations, based on [7], the Navier-Stokes equations without external force and with volumetric mass density $\rho = 1$ are, describing the velocity as (u_1, u_2, u_3) and the spatial coordinates as (x_1, x_2, x_3),

$$(3.9.1) \qquad \frac{\partial^2 X_i}{\partial t^2} = -\sum_{j=1}^{3} \frac{\partial A_j}{\partial x_i}\frac{\partial p}{\partial a_j} + \\ + \nu \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \left(\frac{\partial^2 A_l}{\partial x_k \partial x_k}\frac{\partial u_i}{\partial a_l} + \frac{\partial A_j}{\partial x_k}\frac{\partial A_l}{\partial x_k}\frac{\partial^2 u_i}{\partial a_j \partial a_l}\right),$$

$$(3.9.2) \qquad \frac{\partial A_j}{\partial x_i} \equiv \frac{\partial}{\partial x_i} X_j(x_n, t)|_{x_n = X_n(a_m, s|t)},$$

where a_m is the label given to the fluid particle at time s. Its position and velocity at time t are, respectively, $X_n(a_m, s|t)$ and $u_n(a_m, s|t)$. The respective deduction of these equations we will omit, but the reader can consult [7] for more details.

4 – A new form of Euler and Navier-Stokes equations

The Eulerian (equations (3.1) and (3.8)) and Lagrangian (equations (3.6) and (3.9)) forms are not the unique possible equations for description of fluids. Other equation for modeling of fluids is possible, based on them, with the great advantage of linearity. It is what we will show in this section.

The system (1.3), for the sake of mathematical rigor, needs to be replaced by

(4.1)
$$\begin{cases} \frac{dx}{dt} = u_1(t) \\ \frac{dy}{dt} = u_2(t) \\ \frac{dz}{dt} = u_3(t) \end{cases}$$

emphasizing that the velocity components that appear as the time derivative of the coordinate (x, y, z) are legitimate functions of time, i.e., can be considered as representative of the Lagrangian description, $u_i(t)$, unlike the derivatives of u_i in $\frac{\partial u_i}{\partial t}, \frac{\partial u_i}{\partial x_j}$, $\nabla \cdot u$ and $\nabla^2 u_i$, that are in the Eulerian description, function of (x, y, z, t).

Representing the Eulerian velocity and respective components with the letter E indicated as upper index, and the corresponding Lagrangian components with the letter L, the system (1.4) is rewritten as

(4.2)
$$\begin{cases} \frac{\partial p}{\partial x} + \frac{\partial u_1^E}{\partial t} + u_1^L \frac{\partial u_1^E}{\partial x} + u_2^L \frac{\partial u_1^E}{\partial y} + u_3^L \frac{\partial u_1^E}{\partial z} = \nu \nabla^2 u_1^E + \frac{1}{3} \nu \nabla_1 (\nabla \cdot u^E) + f_1 \\ \frac{\partial p}{\partial y} + \frac{\partial u_2^E}{\partial t} + u_1^L \frac{\partial u_2^E}{\partial x} + u_2^L \frac{\partial u_2^E}{\partial y} + u_3^L \frac{\partial u_2^E}{\partial z} = \nu \nabla^2 u_2^E + \frac{1}{3} \nu \nabla_2 (\nabla \cdot u^E) + f_2 \\ \frac{\partial p}{\partial z} + \frac{\partial u_3^E}{\partial t} + u_1^L \frac{\partial u_3^E}{\partial x} + u_2^L \frac{\partial u_3^E}{\partial y} + u_3^L \frac{\partial u_3^E}{\partial z} = \nu \nabla^2 u_3^E + \frac{1}{3} \nu \nabla_3 (\nabla \cdot u^E) + f_3 \end{cases}$$

being the pressure p and external force f implicitly defined in the Eulerian description. A more concise notation for (4.2) is simply, for $i = 1, 2, 3$,

(4.3)
$$\frac{\partial p}{\partial x_i} + \frac{\partial u_i}{\partial t} + \alpha_1 \frac{\partial u_i}{\partial x} + \alpha_2 \frac{\partial u_i}{\partial y} + \alpha_3 \frac{\partial u_i}{\partial z} = \nu \nabla^2 u_i + \frac{1}{3} \nu \nabla_i (\nabla \cdot u) + f_i,$$

where p, f_i, u and u_i are in Eulerian description and $\alpha_i = \alpha_i(t)$ in Lagrangian description, i.e., $\alpha_i = \frac{dx_i}{dt}$, with the radius vector $r = (x_1, x_2, x_3) \equiv (x, y, z)$ function of time and indicating a motion of a specific particle of fluid starting from position $(x_1^0, x_2^0, x_3^0) \equiv (x_0, y_0, z_0)$.

The equations (4.2) and (4.3) shows us that the nonlinear form disappear, facilitating the obtaining of its solutions, transforming when $\nabla \cdot u = 0$ into a linear and second-order partial differential equation of elliptic type, already well-studied[8]. If $\nu = 0$ (Euler equations) we have equations of first order, obviously, which is also widely studied[9]. We realize that for each possible value of α_i it is possible to obtain different values of u_i, and reciprocally, i.e., there is not an one-one correspondence between α_i and u_i, thus it is convenient choose more easy time functions for the $\alpha_i(t)$, provided that compatible with the physical problem to be studied.

Nevertheless, even though it is very interesting to study other mathematical solutions for the original system (1.4) or the new system (4.2), I understand that the final conclusion made in [2] and [3] remains valid: it is possible to exist velocities in the Eulerian formulation that do not correspond to a real movement of particles of a fluid, according to the Lagrangian formulation. When I wrote this the first time I did not have the equations (4.2) and (4.3), deduced later in [1], but if it is true (as it is) that we should have (1.3) and (4.1) for a motion of fluid particle, then x_i and its respective velocity u_i are closely related, and the initial use of (1.1) in section 1 is valid. This is an excellent question to be examined with examples, which we will see in the next section.

But even when the relationship (1.1) is not required, a general solution for the new Euler equations ($\nu = 0$)

(4.4) $$\frac{\partial p}{\partial x_i} + \frac{\partial u_i}{\partial t} + \alpha_1 \frac{\partial u_i}{\partial x} + \alpha_2 \frac{\partial u_i}{\partial y} + \alpha_3 \frac{\partial u_i}{\partial z} = f_i$$

or

(4.5) $$\frac{\partial p}{\partial x_i} + \frac{Du_i}{Dt} = f_i,$$

in the case which the pressure p and external force $f = (f_1,\ f_2,\ f_3)$ are given and the velocity $u = (u_1,\ u_2,\ u_3)$ is calculated, is

(4.6) $$u_i = u_i^0 + \left(\int_0^t \left(f_i - \frac{\partial p}{\partial x_i} \right) |_L \, dt \right) |_E,$$

using

(4.7) $$\frac{Du_i}{Dt} = \frac{Du_i^E}{Dt} = \left(f_i - \frac{\partial p}{\partial x_i} \right) |_L.$$

u_i^0 is the component i of the initial velocity u^0, $|_L$ represents the use of transformation from Eulerian description to Lagrangian description and $|_E$ represents the inverse transformation used in $|_L$, returning to Eulerian description. We use implicitly $u_i^0 = (u_i^0|_L)|_E$ as well as $u_i = (u_i|_L)|_E$.

So here we conclude that the new Euler equations have a natural physical solution when the pressure and external force are given (or chosen) and the integration in (4.6) is possible, for $i = 1, 2, 3$, solution which varies with the specific movement of particles that is used. Boundary conditions must be in accordance with the solution (4.6) and it is also necessary substitute (4.6) in (4.4) for verification of possible conditions to be obeyed by each u_i^0 and α_i.

In special, when $\left(f_i - \frac{\partial p}{\partial x_i}\right)|_L$ is a function without temporal dependence, a constant function, the solution (4.6) is

(4.8) $$u_i = u_i^0 + \left(f_i - \frac{\partial p}{\partial x_i}\right)|_L\, t,$$

which is an exact solution and it is relatively fast and easy to simulate computationally. Substituting (4.8) in (4.4) we have

(4.9) $$\alpha_1 \frac{\partial u_i^0}{\partial x} + \alpha_2 \frac{\partial u_i^0}{\partial y} + \alpha_3 \frac{\partial u_i^0}{\partial z} = 0,$$

then a condition to be obeyed in this case.

We will see in section 8, Conclusion, an even better form of these equations, where we use

(4.10) $$\frac{D\alpha}{Dt} = \left(\frac{\partial u^E}{\partial t} + \alpha_1 \frac{\partial u^E}{\partial x} + \alpha_2 \frac{\partial u^E}{\partial y} + \alpha_3 \frac{\partial u^E}{\partial z}\right)|_t\, .$$

5 – Verification of physically reasonable solutions

§ 1

Of a point of view purely mathematical, it is not necessary to have the adoption of (1.1). It is possible forgotten that the Euler and Navier-Stokes equations have something relation with motion of fluids, liquids or gases, and accept that they are just equations of high level and difficulty of Pure Mathematics, but in this section we want to keep the bond or link between theses equations and the motion of fluids, and thus the use of (1.1) is born and can be used, as we will see.

If a particle (or some volume) of fluid has the movement governed according to the position vector $r = (x, y, z)$, with a temporal dependence $x = x(t)$, $y = y(t)$, $z = z(t)$, then the respective velocity of this particle (or volume) of fluid is $u = \frac{dr}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$, also, *a priori*, dependent of time (except if all three derivatives are equal to constant).

The first equation of (1.1),

$$\frac{\partial u_i}{\partial x_j} = 0, \ i \neq j, \tag{5.1.1}$$

is valid when we intend to follow the movement of a particle (or group of particles in a small volume) because in a mechanical movement we have by definition

$$u_i = \frac{dx_i}{dt}, \tag{5.1.2}$$

i.e., the component i of velocity is dependent only of component i of position, which is obvious, then we have $\frac{\partial u_i}{\partial x_j} = 0$ if $i \neq j$, according we saw in section 1.

From equation (5.1.2) we conclude that $dx_i = u_i dt$, or

$$\partial x_i = u_i \partial t, \tag{5.1.3}$$

the second equation of (1.1).

Thus we emphasize that if it is not necessary to have some particle or group of particles in the elementary volume $dV = dx\, dy\, dz$ in position (x, y, z) at time t then the use of (1.1), or (5.1.1) and (5.1.3), can be ignored, and we will have a problem purely mathematical.

Even if there is some bond or link between the coordinates, as $x^2 + y^2 + z^2 = R^2$ and $x\dot{x} + y\dot{y} + z\dot{z} = 0$ in a circular motion of constant radius R, the relation (5.1.2) is still true, by definition, and we do not need despise (5.1.1), a calculation facilitator, except if the external force is intrinsically dependent of the more than one spatial coordinate in at least one of the three orthogonal directions and we have $\nabla p \neq f$.

Then, what can be done when it is indispensable to use a determined relation between x, y and z, for example, when the particles need to be moving on a specific surface or manifold as $z = g(x, y)$? We try to first solve the equations using each variable in isolation, following (5.1.1), and at the end we use the dependence $z = g(x, y)$, i.e., the final solution for velocity will be

$$\begin{cases} u_1 = \varphi_1(x, t) \\ u_2 = \varphi_2(y, t) \\ u_3 = \varphi_3(z, t) = \varphi_3(g(x, y), t) = h(x, y, t) \end{cases} \tag{5.1.4}$$

and so we have indeed, in final consequence, $\frac{\partial u_3}{\partial z} = 0$. Obviously, if such procedure is not mathematically possible for some situation or configuration, we should abandon the use of (5.1.1) in this specific case.

We will check now the use of the relations (4.1),

$$(5.1.5)\qquad \begin{cases} \frac{dx}{dt} = u_1(t) \\ \frac{dy}{dt} = u_2(t) \\ \frac{dz}{dt} = u_3(t) \end{cases}$$

origin of the fundamental difference between the traditional equations and the new equations presented here. In fact, when we use and distinguish in a same equation the Eulerian u^E and Lagrangian u^L velocities the use of (1.1) is of secondary importance.

§ 2

Be the example 1

$$(5.2.1)\qquad \begin{cases} x = x_0 + t,\ \frac{dx}{dt} = 1 = u_1^L,\ \frac{Du_1^L}{Dt} = 0 \\ y = y_0 + 2t,\ \frac{dy}{dt} = 2 = u_2^L,\ \frac{Du_2^L}{Dt} = 0 \\ z = z_0 + 3t,\ \frac{dz}{dt} = 3 = u_3^L,\ \frac{Du_3^L}{Dt} = 0 \end{cases}$$

in fact a movement of total acceleration equal to zero, $\frac{Du_1^L}{Dt} = \frac{Du_2^L}{Dt} = \frac{Du_3^L}{Dt} = 0$, each particle starting from a generic initial position (x_0, y_0, z_0).

Suppose that the introduction of external force, internal frictional forces and internal pressure generated a solution for velocity, in the Eulerian formulation, such that, for example,

$$(5.2.2)\qquad \begin{cases} u_1^E = x,\ \frac{Du_1^E}{Dt} = \frac{Dx}{Dt} = \frac{D(x_0+t)}{Dt} = 1 \\ u_2^E = y,\ \frac{Du_2^E}{Dt} = \frac{Dy}{Dt} = \frac{D(y_0+2t)}{Dt} = 2 \\ u_3^E = z,\ \frac{Du_3^E}{Dt} = \frac{Dz}{Dt} = \frac{D(z_0+3t)}{Dt} = 3 \end{cases}$$

The acceleration as used in the Euler and Navier-Stokes equations is

$$(5.2.3)\qquad \begin{cases} \frac{Du_1^E}{Dt} = \frac{\partial u_1^E}{\partial t} + u_1^E \frac{\partial u_1^E}{\partial x} + u_2^E \frac{\partial u_1^E}{\partial y} + u_3^E \frac{\partial u_1^E}{\partial z} = x,\ x(t) = t \not\equiv 1 \\ \frac{Du_2^E}{Dt} = \frac{\partial u_2^E}{\partial t} + u_1^E \frac{\partial u_2^E}{\partial x} + u_2^E \frac{\partial u_2^E}{\partial y} + u_3^E \frac{\partial u_2^E}{\partial z} = y,\ y(t) = 2t \not\equiv 2 \\ \frac{Du_3^E}{Dt} = \frac{\partial u_3^E}{\partial t} + u_1^E \frac{\partial u_3^E}{\partial x} + u_2^E \frac{\partial u_3^E}{\partial y} + u_3^E \frac{\partial u_3^E}{\partial z} = z,\ z(t) = 3t \not\equiv 3 \end{cases}$$

i.e., the use of the expression according to the traditional Euler and Navier-Stokes equations generates a wrong value for the value of the acceleration $\frac{Du^E}{Dt}$.

By other side, using the correct form of the new Euler and Navier-Stokes equations, according (4.2), we have

$$(5.2.4) \qquad \begin{cases} \frac{Du_1^E}{Dt} = \frac{\partial u_1^E}{\partial t} + u_1^L \frac{\partial u_1^E}{\partial x} + u_2^L \frac{\partial u_1^E}{\partial y} + u_3^L \frac{\partial u_1^E}{\partial z} = 1 \\ \frac{Du_2^E}{Dt} = \frac{\partial u_2^E}{\partial t} + u_1^L \frac{\partial u_2^E}{\partial x} + u_2^L \frac{\partial u_2^E}{\partial y} + u_3^L \frac{\partial u_2^E}{\partial z} = 2 \\ \frac{Du_3^E}{Dt} = \frac{\partial u_3^E}{\partial t} + u_1^L \frac{\partial u_3^E}{\partial x} + u_2^L \frac{\partial u_3^E}{\partial y} + u_3^L \frac{\partial u_3^E}{\partial z} = 3 \end{cases}$$

therefore the correct and expected result conform (5.2.2) for the acceleration $\frac{Du^E}{Dt}$, but with the disagreement $\frac{Du^E}{Dt} \neq \frac{Du^L}{Dt}$.

For that to be $\frac{Du^E}{Dt} = \frac{Du^L}{Dt}$ for all time and position it is necessary too, by a logical necessity of consistency between both velocities, that

$$(5.2.5) \qquad u^E(x(t), y(t), z(t), t) = u^L(t),$$

so, from (5.2.1)

$$(5.2.6) \qquad \begin{cases} u_1^E = 1, \ \frac{\partial u_1^E}{\partial t} = \frac{\partial u_1^E}{\partial x_j} = 0 \\ u_2^E = 2, \ \frac{\partial u_2^E}{\partial t} = \frac{\partial u_2^E}{\partial x_j} = 0 \\ u_3^E = 3, \ \frac{\partial u_3^E}{\partial t} = \frac{\partial u_3^E}{\partial x_j} = 0 \end{cases}$$

and now $\frac{Du^E}{Dt} = \frac{Du^L}{Dt} = 0$.

§ 3

Be now the example 2

$$(5.3.1) \qquad \begin{cases} x = x_0 + u_0 t + f\frac{t^2}{2}, \ \frac{dx}{dt} = u_0 + ft = u_1^L, \ \frac{Du_1^L}{Dt} = f \\ y = y_0 + v_0 t + g\frac{t^2}{2}, \ \frac{dy}{dt} = v_0 + gt = u_2^L, \ \frac{Du_2^L}{Dt} = g \\ z = z_0 + w_0 t + h\frac{t^2}{2}, \ \frac{dz}{dt} = w_0 + ht = u_3^L, \ \frac{Du_3^L}{Dt} = h \end{cases}$$

for constants $x_0, y_0, z_0,\ u_0, v_0, w_0,\ f, g, h,$ a movement of constant acceleration (f, g, h).

Suppose again that the introduction of external force, internal frictional forces and internal pressure generated a solution for velocity, in the Eulerian formulation, such that, for example,

$$(5.3.2)\qquad \begin{cases} u_1^E = u_0 + ft,\ \frac{Du_1^E}{Dt} = f \\ u_2^E = v_0 + gt,\ \frac{Du_2^E}{Dt} = g \\ u_3^E = w_0 + ht,\ \frac{Du_3^E}{Dt} = h \end{cases}$$

without dependence of spatial position and with $u^E = u^L$.

The acceleration as used in the Euler and Navier-Stokes equations is

$$(5.3.3)\qquad \begin{cases} \frac{Du_1^E}{Dt} = \frac{\partial u_1^E}{\partial t} + u_1^E \frac{\partial u_1^E}{\partial x} + u_2^E \frac{\partial u_1^E}{\partial y} + u_3^E \frac{\partial u_1^E}{\partial z} = \frac{\partial u_1^E}{\partial t} = f \\ \frac{Du_2^E}{Dt} = \frac{\partial u_2^E}{\partial t} + u_1^E \frac{\partial u_2^E}{\partial x} + u_2^E \frac{\partial u_2^E}{\partial y} + u_3^E \frac{\partial u_2^E}{\partial z} = \frac{\partial u_2^E}{\partial t} = g \\ \frac{Du_3^E}{Dt} = \frac{\partial u_3^E}{\partial t} + u_1^E \frac{\partial u_3^E}{\partial x} + u_2^E \frac{\partial u_3^E}{\partial y} + u_3^E \frac{\partial u_3^E}{\partial z} = \frac{\partial u_3^E}{\partial t} = h \end{cases}$$

i.e., this time the use of the expression according to the traditional Euler and Navier-Stokes equations generates a correct value for the acceleration $\frac{Du^E}{Dt}$ because there is no dependence of position.

Besides this, using the correct form of the new Euler and Navier-Stokes equations, according (4.2), we have

$$(5.3.4)\qquad \begin{cases} \frac{Du_1^E}{Dt} = \frac{\partial u_1^E}{\partial t} + u_1^L \frac{\partial u_1^E}{\partial x} + u_2^L \frac{\partial u_1^E}{\partial y} + u_3^L \frac{\partial u_1^E}{\partial z} = \frac{\partial u_1^E}{\partial t} = f \\ \frac{Du_2^E}{Dt} = \frac{\partial u_2^E}{\partial t} + u_1^L \frac{\partial u_2^E}{\partial x} + u_2^L \frac{\partial u_2^E}{\partial y} + u_3^L \frac{\partial u_2^E}{\partial z} = \frac{\partial u_2^E}{\partial t} = g \\ \frac{Du_3^E}{Dt} = \frac{\partial u_3^E}{\partial t} + u_1^L \frac{\partial u_3^E}{\partial x} + u_2^L \frac{\partial u_3^E}{\partial y} + u_3^L \frac{\partial u_3^E}{\partial z} = \frac{\partial u_3^E}{\partial t} = h \end{cases}$$

therefore the correct and expected result conform (5.3.2) for the acceleration $\frac{Du^E}{Dt}$, this time with the agreement $\frac{Du^E}{Dt} = \frac{Du^L}{Dt}$.

§ 4

We will next use the solution (4.6) of (4.5),

$$(5.4.1)\qquad u_i = u_i^0 + \left(\int_0^t \left(f_i - \frac{\partial p}{\partial x_i}\right)|_L\, dt\right)|_E,$$

solution of the new Euler equations, for the special and easier case that $f_i = \frac{\partial p}{\partial x_i}$, i.e., the external force is conservative, a gradient field, being the pressure its respective potential, and

(5.4.2) $$u_i = u_i^E = u_i^0,\ \frac{Du_i^E}{Dt} = \frac{\partial u_i^E}{\partial t} = 0,$$

and with

(5.4.3) $$\begin{cases} x = x_0 e^{-t},\ \frac{dx}{dt} = -x_0 e^{-t} = u_1^L,\ \frac{Du_1^L}{Dt} = x_0 e^{-t} \\ y = y_0 e^{-t},\ \frac{dy}{dt} = -y_0 e^{-t} = u_2^L,\ \frac{Du_2^L}{Dt} = y_0 e^{-t} \\ z = z_0 e^{-t},\ \frac{dz}{dt} = -z_0 e^{-t} = u_3^L,\ \frac{Du_3^L}{Dt} = z_0 e^{-t} \end{cases}$$

for constants x_0, y_0, z_0, a movement of contraction from (x_0, y_0, z_0) to $(0,0,0)$, with $\frac{Du^L}{Dt} = (x_0, y_0, z_0)e^{-t} = (x(t), y(t), z(t))$.

The acceleration as used in the traditional Euler and Navier-Stokes equations is

(5.4.4)

$$\begin{cases} \frac{Du_1^E}{Dt} = \left(\frac{\partial u_1^E}{\partial t} + u_1^E \frac{\partial u_1^E}{\partial x} + u_2^E \frac{\partial u_1^E}{\partial y} + u_3^E \frac{\partial u_1^E}{\partial z}\right)|_t = \left(u_1^0 \frac{\partial u_1^0}{\partial x} + u_2^0 \frac{\partial u_1^0}{\partial y} + u_3^0 \frac{\partial u_1^0}{\partial z}\right)|_t \\ \frac{Du_2^E}{Dt} = \left(\frac{\partial u_2^E}{\partial t} + u_1^E \frac{\partial u_2^E}{\partial x} + u_2^E \frac{\partial u_2^E}{\partial y} + u_3^E \frac{\partial u_2^E}{\partial z}\right)|_t = \left(u_1^0 \frac{\partial u_2^0}{\partial x} + u_2^0 \frac{\partial u_2^0}{\partial y} + u_3^0 \frac{\partial u_2^0}{\partial z}\right)|_t \\ \frac{Du_3^E}{Dt} = \left(\frac{\partial u_3^E}{\partial t} + u_1^E \frac{\partial u_3^E}{\partial x} + u_2^E \frac{\partial u_3^E}{\partial y} + u_3^E \frac{\partial u_3^E}{\partial z}\right)|_t = \left(u_1^0 \frac{\partial u_3^0}{\partial x} + u_2^0 \frac{\partial u_3^0}{\partial y} + u_3^0 \frac{\partial u_3^0}{\partial z}\right)|_t \end{cases}$$

which shows us the possibility of being valid $\frac{Du_i^E}{Dt} \neq 0$ with $\frac{\partial u_i^E}{\partial t} = 0$.

Being necessary in this case that $\frac{Du_i^E}{Dt} = \frac{\partial u_i^E}{\partial t} = 0$, for $i = 1,2,3$, we have

(5.4.5) $$\begin{cases} u_1^0 \frac{\partial u_1^0}{\partial x} + u_2^0 \frac{\partial u_1^0}{\partial y} + u_3^0 \frac{\partial u_1^0}{\partial z} = 0 \\ u_1^0 \frac{\partial u_2^0}{\partial x} + u_2^0 \frac{\partial u_2^0}{\partial y} + u_3^0 \frac{\partial u_2^0}{\partial z} = 0 \\ u_1^0 \frac{\partial u_3^0}{\partial x} + u_2^0 \frac{\partial u_3^0}{\partial y} + u_3^0 \frac{\partial u_3^0}{\partial z} = 0 \end{cases}$$

which is valid, for example, for initial velocities such that

(5.4.6) $$u_i^0 = k_i \phi_i(ax + by + cz),$$

with

(5.4.7) $\quad k_1\phi_1 a + k_2\phi_2 b + k_3\phi_3 c = 0,$

k_i, a, b, c real numbers, $\phi_i: \mathbb{R} \to \mathbb{R}$ differentiable functions, for $i = 1, 2, 3$. If the condition of incompressibility $\nabla \cdot u = \nabla \cdot u^0 = 0$ is required in the resolution of a given problem then it is also necessary that

(5.4.8) $\quad k_1\phi_1' a + k_2\phi_2' b + k_3\phi_3' c = 0,$

always satisfied when (5.4.7) is true.

With the correct form of the new Euler and Navier-Stokes equations we have, using (5.4.2),

(5.4.9)

$$\begin{cases} \frac{Du_1^E}{Dt} = \left(\frac{\partial u_1^E}{\partial t} + u_1^L \frac{\partial u_1^E}{\partial x} + u_2^L \frac{\partial u_1^E}{\partial y} + u_3^L \frac{\partial u_1^E}{\partial z}\right)\Big|_t = \left(u_1^L \frac{\partial u_1^0}{\partial x} + u_2^L \frac{\partial u_1^0}{\partial y} + u_3^L \frac{\partial u_1^0}{\partial z}\right)\Big|_t = 0 \\ \frac{Du_2^E}{Dt} = \left(\frac{\partial u_2^E}{\partial t} + u_1^L \frac{\partial u_2^E}{\partial x} + u_2^L \frac{\partial u_2^E}{\partial y} + u_3^L \frac{\partial u_2^E}{\partial z}\right)\Big|_t = \left(u_1^L \frac{\partial u_2^0}{\partial x} + u_2^L \frac{\partial u_2^0}{\partial y} + u_3^L \frac{\partial u_2^0}{\partial z}\right)\Big|_t = 0 \\ \frac{Du_3^E}{Dt} = \left(\frac{\partial u_3^E}{\partial t} + u_1^L \frac{\partial u_3^E}{\partial x} + u_2^L \frac{\partial u_3^E}{\partial y} + u_3^L \frac{\partial u_3^E}{\partial z}\right)\Big|_t = \left(u_1^L \frac{\partial u_3^0}{\partial x} + u_2^L \frac{\partial u_3^0}{\partial y} + u_3^L \frac{\partial u_3^0}{\partial z}\right)\Big|_t = 0 \end{cases}$$

which also has by solution, for example,

(5.4.10) $\quad u_i^0 = k_i\phi_i(ax + by + cz),$

supposing $\phi_i: \mathbb{R} \to \mathbb{R}$ differentiable functions and k_i, a, b, c real numbers, for $i = 1, 2, 3$, but this time with

(5.4.11) $\quad a\, u_1^L(t) + b\, u_2^L(t) + c\, u_3^L(t) = 0,$

or equivalently

(5.4.12.1) $\quad u_1^L(t) = -\frac{1}{a}\left(b\, u_2^L(t) + c\, u_3^L(t)\right),\ a \neq 0,$

(5.4.12.2) $\quad u_2^L(t) = -\frac{1}{b}\left(a\, u_1^L(t) + c\, u_3^L(t)\right),\ b \neq 0,$

(5.4.12.3) $\quad u_3^L(t) = -\frac{1}{c}\left(a\, u_1^L(t) + b\, u_2^L(t)\right),\ c \neq 0,$

for all $t \geq 0$, or all $\phi_i{}'$ are constants. For that $\nabla \cdot u = \nabla \cdot u^0 = 0$ it is necessary also be valid (5.4.8) or all ϕ_i need be constant.

According to the solution (5.4.10) and for the chosen movement given by (5.4.3), the condition (5.4.11) imposes that

(5.4.13.1) $\quad x_0 = -\frac{1}{a}(b\, y_0 + c\, z_0),$

(5.4.13.2) $\quad y_0 = -\frac{1}{b}(a\,x_0 + c\,z_0),$

(5.4.13.3) $\quad z_0 = -\frac{1}{c}(a\,x_0 + b\,y_0),$

respectively if $a \neq 0,\ b \neq 0,\ c \neq 0$, therefore each initial position of a specific particle or group of particles need to obey the previous condition, in this case: initial positions on a plane for each family of coefficients (a, b, c).

Note that in this way the Lagrangian solution is which governs the movement of fluids, or rather, explains what happens in the fluid, with respect to velocity. We can choose many different ϕ functions for Eulerian solution of u^E, but the individual motion of the particles or group of particles is the same with each prefixed choice of u^L. Thus, it is unnecessary to choose complicated initial velocities in the Eulerian formulation when the movement in the Lagrangian formulation is simpler, at least when the external force is a conservative field.

As made in § 2, by a logical necessity of consistency between both velocities and for that $\frac{Du^E}{Dt} = \frac{Du^L}{Dt}$ for all time and position it is necessary too that

(5.4.14) $\quad u^E(x(t), y(t), z(t), t) = u^L(t),$

so, from (5.4.3) we have

(5.4.15)
$$\begin{cases} u_1^E = -x,\ \frac{\partial u_1^E}{\partial t} = \frac{\partial u_1^E}{\partial y} = \frac{\partial u_1^E}{\partial z} = 0,\ \frac{\partial u_1^E}{\partial x} = -1 \\ u_2^E = -y,\ \frac{\partial u_2^E}{\partial t} = \frac{\partial u_2^E}{\partial x} = \frac{\partial u_2^E}{\partial z} = 0,\ \frac{\partial u_2^E}{\partial y} = -1 \\ u_3^E = -z,\ \frac{\partial u_3^E}{\partial t} = \frac{\partial u_3^E}{\partial x} = \frac{\partial u_3^E}{\partial y} = 0,\ \frac{\partial u_3^E}{\partial z} = -1 \end{cases}$$

and now $\frac{Du^E}{Dt}\big|_t = \frac{Du^L}{Dt} = \big(x(t), y(t), z(t)\big)$, but it is a compressible motion, with $\nabla \cdot u^E = -3$.

§ 5

In this present case we will analyze the same Lagrangian solution in (5.4.3), but now with time dependent Eulerian solution, i.e., with some or all $\frac{\partial u_i^E}{\partial t} \neq 0$. Again with $\nabla p = f$ and $\frac{Du^E}{Dt} = 0$, the Lagrangian solution is

(5.5.1)
$$\begin{cases} x = x_0 e^{-t},\ \frac{dx}{dt} = -x_0 e^{-t} = u_1^L,\ \frac{Du_1^L}{Dt} = x_0 e^{-t} \\ y = y_0 e^{-t},\ \frac{dy}{dt} = -y_0 e^{-t} = u_2^L,\ \frac{Du_2^L}{Dt} = y_0 e^{-t} \\ z = z_0 e^{-t},\ \frac{dz}{dt} = -z_0 e^{-t} = u_3^L,\ \frac{Du_3^L}{Dt} = z_0 e^{-t} \end{cases}$$

for constants x_0, y_0, z_0, a movement of contraction from (x_0, y_0, z_0) to $(0,0,0)$, with $\frac{Du^L}{Dt} = (x_0, y_0, z_0)e^{-t} = (x(t), y(t), z(t))$.

We have in this case for Eulerian representation in the traditional meaning

(5.5.2) $$\begin{cases} \frac{Du_1^E}{Dt} = \left(\frac{\partial u_1^E}{\partial t} + u_1^E \frac{\partial u_1^E}{\partial x} + u_2^E \frac{\partial u_1^E}{\partial y} + u_3^E \frac{\partial u_1^E}{\partial z}\right)\Big|_t = 0 \\ \frac{Du_2^E}{Dt} = \left(\frac{\partial u_2^E}{\partial t} + u_1^E \frac{\partial u_2^E}{\partial x} + u_2^E \frac{\partial u_2^E}{\partial y} + u_3^E \frac{\partial u_2^E}{\partial z}\right)\Big|_t = 0 \\ \frac{Du_3^E}{Dt} = \left(\frac{\partial u_3^E}{\partial t} + u_1^E \frac{\partial u_3^E}{\partial x} + u_2^E \frac{\partial u_3^E}{\partial y} + u_3^E \frac{\partial u_3^E}{\partial z}\right)\Big|_t = 0 \end{cases}$$

Choosing for respective solution

(5.5.3) $$u_i^E = k_i \phi_i(ax + by + cz + dt),$$

with $\phi_i: \mathbb{R} \to \mathbb{R}$ differentiable functions and k_i, a, b, c real numbers, for $i = 1,2,3$, we have

(5.5.4) $$k_1 \phi_1 a + k_2 \phi_2 b + k_3 \phi_3 c + d = 0,$$

otherwise all ϕ_i are constants. If the condition of incompressibility $\nabla \cdot u = \nabla \cdot u^0 = 0$ is required in the resolution of a given problem then it is also necessary that

(5.5.5) $$k_1 \phi_1' a + k_2 \phi_2' b + k_3 \phi_3' c = 0,$$

always satisfied when (5.5.4) is true.

With the correct form of the new Euler and Navier-Stokes equations we have

(5.5.6) $$\begin{cases} \frac{Du_1^E}{Dt} = \left(\frac{\partial u_1^E}{\partial t} + u_1^L \frac{\partial u_1^E}{\partial x} + u_2^L \frac{\partial u_1^E}{\partial y} + u_3^L \frac{\partial u_1^E}{\partial z}\right)\Big|_t = 0 \\ \frac{Du_2^E}{Dt} = \left(\frac{\partial u_2^E}{\partial t} + u_1^L \frac{\partial u_2^E}{\partial x} + u_2^L \frac{\partial u_2^E}{\partial y} + u_3^L \frac{\partial u_2^E}{\partial z}\right)\Big|_t = 0 \\ \frac{Du_3^E}{Dt} = \left(\frac{\partial u_3^E}{\partial t} + u_1^L \frac{\partial u_3^E}{\partial x} + u_2^L \frac{\partial u_3^E}{\partial y} + u_3^L \frac{\partial u_3^E}{\partial z}\right)\Big|_t = 0 \end{cases}$$

which also has by solution, for example,

(5.5.7) $$u_i^E = k_i \phi_i(ax + by + cz + dt),$$

for $\phi_i: \mathbb{R} \to \mathbb{R}$ differentiable functions, k_i, a, b, c real numbers, $i = 1,2,3$, but this time with

(5.5.8) $$a\, u_1^L(t) + b\, u_2^L(t) + c\, u_3^L(t) + d = 0,$$

or equivalently

(5.5.9.1) $$u_1^L(t) = -\frac{1}{a}(b\, u_2^L(t) + c\, u_3^L(t) + d),\ \ a \neq 0,$$

(5.5.9.2) $$u_2^L(t) = -\frac{1}{b}(a\, u_1^L(t) + c\, u_3^L(t) + d),\ \ b \neq 0,$$

(5.5.9.3) $$u_3^L(t) = -\frac{1}{c}(a\, u_1^L(t) + b\, u_2^L(t) + d),\ \ c \neq 0,$$

for all $t \geq 0$, or all $\phi_i{}'$ are constants. For that $\nabla \cdot u = \nabla \cdot u^0 = 0$ it is necessary also be valid (5.5.5) or all ϕ_i need be constant.

According to the solution (5.5.7) and for the chosen movement given by (5.5.1), the condition (5.5.8) imposes that

(5.5.10.1) $$x_0 = -\frac{1}{a}(b\, y_0 + c\, z_0 - d),$$

(5.5.10.2) $$y_0 = -\frac{1}{b}(a\, x_0 + c\, z_0 - d),$$

(5.5.10.3) $$z_0 = -\frac{1}{c}(a\, x_0 + b\, y_0 - d),$$

respectively if $a \neq 0$, $b \neq 0$, $c \neq 0$, therefore each initial position of a specific particle or group of particles needs to obey the previous condition, in this case: initial positions on a plane for each family of coefficients (a, b, c, d).

Note that a solution in the Lagrangian description may correspond to two (or even more) solutions in the Eulerian description, for example, a steady state solution as well as a non-steady state solution, as can be seen by comparing the solutions in § 4 and § 5, so it is convenient to look for, or pre-define, simpler formats for Eulerian solutions.

On the other hand, as we have already said, for to have logical consistency between both velocities, it is necessary that

(5.5.11) $$u^E(x(t), y(t), z(t), t) = u^L(t)$$

and $\frac{Du^E}{Dt}|_t = \frac{Du^L}{Dt}$ for all time $t \geq 0$, and we came back to the solution obtained in (5.4.15), a steady state solution, i.e.,

(5.5.12) $$\begin{cases} u_1^E = -x,\ \frac{\partial u_1^E}{\partial t} = \frac{\partial u_1^E}{\partial y} = \frac{\partial u_1^E}{\partial z} = 0,\ \frac{\partial u_1^E}{\partial x} = -1 \\ u_2^E = -y,\ \frac{\partial u_2^E}{\partial t} = \frac{\partial u_2^E}{\partial x} = \frac{\partial u_2^E}{\partial z} = 0,\ \frac{\partial u_2^E}{\partial y} = -1 \\ u_3^E = -z,\ \frac{\partial u_3^E}{\partial t} = \frac{\partial u_3^E}{\partial x} = \frac{\partial u_3^E}{\partial y} = 0,\ \frac{\partial u_3^E}{\partial z} = -1 \end{cases}$$

a compressible motion with $\nabla \cdot u^E = -3$ and $\frac{Du^E}{Dt}|_t = \frac{Du^L}{Dt} = \big(x(t), y(t), z(t)\big)$.

§ 6

Lastly, we will see the new Navier-Stokes equations. As the Lagrangian description governs the movement of particles or group of particles, while the Eulerian description is a kind of complicating of the real (or approximate, say) behavior of fluids, at least when the external force is conservative and the pressure is its potential ($\nabla p = f$), we will try an Eulerian solution for velocity using (1.1), i.e., given $u^L = (u_1^L,\ u_2^L,\ u_3^L)$ we will use the form

(5.6.1) $$u_i^E = u_i^E(x_i, t) = \phi_i(x_i)\varphi_i(t)$$

in the equation

(5.6.2) $$\frac{Du_i^E}{Dt} = \nu\, \nabla^2 u_i^E + \frac{1}{3}\nu \frac{\partial}{\partial x_i}(\nabla \cdot u^E),$$

with

(5.6.3) $$\frac{Du_i^E}{Dt} = \frac{\partial u_i^E}{\partial t} + u_1^L \frac{\partial u_i^E}{\partial x} + u_2^L \frac{\partial u_i^E}{\partial y} + u_3^L \frac{\partial u_i^E}{\partial z}$$

and $\nabla \cdot u^E$ without specific value, thus

(5.6.4) $$\phi_i(x_i)\varphi_i'(t) + u_i^L(t)\phi_i'(x_i)\varphi_i(t) = \frac{4}{3}\nu\, \phi_i{}''(x_i)\varphi_i(t),$$

an ordinary differential equation, for $i = 1,2,3$, supposing ϕ_i and φ_i differentiable and continuous functions how much is needed.

By the superposition principle we can also add solutions,

(5.6.5) $$u_i^E = u_i^E(x_i, t) = \sum_{j=1}^{\infty} u_{ij}^E(x_i, t) = \sum_{j=1}^{\infty} \phi_{ij}(x_i)\varphi_{ij}(t),$$

and then

(5.6.6) $$\phi_{ij}(x_i)\varphi_{ij}'(t) + u_i^L(t)\phi_{ij}'(x_i)\varphi_{ij}(t) = \frac{4}{3}\nu\, \phi_{ij}{}''(x_i)\varphi_{ij}(t),$$

but the better use of (1.1) is when we give completely the Lagrangian and Eulerian solutions for velocity (i.e., a choose obeying the required initial and boundary conditions as well as the compressibility condition) and the external force is conservative, such that,

(5.6.7) $$\begin{cases} p = \int_L \left(-\frac{Du^E}{Dt} + \nu\, \nabla^2 u^E + \frac{1}{3}\nu\, \nabla(\nabla \cdot u^E) + f\right) \cdot dl \\ \qquad\qquad u_i^E = u_i^E(x_i, t) \end{cases}$$

for $i = 1, 2, 3$, i.e., the pressure is the unique function which we do not have *a priori* and need be calculated, while the choose components of velocities have the necessity to be logically consistent with the problem in question. In section 6 we will see again this solution.

We now will make the Eulerian solution even easier than (5.6.1) by removing the dependence of time,

(5.6.8) $$u_i^E = u_i^E(x_i) = \phi_i(x_i),$$

with

(5.6.9) $$\frac{Du_i^E}{Dt} = u_1^L \frac{\partial u_i^E}{\partial x} + u_2^L \frac{\partial u_i^E}{\partial y} + u_3^L \frac{\partial u_i^E}{\partial z} = \nu\, \nabla^2 u_i^E + \frac{1}{3}\nu \frac{\partial}{\partial x_i}(\nabla \cdot u^E),$$

$\nabla \cdot u^E$ with free value, and so

(5.6.10) $$u_i^L(t)\phi_i'(x_i) = \frac{4}{3}\nu\, \phi_i''(x_i)$$

or

(5.6.11) $$u_i^L(t) = \frac{4}{3}\nu \frac{\phi_i''(x_i)}{\phi_i'(x_i)} = c_i,$$

a spatial solution which obviously cannot varies in time and for this reason it is necessary that the function $u_i^L(t)$ is a real constant c_i. The solution is exponential in relation to coordinate x_i:

(5.6.12) $$u_i^E = \phi_i(x_i) = k_i e^{3c_i\, x_i/4\nu},$$

which in fact solves (5.6.9) for $k_i,\ c_i,\ \nu > 0$ real constants.

Note that although (5.6.12) is a spatially unlimited function for $x_i \to +\infty$ if $k_i \neq 0$ and $c_i > 0$, the respective Lagrangian solution $u_i^L(t) = c_i$, which indicates a motion of constant velocity, is well behaved, smooth and limited, for all position and all $t \geq 0$. Then this is another case (as in § 2) in that we have a regular motion in the time in Lagrangian description but with possibility of an unlimited solution in Eulerian description. By other side, if $k_i \neq 0$ and $c_i < 0$ the respective component u_i^E decreases with position for $x_i > 0$ and it is unlimited for $x_i \to -\infty$, which also is not compatible with the respective motion of those particles or group of particles, but nevertheless it is a possible solution in Eulerian description.

Also note that in each of the examples in this section, we had initially in general $u^L(t) \neq u^E(x,y,z,t)$, except if $t = 0$ and $x = x_0,\ y = y_0,\ z = z_0$ is the initial position, or some specific set of positions (x,y,z) and (x_0,y_0,z_0) at time t (in special, $x = x(t,x_0),\ y = y(t,y_0),\ z = z(t,z_0)$ according defined in the respective Lagrangian description) or if u^E is not dependent of position (as in § 3), so by the chain rule the correct form of the total acceleration $\frac{Du^E}{Dt}$ in a particle of fluid (or elementary volume dV or group of particles) is

$$(5.6.13) \qquad \frac{Du^E}{Dt} = \frac{\partial u^E}{\partial t} + u_1^L \frac{\partial u^E}{\partial x} + u_2^L \frac{\partial u^E}{\partial y} + u_3^L \frac{\partial u^E}{\partial z},$$

because we have in general

$$(5.6.14) \qquad u_1^L \frac{\partial u^E}{\partial x} + u_2^L \frac{\partial u^E}{\partial y} + u_3^L \frac{\partial u^E}{\partial z} \neq u_1^E \frac{\partial u^E}{\partial x} + u_2^E \frac{\partial u^E}{\partial y} + u_3^E \frac{\partial u^E}{\partial z}.$$

We are using implicitly the initial position $(x_0,\ y_0,\ z_0)$ in the Lagrangian description $u^L(t)$ as constant, although it has the same meaning as in $u^L(t, x_0, y_0, z_0)$.

In the last example of this § 6 for that

$$(5.6.15) \qquad u^E(x(t), y(t), z(t), t) = u^L(t) = \frac{d}{dt}(x(t), y(t), z(t))$$

and $\frac{Du^E}{Dt}|_t = \frac{Du^L}{Dt}$ for all $t \geq 0$ it is necessary to have, for $t = 0$,

$$(5.6.16) \qquad u_i^E(x_0, y_0, z_0, t = 0) = u_i^L(0) = c_i$$

and then, from (5.6.11) and (5.6.12),

$$(5.6.17) \qquad k_i = c_i e^{-3c_i x_i^0/4\nu}$$

and

$$(5.6.18) \qquad u_i^E = c_i e^{3c_i(x_i - x_i^0)/4\nu}$$

where $(x_0, y_0, z_0) \equiv (x_1^0, x_2^0, x_3^0)$ is the respective initial velocity, a motion of constant velocity $c = (c_1, c_2, c_3)$ for each particle or group of particles in Lagrangian description, without compressibility along time, but an exponential function in Eulerian description and with $\nabla \cdot u^E \neq 0$.

Also thinking about other time values, $t > 0$, we cannot accept this solution, and then the unique possible solution here is

$$(5.6.19) \qquad u_i^E\big(x_i(t)\big) = u_i^L(t) = c_i,$$

thus

$$(5.6.20) \qquad x_i = x_i^0$$

and so, no movement,

(5.6.21) $c_i = 0.$

The conclusion in this case is that it is necessary to have time dependence in the velocity u^E.

6 – The question of the breakdown solutions

Without passing through the Lagrangian formulation, given a velocity $u(x,y,z,t)$ at least two times differentiable with respect to spatial coordinates and one respect to time and an integrable external force $f(x,y,z,t)$, perhaps the better expression for the solution of the equation (1.4) is

(6.1)
$$p(x,y,z,t) = \int_L S \cdot dl + \theta(t) = \sum_{i=1}^{3} \int_{P_i^0}^{P_i} S_i dx_i + \theta(t),$$
$$S = (S_1,\ S_2,\ S_3),$$
$$S_i = -\left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}\right) + \nu(\nabla^2 u_i) + \frac{1}{3}\nu\big(\nabla_i(\nabla \cdot u)\big) + f_i,$$

supposing possible the integrations and that the vector $S = -\left[\frac{\partial u}{\partial t} + (u \cdot \nabla)u\right] + \nu\nabla^2 u + \frac{1}{3}\nu\nabla(\nabla \cdot u) + f$ is a gradient function, where it is necessary that

(6.2)
$$\frac{\partial S_i}{\partial x_j} = \frac{\partial S_j}{\partial x_i}.$$

This is the development of the solution of (1.4) for the specific path L going parallely (or perpendicularly) to axes X, Y and Z from $(x_1^0, x_2^0, x_3^0) \equiv (x_0, y_0, z_0)$ to $(x_1, x_2, x_3) \equiv (x, y, z)$, since that the solution (6.1) is valid for any piecewise smooth path L. We can choose $P_1^0 = (x_0, y_0, z_0)$, $P_2^0 = (x, y_0, z_0)$, $P_3^0 = (x, y, z_0)$ for the origin points and $P_1 = (x, y_0, z_0)$, $P_2 = (x, y, z_0)$, $P_3 = (x, y, z)$ for the destination points. $\theta(t)$ is a generic time function, physically and mathematically reasonable, for example with $\theta(0) = 0$ or adjustable for some given condition. Again we have seen that the system of Navier-Stokes equations has no unique solution, only given initial conditions, supposing that there is some solution. We can choose different velocities that have the same initial velocity and also result, in general, in different pressures.

The remark given for the system (1.5), when used in (1.4), leads us to the following conclusion: the integration of the system (1.4), confronting with (1.5), shows that, except for a constant or free term of integration, respectively $A(y,z,t)$, $B(x,z,t)$ and $C(x,y,t)$, anyone of its equations can be used for solve it, and the results must be equals each other, if the velocity u and external force f are given and the pressure p must be calculated. Then again this is a condition to the occurrence of solutions, otherwise there is not any solution, which shows to us the possibility of existence of “breakdown” solutions, as defined in [10].

By other side, using the first condition (1.1), $\frac{\partial u_i}{\partial x_j} = 0$ if $i \neq j$, due to Lagrangian formulation, where $u_i = \frac{dx_i}{dt}$, the original system (1.4) is simplified as

$$(6.3) \qquad \begin{cases} \frac{\partial p}{\partial x} + \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} = \frac{4}{3}\nu \frac{\partial^2 u_1}{\partial x^2} + f_1 \\ \frac{\partial p}{\partial y} + \frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial y} = \frac{4}{3}\nu \frac{\partial^2 u_2}{\partial y^2} + f_2 \\ \frac{\partial p}{\partial z} + \frac{\partial u_3}{\partial t} + u_3 \frac{\partial u_3}{\partial z} = \frac{4}{3}\nu \frac{\partial^2 u_3}{\partial z^2} + f_3 \end{cases}$$

where u_i is a function only of the respective x_i and t, but not x_j if $j \neq i$. When it is required the incompressibility condition, $\nabla \cdot u = \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right) = 0$, then the constant $\frac{4}{3}$ in (6.3) should be replaced by 1.

If the external force has potential, $f = \nabla V$, then the system (6.3) has solution

$$(6.4) \qquad \begin{aligned} p &= \sum_{i=1}^{3} \int_{P_i^0}^{P_i} \left[-\left(\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x_i} \right) + \frac{4}{3}\nu \frac{\partial^2 u_i}{\partial x_i^2} + f_i \right] dx_i + \theta(t) \\ &= V + \sum_{i=1}^{3} \int_{x_i^0}^{x_i} \left[-\left(\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x_i} \right) + \frac{4}{3}\nu \frac{\partial^2 u_i}{\partial x_i^2} \right] dx_i + \theta(t), \end{aligned}$$

$V = \int_L \; f \cdot dl$, which although similar to (6.1) has the solubility guaranteed by the special functional dependence of the components of the vector u, i.e., $u_i = u_i(x_i, t)$, with $\frac{\partial u_i}{\partial x_j} = 0$ if $i \neq j$, supposing u, its derivatives and f integrable vectors. In this case the vector S described in (6.1) is always a gradient function, i.e., the relation (6.2) is satisfied. Note that if f is not an irrotational or gradient vector, i.e., if it does not have a potential, then the system (6.3), with $u_i = u_i(x_i, t)$, it has no solution, the case of “breakdown” solution in [10].

When the incompressibility condition is imposed ($\nabla \cdot u = 0$) we have, using (1.1), a small variety of possible solutions for velocity, of the form

$$(6.5) \qquad u_i(x_i, t) = A_i(t) x_i + B_i(t),$$

$A_i, B_i \in C^\infty([0, \infty))$, with

$$(6.6) \qquad A_1(t) + A_2(t) + A_3(t) = 0,$$

if the coordinates $x_1,\ x_2,\ x_3$ are independent of each other. In this case it is valid $\nabla^2 u = 0$, i.e., the system of equations has a solution for velocity independent of viscosity coefficient, equal to Euler equations, and except when $u = 0$ (for some or all $t \geq 0$) we have always $\int_{\mathbb{R}^3} |u|^2\, dxdydz \to \infty$, the occurrence of unbounded or unlimited energy, which is not difficult to see.

Another class of solutions S for velocity gives more possibility for the construction of the components of velocity u_i, but maintains a bond between $x_1,\ x_2,\ x_3$ and t such that

$$(6.7)\quad S = \{(u_1, u_2, u_3);\ u_i \in C^1(\mathbb{R} \times \mathbb{R}_0^+),\ (x_1, x_2, x_3, t) \in \mathbb{R}^3 \times \mathbb{R}_0^+,\ \nabla \cdot u = 0\},$$

where $\mathbb{R}_0^+ = [0, \infty)$, and there is a scalar function φ_3 with $x_3 = \varphi_3(x_1, x_2, t)$ or similarly $x_1 = \varphi_1(x_2, x_3, t)$ or $x_2 = \varphi_2(x_1, x_3, t)$. The dependence between $x_1,\ x_2,\ x_3$ and t is necessary for that $\nabla \cdot u = 0$ in these points (x_1, x_2, x_3) at each time t, forming a surface or manifold which is the domain of the solutions and which varies in time.

Being correct that (1.1) and (4.1) can be used, which we saw in section 5, the solution (6.4) for pressure can therefore be replaced by

$$(6.8)\qquad \begin{aligned} p &= \sum_{i=1}^3 \int_{P_i^0}^{P_i} \left[-\left(\frac{\partial u_i}{\partial t} + \alpha_i \frac{\partial u_i}{\partial x_i} \right) + \frac{4}{3} \nu \frac{\partial^2 u_i}{\partial x_i^2} + f_i \right] dx_i + \theta(t) \\ &= V + \sum_{i=1}^3 \int_{x_i^0}^{x_i} \left[-\left(\frac{\partial u_i}{\partial t} + \alpha_i \frac{\partial u_i}{\partial x_i} \right) + \frac{4}{3} \nu \frac{\partial^2 u_i}{\partial x_i^2} \right] dx_i + \theta(t) \\ &= V + \sum_{i=1}^3 \left[p_i(x_i, t) - p_i\left(x_i^0, t\right) \right] + \theta(t), \end{aligned}$$

where $\alpha_i = \alpha_i(t)$ is the component i of the velocity in Lagrangian description of a particle of fluid in motion, $u_i = u_i(x_i,\ t)$ is the component i of the velocity in Eulerian description, $p_i(x_i, t) = \int_{x_i^0}^{x_i} \left[-\left(\frac{\partial u_i}{\partial t} + \alpha_i \frac{\partial u_i}{\partial x_i} \right) + \frac{4}{3} \nu \frac{\partial^2 u_i}{\partial x_i^2} \right] dx_i$ and the other meanings already given previously in this article. As we have already seen, when it is required the incompressibility condition then the constant $\frac{4}{3}$ in (6.8) should be replaced by 1 and the general solution (6.5) for velocity with the condition (6.6) remains valid, if the coordinates $x_1,\ x_2,\ x_3$ are independent of each other, as well as (6.7) with possible dependence between $x_1,\ x_2,\ x_3$ and t.

In section 8, Conclusion, we will see other cases of breakdown solution, when the Euler and Navier-Stokes equations have no solution.

7 – The non-uniqueness of solutions

The new equations presented here have clearly non-unique solutions (when there is at least one solution) in the following sense:

1) For the same initial Eulerian velocity, indicated as u^0, we can propose different velocities in the Lagrangian description, u^L, to compose the new equations, also with possibility of collisions between the particles belonging to the different movements described by each u^L. This can result in a rather chaotic Eulerian solution for velocity, in fact many velocities for a same point, and consequently also for the pressure, if it has not previously been chosen.

2) When we analyze the uniqueness of solutions (u^E, p) bearing in mind that the Lagrangian velocity u^L is predetermined, if only the initial velocity u^0 is given we have the non uniqueness of the pair (u^E, p) because we can construct many possible and different velocities u^E, as $u^E = \varphi(t)u^0 + \tau(t), \varphi(0) = 1, \tau(0) = 0$, $\varphi: [0, \infty) \to \mathbb{R}$, $\tau: [0, \infty) \to \mathbb{R}^3$, all smooth functions, and the pressure will be given by (6.8), where we are supposing the use of (1.1), i.e., $u_i^E = u_i^E(x_i, t)$, with $\frac{\partial u_i^E}{\partial x_j} = 0$ if $i \neq j$. Note that in this case we have $\nabla \times u^E = 0$ and the equation has solution, again with many possible pressures.

3) If is given a boundary condition of type $u^E|_{\partial S} = u^\partial$ (Dirichlet condition), with $u^\partial \in C^\infty(\mathbb{R}^3 \times [0, \infty))$ and $u^\partial(x, y, z, t = 0) = u^0$, then we can use the solution for velocity as $u^E = u^\partial$ and also we have the non uniqueness of the pair (u^E, p), because for the pressure to be unique it needs to be known the values of $p_1(x_0, t)$, $p_2(y_0, t)$, $p_3(z_0, t)$, i.e., the pressure is dependent of the values of x_0, y_0, z_0, and moreover $\theta(t)$, according (6.8). Naturally, the velocities u^∂ and u^0 must, themselves, obey to the new equations of Euler and Navier-Stokes, u^∂ for $t \geq 0$ and u^0 for $t = 0$. Note that in our convention the functions $p_1(x_0, t)$, $p_2(y_0, t)$, $p_3(z_0, t)$ denote the pressure value in a generic time $t \geq 0$, respectively at the positions (x_0, y, z), (x, y_0, z), (x, y, z_0), where (x_0, y_0, z_0) is the initial position. In this condition we have $\theta(t = 0) = 0$.

8 – Conclusion

In fact we saw two problems in Euler and Navier-Stokes equations, not only one:

1) the pressure is (or may be) a vector, which was viewed briefly in sections 2 and 3 during the deductions of these equations;

2) the nonlinear characteristic of these equations is not correct for modeling of motion of fluids, because the use of chain rule in $\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$ implies that $u_1 = \frac{dx}{dt}$, $u_2 = \frac{dy}{dt}$ and $u_3 = \frac{dz}{dt}$ are time functions only, without spatial dependence, which we viewed in section 4.

We propose a new form for the Euler ($\nu = 0$) and Navier-Stokes equations, where there is the simultaneous use of Euler and Lagrangian descriptions in a same equation, i.e., for $i = 1, 2, 3$,

(8.1) $$\frac{\partial p}{\partial x_i} + \frac{\partial u_i}{\partial t} + \alpha_1 \frac{\partial u_i}{\partial x} + \alpha_2 \frac{\partial u_i}{\partial y} + \alpha_3 \frac{\partial u_i}{\partial z} = \nu \nabla^2 u_i + \frac{1}{3}\nu \frac{\partial}{\partial x_i}(\nabla \cdot u) + f_i,$$

where p, f_i, u and u_i are in Eulerian description and $\alpha_i = \alpha_i(t)$ in Lagrangian description, i.e., $\alpha_i = \frac{dx_i}{dt}$, according equation (4.3). Of this manner the nonlinear form of these equations disappear, replacing it by linear equations, a second-order equation of elliptic type if $\nu > 0$ or first order equation if $\nu = 0$.

Obviously, using the vector nature of pressure the equation (8.1) needs to be modified to

(8.2) $$\frac{\partial p_i}{\partial x_i} + \frac{\partial u_i}{\partial t} + \alpha_1 \frac{\partial u_i}{\partial x} + \alpha_2 \frac{\partial u_i}{\partial y} + \alpha_3 \frac{\partial u_i}{\partial z} = \nu \nabla^2 u_i + \frac{1}{3}\nu \frac{\partial}{\partial x_i}(\nabla \cdot u) + f_i.$$

In (8.1) it is still necessary to have a resultant conservative field, a gradient vector, specifically for the integrable vector $S = (S_1,\ S_2,\ S_3)$, with

(8.3) $$S_i = \left(\nu \nabla^2 u_i + \frac{1}{3}\nu \frac{\partial}{\partial x_i}(\nabla \cdot u) + f_i\right) - \left(\frac{\partial u_i}{\partial t} + \alpha_1 \frac{\partial u_i}{\partial x} + \alpha_2 \frac{\partial u_i}{\partial y} + \alpha_3 \frac{\partial u_i}{\partial z}\right),$$

whereas in equation (8.2) this is no longer necessary.

In section 4 we conclude that the new Euler equations have a natural physical solution when the pressure and external force are given (or chosen) and the integration in (4.6), which is the mentioned solution,

(8.4) $$u_i = u_i^0 + \left(\int_0^t \left(f_i - \frac{\partial p}{\partial x_i}\right)|_L \, dt\right)|_E,$$

is possible, for $i = 1, 2, 3$, in general a non unique solution varying with the transformations indicated as $|_L$ and $|_E$. Beside this, boundary conditions must be in accordance with this solution, as well as it is necessary the verification of possible conditions to be obeyed by each u_i^0 and $\alpha_i(t)$, substituting the solution in the equation, for that the mentioned solution effectively satisfies the equation of a mathematical point of view.

The functions α describe the velocity of the particles of the fluid over time, so the importance of them can be considered greater than that of velocity u, that is, it is convenient to choose initial velocities u^0 as simple as possible that are compatible with the selected movement described by the α functions, in special: $u^0(x_0, y_0, z_0) = \alpha(t = 0, x_0, y_0, z_0)$. Without the compromise of the equality in time of the Eulerian and Lagrangian descriptions, it is even possible that different velocities u, for example $u' \neq u''$, correspond to the same motion described by α, and we have $div\, u' = 0$ and $div\, u'' \neq 0$. So, seems that the incompressibility condition is not of priority importance for the description of motion of fluids. Note that similarly to what we have already said in section 5, we use implicitly the initial position (x_0, y_0, z_0) in the function $\alpha(t)$ as constant, although it has the same meaning as in $\alpha(t, x_0, y_0, z_0)$. Other constant parameters also can be included, of course: $R, \theta_0, \omega, \nu, \rho$, etc., able to describe a very large class of motions.

It is also possible an easier form for the Euler ($\nu = 0$) and Navier-Stokes equations, that is

$$\text{(8.5)} \qquad \frac{\partial p_i}{\partial x_i} + \frac{D\alpha_i}{Dt} = \nu\nabla^2 u_i + \frac{1}{3}\nu\frac{\partial}{\partial x_i}(\nabla \cdot u) + f_i,$$

where we can substitute p_i by p if $p_1 = p_2 = p_3 = p$ is scalar pressure. Here $\frac{D\alpha_i}{Dt}$ is, in fact, a function only of time (and possibly constant parameters), without explicit dependence of x, y, z. The new forms for these equations are most didactic, because they can remind us of the need to be valid

$$\text{(8.6)} \qquad u^E(x(t), y(t), z(t), t) = u^L(t) = \alpha(t) = \frac{d}{dt}(x(t), y(t), z(t))$$

and

$$\text{(8.7)} \qquad \frac{Du^E}{Dt}\Big|_t = \frac{Du^L}{Dt} = \frac{D\alpha}{Dt} = \left(\frac{\partial u^E}{\partial t} + \alpha_1\frac{\partial u^E}{\partial x} + \alpha_2\frac{\partial u^E}{\partial y} + \alpha_3\frac{\partial u^E}{\partial z}\right)\Big|_t$$

when we analyze a fluid motion, a physical system, not only the solution of a problem purely mathematical, without application.

Now, to solve the equations of Navier-Stokes, and especially the Euler equations, is no more difficult than solve the traditional equations of mathematical physics, as heat equation, wave equation, Laplace and Poisson equations, etc., all of them linear differential equations. Despite this, in case of scalar pressure, if $\nu = 0$ and the external force is non conservative there is no solution for Euler equations, as well as if the initial velocity is gradient ($u^0 = \nabla\phi^0,\ \nabla \times u^0 = 0$) and the external force is non conservative, which leads us to the case of breakdown solution described in [10], when the pressure is a scalar function, because is not possible the calculation of pressure, according rule (6.2) viewed in section 6.

Note that the application of a non conservative force in fluid is naturally possible and there will always be some movement, even starting from rest. So that this is not a paradoxical situation it seems certain that the pressure in this case cannot be scalar, but rather vector, and thus the equations returns to solution in all cases (assuming all derivatives are possible, etc.). It is as indicated in (8.2) and (8.5). With the use of vector pressure the conditions mentioned for systems (1.4) and (1.5) also becomes unnecessary.

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23 – Describing a fluid in three-dimensional circular motion with at most one spatial variable by rectangular coordinate

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Abstract: Describe a fluid in three-dimensional circular motion with at most one spatial variable by rectangular coordinate, beyond time, and concludes on the breakdown of Euler and Navier-Stokes solutions and the necessity of use of vector pressure.

In [1] we showed that the three-dimensional Euler ($\nu = 0$) and Navier-Stokes equations in rectangular coordinates need to be adopted as

(1) $$\frac{\partial p}{\partial x_i} + \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i + \frac{1}{3} \nu \nabla_i (\nabla \cdot u) + f_i,$$

for $i = 1,2,3$, where $\alpha_j = \frac{dx_j}{dt}$ is the velocity in Lagrangian description and u_i and the partial derivatives of u_i are in Eulerian description, as well as the scalar pressure p and density of external force f_i. The coefficient of viscosity is ν and by ease we prefer to use the mass density $\rho = 1$ (otherwise substitute p by p/ρ and ν by ν/ρ).

An alternative equation is

(2) $$\frac{\partial p_i}{\partial x_i} + \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i + \frac{1}{3} \nu \nabla_i (\nabla \cdot u) + f_i,$$

thus making the pressure a vector: $p = (p_1, p_2, p_3)$. In both equations is valid

(3) $$\frac{Du_i}{Dt} = \frac{Du_i^E}{Dt} = \frac{Du_i^L}{Dt} = \left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j} \right) |_L,$$

where the upper letter E refers to Eulerian velocity and L to Lagrangian velocity. The symbol $|_L$ means the respective calculation in Lagrangian description, substituting each x_i as a function of time, initial value and eventually some parameters.

A condition indicated by us in [1] were

(4) $$\begin{cases} \frac{\partial u_i}{\partial x_j} = 0, \ i \neq j, \\ \partial x_i = u_i \partial t \end{cases}$$

because we have, by definition,

(5) $$u_i = \frac{dx_i}{dt},$$

in Lagrangian description, and for this reason the velocity u_i, *a priori*, is not dependent of others variables x_j, with $x_j \neq x_i$. More than a rigorous mathematical proof, this is a practical approach, which simplifies the original system.

It is very easy to accept the first equation of (4) when there is no link between the spatial coordinates during the movement of the fluid over time, but in a circular motion, for example, it seems to be no longer valid.

Let a circular motion of radius R, centered at (x_C, y_C) and with constant angular velocity $\omega > 0$ described by the equations:

(6) $$\begin{cases} x = x_C + R\cos(\theta_0 + \omega t) \\ y = y_C + R\sin(\theta_0 + \omega t) \end{cases}$$

and consequently

(7) $$(x - x_C)^2 + \left(y - y_C\right)^2 = R^2.$$

Then the velocity components are

(8) $$\begin{cases} \alpha_1 = u_1^L = \dot{x} = -\omega R\sin(\theta_0 + \omega t) = -\omega(y - y_C) = u_1^E \\ \alpha_2 = u_2^L = \dot{y} = +\omega R\cos(\theta_0 + \omega t) = +\omega(x - x_C) = u_2^E \end{cases}$$

and the acceleration components are

(9) $$\begin{cases} \frac{Du_1^L}{Dt} = \ddot{x} = -\omega^2 R\cos(\theta_0 + \omega t) = -\omega^2(x - x_C) = \frac{Du_1^E}{Dt} \\ \frac{Du_2^L}{Dt} = \ddot{y} = -\omega^2 R\sin(\theta_0 + \omega t) = -\omega^2(y - y_C) = \frac{Du_2^E}{Dt} \end{cases}$$

Supposing that the particles of fluid obey the motion described by (6) to (9), we have

(10) $$\begin{cases} \frac{\partial u_1}{\partial y} = -\omega, \quad \frac{\partial u_1}{\partial x} = 0 \\ \frac{\partial u_2}{\partial x} = +\omega, \quad \frac{\partial u_2}{\partial y} = 0 \end{cases}$$

apparently in disagree with (4) if $\omega \neq 0$. But, as x is a function of y and reciprocally, in this circular motion according (7), again (4) turns valid, for any signal of x and y. For to complete a three-dimensional description, we define $z = z_0$, without dependence of time, and $u_3 = 0$.

This is a motion of velocity without potential, because $\frac{\partial u_i}{\partial x_j} \neq \frac{\partial u_j}{\partial x_i}$ for some $i \neq j$, but if $f = (f_1, f_2, f_3)$ has potential we have $\frac{\partial s_i}{\partial x_j} = \frac{\partial s_j}{\partial x_i}$ for all $i, j = 1,2,3$, with

$$(11)\quad S_i = -\frac{\partial u_i}{\partial t} - \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j} + \nu\nabla^2 u_i + \frac{1}{3}\nu\frac{\partial}{\partial x_i}(\nabla\cdot u) + f_i,$$

then the system (1) has solution.

A calculation for the scalar pressure of this motion is

$$(12)\quad \begin{aligned} p &= \int_L (S_1, S_2, S_3)\cdot dl = \int_L \left(-\frac{Du}{Dt} + f\right)\cdot dl \\ &= \omega^2\left[\left(\frac{x^2}{2} - x_C x\right)\Big|_{x_0}^{x} + \left(\frac{y^2}{2} - y_C y\right)\Big|_{y_0}^{y}\right] + U - U_0 + \theta(t) \\ &= \omega^2\left[\left(\frac{x^2}{2} - x_C x\right) - \left(\frac{{x_0}^2}{2} - x_C x_0\right) + \left(\frac{y^2}{2} - y_C y\right) - \left(\frac{{y_0}^2}{2} - y_C y_0\right)\right] + \\ &\quad U - U_0 + \theta(t), \end{aligned}$$

where $f = \nabla U$, $U_0 = U(x_0, y_0, z_0, t)$ and L is any smooth path linking a point (x_0, y_0, z_0) to (x, y, z). We can ignore the use of x_0, y_0, z_0 and U_0, and use only the free function for time, $\theta(t)$, which on the other hand can include the terms in x_0, y_0 and z_0, and nevertheless this solution shows us that the pressure is not uniquely well determined, therefore we get to the negative answer to Smale's 15th problem, according already seen in [2] and [3], even if we assign the velocity value on some surface that we wish and even if $\theta(t)$ and U does not depend explicitly on the variable time t. In this motion the pressure is dependent, besides of x, y and U, without any problematic question, and x_C, y_C and ω, specific parameters of the movement conditions of a particle, of $\theta(t)$, U_0 and more three parameters, x_0, y_0 and z_0, then there is not uniqueness of solution.

Another calculation for pressure is possible due to fact that we can describe the acceleration $\frac{Du}{Dt}$ of a particle of fluid as a function only of time, $\frac{D\alpha}{Dt}$, without the variables x, y, z, and then

$$(13)\quad \begin{aligned} p &= -\frac{D\alpha}{Dt}\cdot\int_L dl + U - U_0 + \theta(t) \\ &= +\omega^2 R[\cos(\theta_0 + \omega t)(x - x_0) + \sin(\theta_0 + \omega t)(y - y_0)] \\ &\quad + U - U_0 + \theta(t), \end{aligned}$$

with

$$(14)\quad \begin{cases} \frac{\partial p}{\partial x} = +\omega^2 R\cos(\theta_0 + \omega t) + f_1 = +\omega^2(x - x_C) + f_1 \\ \frac{\partial p}{\partial y} = +\omega^2 R\sin(\theta_0 + \omega t) + f_2 = +\omega^2(y - y_C) + f_2 \\ \frac{\partial p}{\partial z} = f_3 \end{cases}$$

in fact derivatives such as can be obtained from (12).

Note that in order to continue using the traditional form of the Euler and Navier-Stokes equations we will have non-linear equations, which can make it difficult to obtain the solutions and bring all the difficulties that we know. To make sense to use the velocity in Eulerian description rather than the Lagrangian description in α_j it is necessary that, for all $t \geq 0$,

(15) $$u^E(x(t), y(t), z(t), t) = \alpha(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = u^L(t),$$

omitting the use of possible parameters of motion, then nothing more natural than the definitive substitution of the terms $\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j}$, as well as $\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}$ in the traditional form, by $\frac{Du_i^L}{Dt}$ or $\frac{D\alpha_i}{Dt}$. This brings a great and important simplification in the equations, and to return to having the position as reference it is enough to use the conversion or definition adopted for $x(t), y(t)$ and $z(t)$, including the possible additional parameters, for example, substituting initial positions in function of position and time, etc.

Thus, more appropriate Euler ($\nu = 0$) and Navier-Stokes equations with scalar pressure are, in index notation,

(16) $$\frac{\partial p}{\partial x_i} + \frac{D\alpha_i}{Dt} = \nu \nabla^2 u_i + \frac{1}{3} \nu \frac{\partial}{\partial x_i}(\nabla \cdot u) + f_i.$$

If $\nu = 0$ and f is not conservative then there is no solution for Euler equations, as well as if u is conservative and f is not conservative there is no solution for Navier-Stokes equations, which now it is very clear to see from (16) and it is according [4]. More specifically, if u^0, the initial velocity, is conservative (irrotational or potential flow) and f is not conservative then there is no solution for Navier-Stokes equations, because it is impossible to obtain the pressure. This then solve [5] for the cases (C) and (D), the breakdown of solutions, for both u^0 and f belonging to Schwartz Space in case (C), and smooth functions with period 1 in the three orthogonal directions e_1, e_2, e_3 in case (D). As u^0 need obey to the incompressibility condition, $\nabla \cdot u^0 = 0$, with $\nabla \times u^0 = 0$ and $u^0 = \nabla \varphi^0$, where φ^0 is the potential of u^0, we have $\nabla^2 u^0 = 0$ and $\nabla^2 \varphi^0 = 0$, i.e., u^0 and φ^0 are harmonic functions, unlimited functions except the constants, including zero. As u^0 need be limited, we choose $u^0 = 0$ for case (C) (where it is necessary that $\int_{\mathbb{R}^3} |u^0|^2 dx\, dy\, dz$ is finite) and any constant for case (D), of spatially periodic solutions. In case (D) the external force need belonging to Schwartz Space with relation to time.

Note that the application of a non conservative force in fluid is naturally possible and there will always be some movement, even starting from rest. So that this is not a paradoxical situation it seems certain that the pressure in this case cannot be scalar, but rather vector, and thus the equation returns to solution in all cases (assuming all derivatives are possible, etc.). It is as indicated in (2), or substituting p by p_i in (16).

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24 – Describing a Fluid Motion with 3-D Rectangular Coordinates

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Abstract: We describe a fluid in three-dimensional motion with at most one spatial variable by rectangular coordinate, beyond time, and conclude on the breakdown of Euler and Navier-Stokes solutions and the necessity of use of vector pressure.

Keywords: Euler equations, Navier-Stokes equations, Lagrangian description, Eulerian description, Bernoulli's law, breakdown solutions, vector pressure.

1 – Introduction

In [1] we showed that the three-dimensional Euler ($\nu = 0$) and Navier-Stokes equations in rectangular coordinates need to be adopted as

(1) $$\frac{\partial p}{\partial x_i} + \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i + \frac{1}{3}\nu \frac{\partial}{\partial x_i}(\nabla \cdot u) + f_i,$$

for $i = 1,2,3$, where $\alpha_j = \frac{dx_j}{dt}$ is the velocity in Lagrangian description and u_i and the partial derivatives of u_i are in Eulerian description, as well as the scalar pressure p and density of external force f_i. The coefficient of viscosity is ν and by ease we prefer to use the mass density $\rho = 1$ (otherwise substitute p by p/ρ and ν by ν/ρ).

An alternative equation is

(2) $$\frac{\partial p_i}{\partial x_i} + \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i + \frac{1}{3}\nu \frac{\partial}{\partial x_i}(\nabla \cdot u) + f_i,$$

thus making the pressure a vector: $p = (p_1, p_2, p_3)$. In both equations is valid

(3) $$\frac{Du_i}{Dt} = \frac{Du_i^E}{Dt} = \frac{Du_i^L}{Dt} = \left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j}\right)\Big|_L,$$

where the upper letter E refers to Eulerian velocity (u) and L to Lagrangian velocity (α). The symbol $|_L$ means the respective calculation in Lagrangian description, substituting each x_i as a function of time, initial value and eventually some parameters. With the notation $\frac{D}{Dt}$ we want, in principle, to make explicit that we are calculating a total derivative in relation to time, and the result is a function exclusively of time (and possibly a set of parameters), without the spatial coordinates x, y, z, but when for some reason we need to leave the result as a function of the spatial coordinates we can also do it.

A condition indicated by us in [1] were

$$(4) \qquad \begin{cases} \frac{\partial u_i}{\partial x_j} = 0, \; i \neq j, \\ \partial x_i = u_i \partial t \end{cases}$$

because we have, by definition,

$$(5) \qquad u_i = \frac{dx_i}{dt},$$

in Lagrangian description, and for this reason the velocity u_i, *a priori*, is not dependent of others variables x_j, with $x_j \neq x_i$. More than a rigorous mathematical proof, this is a practical approach, which simplifies the original system.

It is very easy to accept the first equation of (4) when there is no link between the spatial coordinates during the movement of the fluid over time, but in a circular motion, for example, it seems to be no longer valid. In order to show how it is possible to describe a motion with a single independent spatial variable by rectangular coordinate, $u_i = \varphi_i(x_i, t)$, we will describe in section 2 a circular motion and in section 3 a quite general movement.

The section 4 will be our Conclusion, concluding on the breakdown solutions and the necessity of use of vector pressure.

2 – Circular Motion

Let a circular motion of radius R, centered at $(x_C,\ y_C)$ and with constant angular velocity $\omega > 0$ described by the equations:

$$(6) \qquad \begin{cases} x = x_C + R\cos(\theta_0 + \omega t) \\ y = y_C + R\sin(\theta_0 + \omega t) \end{cases}$$

and consequently

$$(7) \qquad (x - x_C)^2 + (y - y_C)^2 = R^2.$$

Then the velocity components are

$$(8) \qquad \begin{cases} \alpha_1 = u_1^L = \dot{x} = -\omega R \sin(\theta_0 + \omega t) = -\omega(y - y_C) = u_1^E \\ \alpha_2 = u_2^L = \dot{y} = +\omega R \cos(\theta_0 + \omega t) = +\omega(x - x_C) = u_2^E \end{cases}$$

and the acceleration components are

(9) $$\begin{cases}\frac{Du_1^L}{Dt} = \ddot{x} = -\omega^2 R\cos(\theta_0+\omega t) = -\omega^2(x-x_C) = \frac{Du_1^E}{Dt}\\ \frac{Du_2^L}{Dt} = \ddot{y} = -\omega^2 R\sin(\theta_0+\omega t) = -\omega^2(y-y_C) = \frac{Du_2^E}{Dt}\end{cases}$$

Supposing that the particles of fluid obey the motion described by (6) to (9), we have

(10) $$\begin{cases}\frac{\partial u_1}{\partial y} = -\omega, \quad \frac{\partial u_1}{\partial x} = 0\\ \frac{\partial u_2}{\partial x} = +\omega, \quad \frac{\partial u_2}{\partial y} = 0\end{cases}$$

apparently in disagree with (4) if $\omega \neq 0$. But, as x is a function of y and reciprocally, in this circular motion according (7), again (4) turns valid, for any signal of x and y. For to complete a three-dimensional description, we define $z = z_0$, without dependence of time, and $u_3 = 0$.

This is a motion of velocity without potential, because $\frac{\partial u_i}{\partial x_j} \neq \frac{\partial u_j}{\partial x_i}$ for some $i \neq j$, but if $f = (f_1, f_2, f_3)$ has potential we have $\frac{\partial S_i}{\partial x_j} = \frac{\partial S_j}{\partial x_i}$ for all $i, j = 1,2,3$, with

(11) $$S_i = -\frac{\partial u_i}{\partial t} - \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j} + \nu\nabla^2 u_i + \frac{1}{3}\nu\frac{\partial}{\partial x_i}(\nabla\cdot u) + f_i,$$

then the system (1) has solution.

A calculation for the scalar pressure of this motion is

(12) $$\begin{aligned} p &= \int_L (S_1, S_2, S_3)\cdot dl = \int_L \left(-\frac{Du}{Dt} + f\right)\cdot dl \\ &= \omega^2\left[\left(\frac{x^2}{2} - x_C x\right)\Big|_{x_0}^{x} + \left(\frac{y^2}{2} - y_C y\right)\Big|_{y_0}^{y}\right] + U - U_0 + \theta(t) \\ &= \omega^2\left[\left(\frac{x^2}{2} - x_C x\right) - \left(\frac{{x_0}^2}{2} - x_C x_0\right) + \left(\frac{y^2}{2} - y_C y\right) - \left(\frac{{y_0}^2}{2} - y_C y_0\right)\right] + \\ &\quad U - U_0 + \theta(t), \end{aligned}$$

where $f = \nabla U$, $U_0 = U(x_0, y_0, z_0, t)$ and L is any smooth path linking a point (x_0, y_0, z_0) to (x, y, z). We can ignore the use of x_0, y_0, z_0 and U_0, and use only the free function for time, $\theta(t)$, which on the other hand can include the terms in x_0, y_0 and z_0, and nevertheless this solution shows us that the pressure is not uniquely well determined, therefore we get to the negative answer to Smale's 15th problem, according already seen in [2] and [3], even if we assign the velocity value on some surface that we wish and even if $\theta(t)$ and U does not depend explicitly on the variable time t. In this motion the pressure is dependent, besides of x, y and U, without any problematic question, and x_C, y_C and ω, specific parameters of the movement conditions of a particle, of $\theta(t)$, U_0 and more three parameters, x_0, y_0 and z_0, then there is not uniqueness of solution.

Another calculation for pressure is possible due to fact that we can describe the acceleration $\frac{Du}{Dt}$ of a particle of fluid as a function only of time, $\frac{D\alpha}{Dt}$, without the variables x, y, z, and then

$$\text{(13)}\quad \begin{aligned} p &= -\frac{D\alpha}{Dt} \cdot \int_L dl + U - U_0 + \theta(t) \\ &= +\omega^2 R[\cos(\theta_0 + \omega t)(x - x_0) + \sin(\theta_0 + \omega t)(y - y_0)] \\ &\quad + U - U_0 + \theta(t), \end{aligned}$$

with

$$\text{(14)}\quad \begin{cases} \frac{\partial p}{\partial x} = +\omega^2 R \cos(\theta_0 + \omega t) + f_1 = +\omega^2 (x - x_C) + f_1 \\ \frac{\partial p}{\partial y} = +\omega^2 R \sin(\theta_0 + \omega t) + f_2 = +\omega^2 (y - y_C) + f_2 \\ \frac{\partial p}{\partial z} = f_3 \end{cases}$$

in fact derivatives such as can be obtained from (12).

Note that in order to continue using the traditional form of the Euler and Navier-Stokes equations we will have non-linear equations, which can make it difficult to obtain the solutions and bring all the difficulties that we know. To make sense to use the velocity in Eulerian description rather than the Lagrangian description in α_j it is necessary that, for all $t \geq 0$,

$$\text{(15)}\quad u^E(x(t), y(t), z(t), t) = \alpha(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = u^L(t),$$

omitting the use of possible parameters of motion, then nothing more natural than the definitive substitution of the terms $\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j}$, as well as $\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}$ in the traditional form, by $\frac{Du_i^L}{Dt}$ or $\frac{D\alpha_i}{Dt}$. This brings a great and important simplification in the equations, and to return to having the position as reference it is enough to use the conversion or definition adopted for $x(t), y(t)$ and $z(t)$, including the possible additional parameters, for example, substituting initial positions in function of position and time, etc.

Thus, more appropriate Euler ($\nu = 0$) and Navier-Stokes equations with scalar pressure are, in index notation,

$$\text{(16)}\quad \frac{\partial p}{\partial x_i} + \frac{D\alpha_i}{Dt} = \nu \nabla^2 u_i + \frac{1}{3} \nu \frac{\partial}{\partial x_i} (\nabla \cdot u) + f_i.$$

3 – Generic three-dimensional motion

Suppose that a particle of fluid moves according to equation

(17) $x_i = A_i(t)x_i^0 + B_i(t),$

$A_i(0) = 1,\ B_i(0) = 0,\ A_i, B_i \in C^\infty\big([0,\infty)\big),\ i = 1,2,3,\ (x_1, x_2, x_3) \equiv (x, y, z)$,where $(x_1^0, x_2^0, x_3^0) \equiv (x_0, y_0, z_0)$ is the initial position of this particle in relation to three-orthogonal system of reference considered at rest.

Your velocity in relation to this system is, for $i = 1,2,3$,

(18) $\dot{x}_i = \frac{d}{dt} x_i = u_i^L = \alpha_i = A_i'(t)x_i^0 + B_i'(t),$

with acceleration

(19) $\ddot{x}_i = \frac{d}{dt}\dot{x}_i = \frac{D}{Dt}u_i^L = \frac{D}{Dt}\alpha_i = A_i''(t)x_i^0 + B_i''(t).$

We are using both the superior point $(\dot{x})$ and the prime mark (A'), and respective repetitions, for indicate differentiations in relation to time.

We are going to transform Lagrangian velocity into Eulerian velocity through transformation

(20) $x_i^0 = \frac{x_i - B_i(t)}{A_i(t)},$

which results in

(21) $$u_i^E = A_i'(t)x_i^0 + B_i'(t) = A_i'(t)\frac{x_i - B_i(t)}{A_i(t)} + B_i'(t)$$
$$= \frac{A_i'(t)}{A_i(t)}x_i - \frac{A_i'(t)B_i(t)}{A_i(t)} + B_i'(t)$$

and

(22) $$\frac{Du_i^E}{Dt} = A_i''(t)x_i^0 + B_i''(t) = A_i''(t)\frac{x_i - B_i(t)}{A_i(t)} + B_i''(t)$$
$$= \frac{A_i''(t)}{A_i(t)}x_i - \frac{A_i''(t)B_i(t)}{A_i(t)} + B_i''(t).$$

We see that both u_i^E and $\frac{Du_i^E}{Dt}$ are linear functions in x_i or only functions of time if $A_i(t) \equiv 1$. We still want the limits $\lim \frac{A_i'(t)}{A_i(t)}$ and $\lim \frac{A_i''(t)}{A_i(t)}$ to be finite for all $t \geq 0$, otherwise we will have infinite velocities or accelerations in these instants of infinity if the corresponding values in Lagrangian description also are. When $A_i(t) = 0$ the values respect to Eulerian description are equal to the corresponding Lagrangian description.

The expression (22) is also obtained through the chain rule

(23) $$\frac{Du_i^E}{Dt} = \frac{\partial u_i^E}{\partial t} + \sum_{j=1}^{3} \alpha_j \frac{\partial u_i^E}{\partial x_j},$$

being

(24) $$\frac{\partial u_i^E}{\partial t} = \frac{A_i'' A_i - \left(A_i'\right)^2}{A_i^2} x_i - \left(\frac{A_i' B_i}{A_i}\right)' + B_i'',$$

(25) $$\left(\frac{A_i' B_i}{A_i}\right)' = \frac{A_i'' B_i + A_i' B_i'}{A_i} - \left(\frac{A_i'}{A_i}\right)^2 B_i$$

and

(26) $$\sum_{j=1}^{3} \alpha_j \frac{\partial u_i^E}{\partial x_j} = \alpha_i \frac{\partial u_i^E}{\partial x_i} = \left(A_i' x_i^0 + B_i'\right) \frac{A_i'}{A_i}.$$

With movements where there is some linear relation between the spatial coordinates, as

(27) $$x_i = A_{i1}(t) x_1^0 + A_{i2}(t) x_2^0 + A_{i3}(t) x_3^0 + B_i(t),$$

$A_{ij}(t), B_i(t) \in C^{\infty}([0,\infty))$ for $i, j = 1,2,3$, we can transform

(28.1) $A_i(t) \mapsto A_{ii}(t) x_i^0$
(28.2) $B_i(t) \mapsto A_{ij}(t) x_j^0 + A_{ik}(t) x_k^0 + B_i(t)$

into the previous equations (17) to (26), with $j < k,\ i \neq j \neq k,\ i,j,k = 1,2,3$, and we will arrive at results similar to those already obtained.

If the relation between the coordinates is more complicated, not just linear, for example when the particles need follow a specific family of surfaces of type $z = g(x, y)$ (omitting other possible parameters), for g smooth function, then we can abandon the dependency of position, at least in one coordinate, as

(29) $$z = g(x, y) = g\big(x(t), y(t)\big) = h(t),$$

and therefore

(30) $$\begin{cases} u_1 = \varphi_1(x, t) \\ u_2 = \varphi_2(y, t) \\ u_3 = \varphi_3(z, t) = \varphi_3(h(t), t) = \alpha_3(t) \end{cases}$$

Thus, (4) holds in an infinity of cases and the Euler and Navier-Stokes equations has solution in this way (if f is conservative).

Note that in both examples, sections 2 and 3, the solutions for velocity are at most linear in relation to spatial coordinates, and then there is no necessity of calculation of second derivatives of velocity, i.e., $\nabla^2 u = 0$ for any viscosity coefficient and the Navier-Stokes equations are reduced to the Euler equations. In general terms we have, from (21),

(31) $$u_i^0 = \frac{A_i'(0)}{A_i(0)} x_i - \frac{A_i'(0) B_i(0)}{A_i(0)} + B_i'(0),$$

where we suppose that $\lim_{t \to 0} \frac{A_i'(t)}{A_i(t)}$ is finite for $i = 1,2,3$. If it is necessary that $\nabla \cdot u = \nabla \cdot u^0 = 0$ (incompressible fluids) then it must be valid, for all $t \geq 0$, the relation

(32) $$\frac{A_1'(t)}{A_1(t)} + \frac{A_2'(t)}{A_2(t)} + \frac{A_3'(t)}{A_3(t)} = 0.$$

In all functions of time $A_i(t)$, $A_{ij}(t)$ and $B_i(t)$ are implicit the inclusion of constant parameters of movement, as $R, \theta_0, \omega, x_C, y_C$, etc.

The scalar pressure is equal to

(33) $$p = \int_L (S_1, S_2, S_3) \cdot dl = \int_L \left(-\frac{Du}{Dt} + f \right) \cdot dl$$
$$= \sum_{i=1}^3 \int_{x_i^0}^{x_i} \left(-\frac{Du_i}{Dt} \right) dx_i + U - U_o + \theta(t),$$

if f is a conservative external force, $f = \nabla U$, with

(34) $$S_i = -\frac{Du_i}{Dt} + f_i$$

and

(35) $$\frac{\partial S_i}{\partial x_j} = \frac{\partial S_j}{\partial x_i}, \text{ for } i, j = 1,2,3, \text{ i.e., } \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i},$$

and then there is solution for Euler equations in this case.

As we have seen previously, the calculation of pressure is not unique and we can use $\frac{Du_i}{Dt}$ as a function of x_i and t or only of t. The simpler calculation gives

(36) $$p = -\sum_{i=1}^3 \left[A_i''(t) x_i^0 + B_i''(t) \right] \left(x_i - x_i^0 \right) + U - U_o + \theta(t),$$

using

(37) $\frac{Du_i}{Dt} = \frac{D\alpha_i}{Dt} = A_i''(t)x_i^0 + B_i''(t),$

according (19). The pressure is not dependent only of position and time, but also initial position, although there is a one-to-one correspondence between initial position with time and position, according (17) and (20).

See that we use $A_i(t) \not\equiv 0$ because any particle start from some position and it is not possible all particles start from the same position, but if $A_i(t) = 0$ for some $t > 0$ use for (18) to (37) the results equivalents to $A_i'(t) = A_i''(t) = 0$ and $A_i(t) = 1$, except (20) which is no sense in this case, and (17) will be $x_i = B_i(t)$.

Another calculation for scalar pressure gives, from (33) and using

(38) $\frac{Du_i}{Dt} dx_i = \frac{dx_i}{dt} du_i = u_i du_i,$

the interesting result

(39) $$\begin{aligned} p &= -\sum_{i=1}^{3} \tfrac{1}{2}\left(u_i^2 - u_i^{0\,2}\right) + U - U_o + \theta(t) \\ &= -\tfrac{1}{2}(u^2 - u^{0\,2}) + U - U_o + \theta(t), \end{aligned}$$

as the Bernoulli's law with $\frac{\partial \phi}{\partial t} = 0,\ u = \nabla\phi$.

4 – Conclusion

From equation (16),

(40) $\frac{\partial p}{\partial x_i} + \frac{D\alpha_i}{Dt} = \nu\nabla^2 u_i + \frac{1}{3}\nu\frac{\partial}{\partial x_i}(\nabla\cdot u) + f_i,$

we realize that if $\nu = 0$ and f is not conservative then there is no solution for Euler equations, as well as if u is conservative and f is not conservative there is no solution for Navier-Stokes equations, which now it is very clear to see and it is complementing [4]. More specifically, if u^0, the initial velocity, is conservative (irrotational or potential flow) and f is not conservative then there is no solution for Navier-Stokes equations, because it is impossible to obtain the pressure. This then solve [5] for the cases (C) and (D), the breakdown of solutions, for both u^0 and f belonging to Schwartz Space in case (C), and smooth functions with period 1 in the three orthogonal directions e_1, e_2, e_3 in case (D). As u^0 need obey to the incompressibility condition, $\nabla\cdot u^0 = 0$, with $\nabla\times u^0 = 0$ and $u^0 = \nabla\varphi^0$, where φ^0 is the potential of u^0, we have $\nabla^2 u^0 = 0$ and $\nabla^2\varphi^0 = 0$, i.e., u^0 and φ^0 are harmonic functions, unlimited functions except the constants, including zero. As u^0 need be limited, we choose $u^0 = 0$ for case (C) (where it is necessary that $\int_{\mathbb{R}^3} |u^0|^2 dx\, dy\, dz$ is finite) and any constant for case (D), of spatially periodic solutions. In case (D) the external force need belonging to Schwartz Space with relation to time.

Note that the application of a non conservative force in fluid is naturally possible and there will always be some movement, even starting from rest. So that this is not a paradoxical situation it seems certain that the pressure in this case cannot be scalar, but rather vector, and thus the equation returns to solution in all cases (assuming all derivatives are possible, etc.). It is as indicated in (2), or substituting p by p_i in (16).

According to what we saw in this article, solve the Navier-Stokes equations can be synonymous to solve the Euler equations and we can take advantage of this facility. For the time being, I do not know any reason for having to a more complicated solution than the one described here, when the use of $\nabla^2 u \neq 0$ is necessary, except if the compromise with the motion of particles is forgotten or we intend to describe a spatially periodic solution in Fourier series or the pressure is given and is not $\nabla p = f$ or, the worst, the velocity is not smooth (C^∞) and there are boundary conditions. Nevertheless, even in the most complicated cases, the movement of particles can be transformed into functions exclusively of time. Perhaps naval or aeronautical engineers have other motives, but with a greater rigor, involving temperature and the collision of particles, other equations must be constructed.

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25 – Describing a 3-D Fluid Motion with Rectangular, Cylindrical and Spherical Coordinates

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Abstract: We describe a fluid motion in three dimensions with rectangular, cylindrical and spherical coordinates.

Keywords: Euler equations, Navier-Stokes equations, Lagrangian description, Eulerian description, Bernoulli's law, rectangular coordinates, cylindrical coordinates, spherical coordinates.

1 – Introduction

In [1] we showed that the three-dimensional Euler ($\nu = 0$) and Navier-Stokes equations in rectangular coordinates need to be adopted as

(1) $$\frac{\partial p}{\partial x_i} + \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i + \frac{1}{3} \nu \frac{\partial}{\partial x_i} (\nabla \cdot u) + f_i,$$

for $i = 1,2,3$, where $\alpha_j = \frac{dx_j}{dt}$ is the velocity in Lagrangian description and u_i and the partial derivatives of u_i are in Eulerian description, as well as the scalar pressure p and density of external force f_i. The coefficient of viscosity is ν and by ease we prefer to use the mass density $\rho = 1$ (otherwise substitute p by p/ρ and ν by ν/ρ).

An alternative equation is

(2) $$\frac{\partial p_i}{\partial x_i} + \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i + \frac{1}{3} \nu \frac{\partial}{\partial x_i} (\nabla \cdot u) + f_i,$$

thus making the pressure a vector: $p = (p_1, p_2, p_3)$. In both equations is valid

(3) $$\frac{Du_i}{Dt} = \frac{Du_i^E}{Dt} = \frac{Du_i^L}{Dt} = \left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j} \right) |_L,$$

where the upper letter E refers to Eulerian velocity (u) and L to Lagrangian velocity (α). The symbol $|_L$ means the respective calculation in Lagrangian description, substituting each x_i as a function of time, initial value and eventually some parameters. With the notation $\frac{D}{Dt}$ we want, in principle, to make explicit that we are calculating a total derivative in relation to time, and the result is a function exclusively of time (and possibly a set of constant parameters and initial position), without the spatial coordinates x, y, z, but when for some reason we need to leave the result as a function of the spatial coordinates we can also do it. It is like when we want to write $\frac{Du_i^E}{Dt} = \frac{Du_i^L}{Dt}\Big|_E$, where the symbol $\Big|_E$ means the respective calculation in Eulerian description.

A condition indicated by us in [1] were

(4) $$\begin{cases} \frac{\partial u_i}{\partial x_j} = 0, \ i \neq j, \\ \partial x_i = u_i \partial t \end{cases}$$

because we have, by definition,

(5) $$u_i = \frac{dx_i}{dt},$$

in Lagrangian description, and for this reason the velocity component u_i, *a priori*, is not dependent of others variables x_j, with $x_j \neq x_i$. More than a rigorous mathematical proof, this is a practical approach, which simplifies the original system.

We will describe in section 2 a circular motion with uniform angular velocity and in section 3 a quite general movement, both in rectangular coordinates. In the section 4 we will write the Euler and Navier-Stokes equations in cylindrical coordinates and in section 5 in spherical coordinates. The section 6 will be our Conclusion, concluding again on the breakdown solutions and the necessity of use of vector pressure.

2 – Circular Motion in Rectangular Coordinates

Let a circular motion of radius R, centered at $(x_C,\ y_C)$ and with constant angular velocity $\omega > 0$ described by the equations:

(6) $$\begin{cases} x = x_C + R\cos(\theta_0 + \omega t) \\ y = y_C + R\sin(\theta_0 + \omega t) \end{cases}$$

and consequently

(7) $$(x - x_C)^2 + \left(y - y_C\right)^2 = R^2.$$

Then the velocity components are

(8) $$\begin{cases} \alpha_1 = u_1^L = \dot{x} = -\omega R\sin(\theta_0 + \omega t) = -\omega(y - y_C) = u_1^E \\ \alpha_2 = u_2^L = \dot{y} = +\omega R\cos(\theta_0 + \omega t) = +\omega(x - x_C) = u_2^E \end{cases}$$

and the acceleration components are

(9) $$\begin{cases} \frac{Du_1^L}{Dt} = \ddot{x} = -\omega^2 R\cos(\theta_0 + \omega t) = -\omega^2(x - x_C) = \frac{Du_1^E}{Dt} \\ \frac{Du_2^L}{Dt} = \ddot{y} = -\omega^2 R\sin(\theta_0 + \omega t) = -\omega^2(y - y_C) = \frac{Du_2^E}{Dt} \end{cases}$$

Supposing that the particles of fluid obey the motion described by (6) to (9), we have

(10) $$\begin{cases} \frac{\partial u_1}{\partial y} = -\omega, \quad \frac{\partial u_1}{\partial x} = 0 \\ \frac{\partial u_2}{\partial x} = +\omega, \quad \frac{\partial u_2}{\partial y} = 0 \end{cases}$$

apparently in disagree with (4) if $\omega \neq 0$. But, as x is a function of y and reciprocally, in this circular motion according (7), again (4) turns valid, for any signal of x and y. For to complete a three-dimensional description, we define $z = z_0$, without dependence of time, and $u_3 = 0$.

This is a motion of velocity without potential, because $\frac{\partial u_i}{\partial x_j} \neq \frac{\partial u_j}{\partial x_i}$ for some $i \neq j$, but if $f = (f_1, f_2, f_3)$ has potential we have $\frac{\partial S_i}{\partial x_j} = \frac{\partial S_j}{\partial x_i}$ for all $i, j = 1,2,3$, with

(11) $$S_i = -\frac{\partial u_i}{\partial t} - \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j} + \nu \nabla^2 u_i + \frac{1}{3}\nu \frac{\partial}{\partial x_i}(\nabla \cdot u) + f_i,$$

then the system (1) has solution.

A calculation for the scalar pressure of this motion is

(12) $$\begin{aligned} p &= \int_L (S_1, S_2, S_3) \cdot dl = \int_L \left(-\frac{Du}{Dt} + f\right) \cdot dl \\ &= \omega^2 \left[\left(\frac{x^2}{2} - x_C x\right)\Big|_{x_0}^{x} + \left(\frac{y^2}{2} - y_C y\right)\Big|_{y_0}^{y}\right] + U - U_0 + q(t) \\ &= \omega^2 \left[\left(\frac{x^2}{2} - x_C x\right) - \left(\frac{x_0{}^2}{2} - x_C x_0\right) + \left(\frac{y^2}{2} - y_C y\right) - \left(\frac{y_0{}^2}{2} - y_C y_0\right)\right] + \\ &\quad U - U_0 + q(t), \end{aligned}$$

where $f = \nabla U$, $U_0 = U(x_0, y_0, z_0, t)$ and L is any smooth path linking a point (x_0, y_0, z_0) to (x, y, z). We can ignore the use of x_0, y_0, z_0 and U_0, and use only the free function for time, $q(t)$, which on the other hand can include the terms in x_0, y_0 and z_0, and nevertheless this solution shows us that the pressure is not uniquely well determined, therefore we get to the negative answer to Smale's 15th problem, according already seen in [2] and [3], even if we assign the velocity value on some surface that we wish and even if $q(t)$ and U does not depend explicitly on the variable time t. In this motion the pressure is dependent, besides of x, y and U, without any problematic question, and x_C, y_C and ω, specific parameters of the movement conditions of a particle, of $q(t)$, U_0 and more three parameters, x_0, y_0 and z_0, then there is not uniqueness of solution.

Another calculation for pressure is possible due to fact that we can describe the acceleration $\frac{Du}{Dt}$ of a particle of fluid as a function only of time, $\frac{D\alpha}{Dt}$, without the variables x, y, z, and then

$$(13)\quad \begin{aligned} p &= -\frac{D\alpha}{Dt}\cdot\int_L dl + U - U_0 + q(t) \\ &= +\omega^2 R[\cos(\theta_0+\omega t)(x-x_0)+\sin(\theta_0+\omega t)(y-y_0)] \\ &\quad + U - U_0 + q(t), \end{aligned}$$

with

$$(14)\quad \begin{cases} \frac{\partial p}{\partial x} = +\omega^2 R\cos(\theta_0+\omega t) + f_1 = +\omega^2(x-x_C)+f_1 \\ \frac{\partial p}{\partial y} = +\omega^2 R\sin(\theta_0+\omega t) + f_2 = +\omega^2(y-y_C)+f_2 \\ \frac{\partial p}{\partial z} = f_3 \end{cases}$$

in fact derivatives such as can be obtained from (12).

Note that in order to continue using the traditional form of the Euler and Navier-Stokes equations we will have non-linear equations, which can make it difficult to obtain the solutions and bring all the difficulties that we know. To make sense to use the velocity in Eulerian description rather than the Lagrangian description in α_j it is necessary that, for all $t \geq 0$,

$$(15)\quad u^E(x(t), y(t), z(t), t) = \alpha(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = u^L(t),$$

omitting the use of possible parameters of motion, then nothing more natural than the definitive substitution of the terms $\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3}\alpha_j\frac{\partial u_i}{\partial x_j}$, as well as $\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3}u_j\frac{\partial u_i}{\partial x_j}$ in the traditional form, by $\frac{Du_i^L}{Dt}$ or $\frac{D\alpha_i}{Dt}$. This brings a great and important simplification in the equations, and to return to having the position as reference it is enough to use the conversion or definition adopted for $x(t), y(t)$ and $z(t)$, including the possible additional parameters, for example, substituting initial positions in function of position and time, etc.

Thus, more appropriate Euler ($\nu = 0$) and Navier-Stokes equations with scalar pressure are, in index notation,

$$(16)\quad \frac{\partial p}{\partial x_i} + \frac{D\alpha_i}{Dt} = \nu\nabla^2 u_i + \frac{1}{3}\nu\frac{\partial}{\partial x_i}(\nabla\cdot u) + f_i.$$

3 – Generic three-dimensional motion in Rectangular Coordinates

Suppose that a particle of fluid moves according to equation

(17) $x_i = A_i(t)x_i^0 + B_i(t),$

$A_i(0) = 1,\ B_i(0) = 0,\ A_i, B_i \in C^\infty\big([0,\infty)\big),\ i = 1,2,3,\ (x_1, x_2, x_3) \equiv (x, y, z)$,where $(x_1^0, x_2^0, x_3^0) \equiv (x_0, y_0, z_0)$ is the initial position of this particle in relation to three-orthogonal system of reference considered at rest.

Your velocity in relation to this system is, for $i = 1,2,3$,

(18) $\dot{x}_i = \frac{d}{dt}x_i = u_i^L = \alpha_i = A_i'(t)x_i^0 + B_i'(t),$

with acceleration

(19) $\ddot{x}_i = \frac{d}{dt}\dot{x}_i = \frac{D}{Dt}u_i^L = \frac{D}{Dt}\alpha_i = A_i''(t)x_i^0 + B_i''(t).$

We are using both the superior point $(\dot{x})$ and the prime mark (A'), and respective repetitions, for indicate differentiations in relation to time.

We are going to transform Lagrangian velocity into Eulerian velocity through transformation

(20) $x_i^0 = \frac{x_i - B_i(t)}{A_i(t)},$

which results in

(21) $$u_i^E = u_i^L|_E = \Big(A_i'(t)x_i^0 + B_i'(t)\Big)|_E = A_i'(t)\frac{x_i - B_i(t)}{A_i(t)} + B_i'(t)$$
$$= \frac{A_i'(t)}{A_i(t)}x_i - \frac{A_i'(t)B_i(t)}{A_i(t)} + B_i'(t)$$

and

(22) $$\frac{Du_i^E}{Dt} = \frac{Du_i^L}{Dt}|_E = \Big(A_i''(t)x_i^0 + B_i''(t)\Big)|_E = A_i''(t)\frac{x_i - B_i(t)}{A_i(t)} + B_i''(t)$$
$$= \frac{A_i''(t)}{A_i(t)}x_i - \frac{A_i''(t)B_i(t)}{A_i(t)} + B_i''(t).$$

We see that both u_i^E and $\frac{Du_i^E}{Dt}$ are linear functions in x_i or only functions of time if $A_i(t) \equiv 1$. We still want the limits $\lim \frac{A_i'(t)}{A_i(t)}$ and $\lim \frac{A_i''(t)}{A_i(t)}$ to be finite for all $t \geq 0$, otherwise we will have infinite velocities or accelerations in these instants of infinity if the corresponding values in Lagrangian description also are. When $A_i(t) = 0$ the values respect to Eulerian description are equal to the corresponding Lagrangian description.

The expression (22) is also obtained through the chain rule

(23) $$\frac{Du_i^E}{Dt} = \frac{\partial u_i^E}{\partial t} + \sum_{j=1}^{3} \alpha_j \frac{\partial u_i^E}{\partial x_j},$$

being

(24) $$\frac{\partial u_i^E}{\partial t} = \frac{A_i'' A_i - (A_i')^2}{A_i^2} x_i - \left(\frac{A_i' B_i}{A_i}\right)' + B_i'',$$

(25) $$\left(\frac{A_i' B_i}{A_i}\right)' = \frac{A_i'' B_i + A_i' B_i'}{A_i} - \left(\frac{A_i'}{A_i}\right)^2 B_i$$

and

(26) $$\sum_{j=1}^{3} \alpha_j \frac{\partial u_i^E}{\partial x_j} = \alpha_i \frac{\partial u_i^E}{\partial x_i} = \left(A_i' x_i^0 + B_i'\right) \frac{A_i'}{A_i}.$$

With movements where there is some linear relation between the spatial coordinates, as

(27) $$x_i = A_{i1}(t) x_1^0 + A_{i2}(t) x_2^0 + A_{i3}(t) x_3^0 + B_i(t),$$

$A_{ij}(t), B_i(t) \in C^{\infty}([0, \infty))$ for $i, j = 1,2,3$, we can transform

(28.1) $$A_i(t) \mapsto A_{ii}(t) x_i^0$$
(28.2) $$B_i(t) \mapsto A_{ij}(t) x_j^0 + A_{ik}(t) x_k^0 + B_i(t)$$

into the previous equations (17) to (26), with $j < k,\ i \neq j \neq k,\ i, j, k = 1,2,3$, and we will arrive at results similar to those already obtained.

If the relation between the coordinates is more complicated, not just linear, for example when the particles need follow a specific family of surfaces of type $z = g(x, y)$ (omitting other possible parameters), for g smooth function, then we can abandon the dependency of position, at least in one coordinate, as

(29) $$z = g(x, y) = g\big(x(t), y(t)\big) = h(t),$$

and therefore

$$(30)\quad \begin{cases} u_1 = \varphi_1(x,t) \\ u_2 = \varphi_2(y,t) \\ u_3 = \varphi_3(z,t) = \varphi_3(h(t),t) = \alpha_3(t) \end{cases}$$

Thus, (4) holds in an infinity of cases and the Euler and Navier-Stokes equations has solution in this way (if the external force f is conservative).

Note that in both examples, sections 2 and 3, the solutions for velocity are at most linear in relation to spatial coordinates, and then there is no necessity of calculation of second derivatives of velocity, i.e., $\nabla^2 u = 0$ for any viscosity coefficient and the Navier-Stokes equations are reduced to the Euler equations. In general terms we have, from (21),

$$(31)\quad u_i^0 = \frac{A_i'(0)}{A_i(0)} x_i - \frac{A_i'(0) B_i(0)}{A_i(0)} + B_i'(0),$$

where we suppose that $\lim_{t\to 0} \frac{A_i'(t)}{A_i(t)}$ is finite for $i = 1,2,3$. If it is necessary that $\nabla \cdot u = \nabla \cdot u^0 = 0$ (incompressible fluids) then it must be valid, for all $t \geq 0$, the relation

$$(32)\quad \frac{A_1'(t)}{A_1(t)} + \frac{A_2'(t)}{A_2(t)} + \frac{A_3'(t)}{A_3(t)} = 0.$$

In all functions of time $A_i(t)$, $A_{ij}(t)$ and $B_i(t)$ are implicit the inclusion of constant parameters of movement, as $R, \theta_0, \omega, x_C, y_C, \nu$, etc.

The scalar pressure is equal to

$$(33)\quad p = \int_L (S_1, S_2, S_3) \cdot dl = \int_L \left(-\frac{Du}{Dt} + f\right) \cdot dl$$
$$= \sum_{i=1}^{3} \int_{x_i^0}^{x_i} \left(-\frac{Du_i}{Dt}\right) dx_i + U - U_o + q(t),$$

if f is a conservative external force, $f = \nabla U$, with

$$(34)\quad S_i = -\frac{Du_i}{Dt} + f_i$$

and

$$(35)\quad \frac{\partial S_i}{\partial x_j} = \frac{\partial S_j}{\partial x_i},\ \text{for } i,j = 1,2,3\text{, i.e., } \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i},$$

and then there is solution for Euler equations in this case.

As we have seen previously, the calculation of pressure is not unique and we can use $\frac{Du_i}{Dt}$ as a function of x_i and t or only of t. The simpler calculation gives

(36) $$p = -\sum_{i=1}^{3}\left[A_i''(t)x_i^0 + B_i''(t)\right]\left(x_i - x_i^0\right) + U - U_o + q(t),$$

using

(37) $$\frac{Du_i}{Dt} = \frac{D\alpha_i}{Dt} = A_i''(t)x_i^0 + B_i''(t),$$

according (19). The pressure is not dependent only of position and time, but also initial position, although there is a one-to-one correspondence between initial position with time and position, according (17) and (20).

See that we use $A_i(t) \not\equiv 0$ because any particle start from some position and it is not possible all particles start from the same position, but if $A_i(t) = 0$ for some $t > 0$ use for (18) to (37) the results equivalents to $A_i'(t) = A_i''(t) = 0$ and $A_i(t) = 1$, except (20) which is no sense in this case, and (17) will be $x_i = B_i(t)$.

Another calculation for scalar pressure gives, from (33) and using

(38) $$\frac{Du_i}{Dt}dx_i = \frac{dx_i}{dt}du_i = u_i du_i,$$

the interesting result

(39) $$\begin{aligned} p &= -\sum_{i=1}^{3}\frac{1}{2}\left(u_i^2 - u_i^{0\,2}\right) + U - U_o + q(t) \\ &= -\frac{1}{2}\left(u^2 - u^{0\,2}\right) + U - U_o + q(t), \end{aligned}$$

as the Bernoulli's law with $\frac{\partial\phi}{\partial t} = 0$, $u = \nabla\phi$, compatible with the velocity u in Lagrangian description, $u = u^L = \alpha$, but whose value may be converted to Eulerian description too, using $u_i^E = u_i^L|_E$ as (21).

4 – Cylindrical Coordinates $(\boldsymbol{r}, \boldsymbol{\varphi}, \boldsymbol{z})$

Using the transformations

(40) $$\begin{cases} r = \sqrt{x^2 + y^2} \\ \varphi = \arctan\frac{y}{x} \end{cases}$$

and the inverse transformations

(41) $$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

for radius r, azimuthal angle φ and elevation z as shown in figure 1, it is possible write the Euler and Navier-Stokes Equations in cylindrical coordinates. Note that the inverse tangent denoted in $\varphi = \arctan\frac{y}{x}$ must be suitably defined, taking into account the correct quadrant of (x, y).

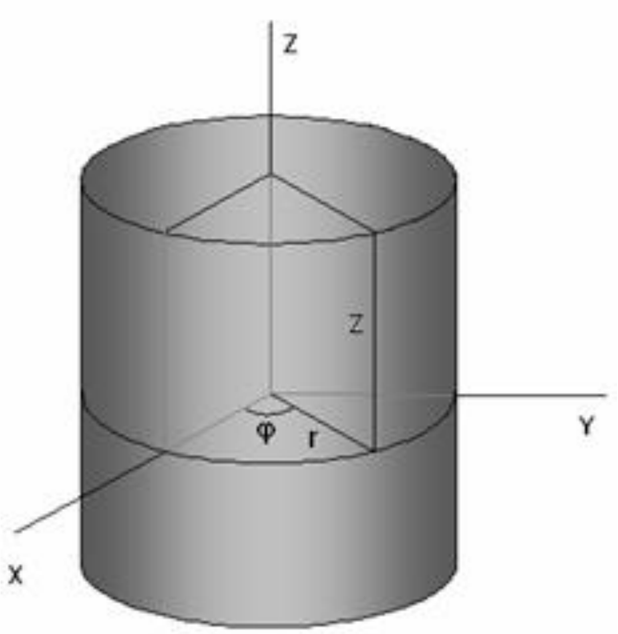

Fig. 1 – Cylindrical coordinates (*r*,*φ*, *z*).

Based on Landau and Lifshitz[4], for viscous incompressible fluid we have for the three components the new form

(42.1) $\frac{\partial p}{\partial r} + \frac{\partial u_r}{\partial t} + (u^L \cdot \nabla)u_r - \frac{u_\varphi^L u_\varphi}{r} = \nu\left(\nabla^2 u_r - \frac{2}{r^2}\frac{\partial u_\varphi}{\partial \varphi} - \frac{u_r}{r^2}\right) + f_r,$

(42.2) $\frac{1}{r}\frac{\partial p}{\partial \varphi} + \frac{\partial u_\varphi}{\partial t} + (u^L \cdot \nabla)u_\varphi + \frac{u_\varphi^L u_r}{r} = \nu\left(\nabla^2 u_\varphi + \frac{2}{r^2}\frac{\partial u_r}{\partial \varphi} - \frac{u_\varphi}{r^2}\right) + f_\varphi,$

(42.3) $\frac{\partial p}{\partial z} + \frac{\partial u_z}{\partial t} + (u^L \cdot \nabla)u_z = \nu\,\nabla^2 u_z + f_z,$

where the Lagrangian velocity is

(43) $u^L = \left(u_r^L, u_\varphi^L, u_z^L\right),$

the Eulerian velocity is

(44) $u^E = u = \left(u_r, u_\varphi, u_z\right),$

and, for $v: \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}$ smooth scalar function,

(45) $$(u^L \cdot \nabla)v = u_r^L \frac{\partial v}{\partial r} + \frac{u_\varphi^L}{r}\frac{\partial v}{\partial \varphi} + u_z^L \frac{\partial v}{\partial z},$$

(46) $$\nabla^2 v = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \varphi^2} + \frac{\partial^2 v}{\partial z^2}.$$

The incompressibility condition (equation of continuity) is

(47) $$\nabla \cdot u = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z} = 0.$$

You can consult in internet for a brief comparison the links https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes_equations and https://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates. In the first link the terms $\frac{1}{3}\nu\frac{\partial}{\partial r}\nabla\cdot u$, $\frac{1}{3}\nu\frac{1}{r}\frac{\partial}{\partial \varphi}\nabla\cdot u$ and $\frac{1}{3}\nu\frac{\partial}{\partial z}\nabla\cdot u$ are added respectively in right side of each one of equations (42) for viscous compressible fluid.

Using the substitutions

(48) $$\begin{cases} x_1, x \mapsto r \\ x_2, y \mapsto \varphi \\ x_3, z \mapsto z \end{cases}$$

and, respectively,

(49) $$\begin{cases} u_1, \alpha_1, A_1, B_1, A_{1j} \mapsto u_r, \alpha_r, A_r, B_r, A_{rj} \\ u_2, \alpha_2, A_2, B_2, A_{2j} \mapsto u_\varphi, \alpha_\varphi, A_\varphi, B_\varphi, A_{\varphi j} \\ u_3, \alpha_3, A_3, B_3, A_{3j} \mapsto u_z, \alpha_z, A_z, B_z, A_{zj} \end{cases}$$

in section 3, equations (17) to (22) and (27) to (31) for rectangular coordinates, it is possible obtain similar relations for cylindrical coordinates, such that

(50) $$\begin{cases} r = A_r(t)\, r_0 + B_r(t) \\ \varphi = A_\varphi(t)\, \varphi_0 + B_\varphi(t) \\ z = A_z(t)\, z_0 + B_z(t) \end{cases}$$

supposing the time functions are smooth, with the initial conditions

(51) $$\begin{cases} A_r(0) = A_\varphi(0) = A_z(0) = 1 \\ B_r(0) = B_\varphi(0) = B_z(0) = 0 \end{cases}$$

Differentiating in relation to time we have the velocity components in Lagrangian description

(52) $$\begin{cases} \dot{r} = \frac{d}{dt} r = u_r^L = \alpha_r = A_r^{'}(t)\, r_0 + B_r^{'}(t) \\ \dot{\varphi} = \frac{d}{dt} \varphi = u_\varphi^L = \alpha_\varphi = A_\varphi^{'}(t)\, \varphi_0 + B_\varphi^{'}(t) \\ \dot{z} = \frac{d}{dt} z = u_z^L = \alpha_z = A_z^{'}(t)\, z_0 + B_z^{'}(t) \end{cases}$$

and using the initial conditions parameters obtained from (50)

(53) $$\begin{cases} r_0 = \frac{r - B_r(t)}{A_r(t)} \\ \varphi_0 = \frac{\varphi - B_\varphi(t)}{A_\varphi(t)} \\ z_0 = \frac{z - B_z(t)}{A_z(t)} \end{cases}$$

in the Lagrangian velocity components (52) we have the Eulerian velocity components

(54) $$\begin{cases} u_r = u_r^E = u_r^L|_E = \frac{A_r'(t)}{A_r(t)} r - \frac{A_r'(t) B_r(t)}{A_r(t)} + B_r'(t) \\ u_\varphi = u_\varphi^E = u_\varphi^L|_E = \frac{A_\varphi'(t)}{A_\varphi(t)} \varphi - \frac{A_\varphi'(t) B_\varphi(t)}{A_\varphi(t)} + B_\varphi'(t) \\ u_z = u_z^E = u_z^L|_E = \frac{A_z'(t)}{A_z(t)} z - \frac{A_z'(t) B_z(t)}{A_z(t)} + B_z'(t) \end{cases}$$

Differentiating (52) in relation to time we have the Lagrangian acceleration components

(55) $$\begin{cases} \ddot{r} = \frac{d}{dt} \dot{r} = \frac{D}{Dt} u_r^L = \frac{D}{Dt} \alpha_r = A_r^{''}(t)\, r_0 + B_r^{''}(t) \\ \ddot{\varphi} = \frac{d}{dt} \dot{\varphi} = \frac{D}{Dt} u_\varphi^L = \frac{D}{Dt} \alpha_\varphi = A_\varphi^{''}(t)\, \varphi_0 + B_\varphi^{''}(t) \\ \ddot{z} = \frac{d}{dt} \dot{z} = \frac{D}{Dt} u_z^L = \frac{D}{Dt} \alpha_z = A_z^{''}(t)\, z_0 + B_z^{''}(t) \end{cases}$$

and using again the initial conditions parameters (53) now in (55) we have the Eulerian acceleration components

(56) $$\begin{cases} \frac{Du_r^E}{Dt} = \frac{Du_r^L}{Dt}\Big|_E = \frac{A_r^{''}(t)}{A_r(t)} r - \frac{A_r^{''}(t) B_r(t)}{A_r(t)} + B_r^{''}(t) \\ \frac{Du_\varphi^E}{Dt} = \frac{Du_\varphi^L}{Dt}\Big|_E = \frac{A_\varphi^{''}(t)}{A_\varphi(t)} \varphi - \frac{A_\varphi^{''}(t) B_\varphi(t)}{A_\varphi(t)} + B_\varphi^{''}(t) \\ \frac{Du_z^E}{Dt} = \frac{Du_z^L}{Dt}\Big|_E = \frac{A_z^{''}(t)}{A_z(t)} z - \frac{A_z^{''}(t) B_z(t)}{A_z(t)} + B_z^{''}(t) \end{cases}$$

Being true that numerically we have

(57) $$\begin{cases} \frac{Du_r}{Dt} = \frac{\partial u_r}{\partial t} + (u^L \cdot \nabla) u_r - \frac{u_\varphi^L u_\varphi}{r} \\ \frac{Du_\varphi}{Dt} = \frac{\partial u_\varphi}{\partial t} + (u^L \cdot \nabla) u_\varphi + \frac{u_\varphi^L u_r}{r} \\ \frac{Du_z}{Dt} = \frac{\partial u_z}{\partial t} + (u^L \cdot \nabla) u_z \end{cases}$$

when all functions are converted into time functions, where

(58) $$\begin{cases} (u^L \cdot \nabla) u_r = u_r^L \frac{\partial u_r}{\partial r} \\ (u^L \cdot \nabla) u_\varphi = \frac{u_\varphi^L}{r} \frac{\partial u_\varphi}{\partial \varphi} \\ (u^L \cdot \nabla) u_z = u_z^L \frac{\partial u_z}{\partial z} \end{cases}$$

for

(59) $$\frac{\partial u_r}{\partial \varphi} = \frac{\partial u_r}{\partial z} = \frac{\partial u_\varphi}{\partial r} = \frac{\partial u_\varphi}{\partial z} = \frac{\partial u_z}{\partial r} = \frac{\partial u_z}{\partial \varphi} = 0,$$

according (54), we write the Navier-Stokes equations (42) in cylindrical coordinates as

(60) $$\begin{cases} \frac{\partial p}{\partial r} + \frac{Du_r}{Dt} = \nu \left(\nabla^2 u_r - \frac{2}{r^2} \frac{\partial u_\varphi}{\partial \varphi} - \frac{u_r}{r^2} \right) + f_r \\ \frac{1}{r} \frac{\partial p}{\partial \varphi} + \frac{Du_\varphi}{Dt} = \nu \left(\nabla^2 u_\varphi - \frac{u_\varphi}{r^2} \right) + f_\varphi \\ \frac{\partial p}{\partial z} + \frac{Du_z}{Dt} = \nu \, \nabla^2 u_z + f_z \end{cases}$$

From (46) and (54) we have

(61) $$\begin{cases} \nabla^2 u_r = \frac{A_r'(t)}{A_r(t)} \frac{1}{r} \\ \nabla^2 u_\varphi = \nabla^2 u_z = 0 \end{cases}$$

and then in this case the Navier-Stokes equations are

(62) $$\begin{cases} \frac{\partial p}{\partial r} + \frac{Du_r}{Dt} = \nu \left(\frac{A_r'(t)}{A_r(t)} \frac{1}{r} - \frac{2}{r^2} \frac{A_\varphi'(t)}{A_\varphi(t)} - \frac{u_r}{r^2} \right) + f_r \\ \frac{\partial p}{\partial \varphi} + r \frac{Du_\varphi}{Dt} = -\nu \frac{u_\varphi}{r} + r f_\varphi \\ \frac{\partial p}{\partial z} + \frac{Du_z}{Dt} = f_z \end{cases}$$

Defining

(63) $$\begin{cases} S_r = -\frac{Du_r}{Dt} + \nu\left(\frac{A_r^{'}(t)}{A_r(t)}\frac{1}{r} - \frac{2}{r^2}\frac{A_\varphi^{'}(t)}{A_\varphi(t)} - \frac{u_r}{r^2}\right) + f_r \\ S_\varphi = -r\frac{Du_\varphi}{Dt} - \nu\frac{u_\varphi}{r} + rf_\varphi \\ S_z = -\frac{Du_z}{Dt} + f_z \end{cases}$$

to have some solution to the system (62) it is necessary that

(64) $$\begin{cases} \frac{\partial S_r}{\partial \varphi} = \frac{\partial S_\varphi}{\partial r} \Rightarrow \frac{\partial f_r}{\partial \varphi} = -\frac{Du_\varphi}{Dt} + \nu\frac{u_\varphi}{r^2} + f_\varphi + r\frac{\partial f_\varphi}{\partial r} \\ \frac{\partial S_r}{\partial z} = \frac{\partial S_z}{\partial r} \Rightarrow \frac{\partial f_r}{\partial z} = \frac{\partial f_z}{\partial r} \\ \frac{\partial S_\varphi}{\partial z} = \frac{\partial S_z}{\partial \varphi} \Rightarrow r\frac{\partial f_\varphi}{\partial z} = \frac{\partial f_z}{\partial \varphi} \end{cases}$$

so there is not always a solution to the Euler and Navier-Stokes equations in cylindrical coordinates, according to the above system, as too occurs in the case of rectangular coordinates.

When there is some solution for the system (62), given the velocity and external force, a solution for pressure is then as in the rectangular coordinates case,

(65) $$p = \int_{A_0}^{A} S \cdot dl + q(t),$$

for $A = (r, \varphi, z)$, $A_0 = (r_0, \varphi_0, z_0)$, $\left(\frac{\partial p}{\partial r}, \frac{\partial p}{\partial \varphi}, \frac{\partial p}{\partial z}\right) = (S_r, S_\varphi, S_z) = S$ and with the line differential element equal to $dl = (dr,\ r\,d\varphi,\ dz)$, being also possible the use of the vector S transformed as time function only, the Lagrangian description. The value of line integral is independent of path. $q(t)$ is any smooth and limited time function, a physically reasonable time function.

5 – Spherical Coordinates $(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\varphi})$

Using the transformations

(66) $$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arccos\frac{z}{r} \\ \varphi = \arctan\frac{y}{x} \end{cases}$$

and the inverse transformations

(67) $$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

where

$0 \leq \theta \leq \pi$, θ is the polar angle or colatitude,

$0 \leq \varphi \leq 2\pi$, φ is the azimuthal angle,

$r \geq 0$ is the radius,

as shown in figure 2, it is possible write the Euler and Navier-Stokes equations in spherical coordinates.

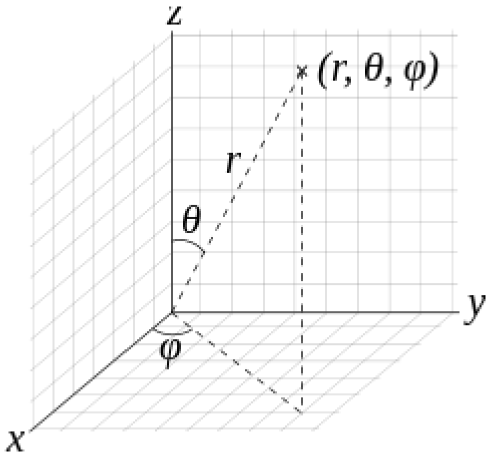

Fig. 2 - Spherical coordinates (r,θ,φ) as commonly used in *physics* (ISO convention): radial distance r, polar angle θ (theta), and azimuthal angle φ (phi). The symbol ρ (rho) is often used instead of r.

Based on Landau and Lifshitz[4], for viscous incompressible fluid we have for the three components the new form

(68.1) $$\frac{\partial p}{\partial r} + \frac{\partial u_r}{\partial t} + (u^L \cdot \nabla)u_r - \frac{u_\theta^L u_\theta + u_\varphi^L u_\varphi}{r} = \nu\left(\nabla^2 u_r - \frac{2}{r^2 \sin^2\theta}\frac{\partial(u_\theta \sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} - \frac{2u_r}{r^2}\right) + f_r$$

(68.2) $$\frac{1}{r}\frac{\partial p}{\partial\theta} + \frac{\partial u_\theta}{\partial t} + (u^L \cdot \nabla)u_\theta + \frac{u_\theta^L u_r}{r} - \frac{u_\varphi^L u_\varphi \cot\theta}{r} = \nu\left(\nabla^2 u_\theta - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\varphi}{\partial\varphi} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta}\right) + f_\theta$$

(68.3) $$\frac{1}{r\sin\theta}\frac{\partial p}{\partial\varphi} + \frac{\partial u_\varphi}{\partial t} + (u^L \cdot \nabla)u_\varphi + \frac{u_\varphi^L u_r}{r} + \frac{u_\varphi^L u_\theta \cot\theta}{r} = \nu\left(\nabla^2 u_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\varphi} - \frac{u_\varphi}{r^2\sin^2\theta}\right) + f_\varphi$$

where the Lagrangian velocity is

(69) $$u^L = \left(u_r^L, u_\theta^L, u_\varphi^L\right),$$

the Eulerian velocity is

(70) $$u^E = u = \left(u_r, u_\theta, u_\varphi\right),$$

and for $v: \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}$ smooth scalar function,

(71) $$(u^L \cdot \nabla)v = u_r^L \frac{\partial v}{\partial r} + \frac{u_\theta^L}{r} \frac{\partial v}{\partial \theta} + \frac{u_\varphi^L}{r \sin\theta} \frac{\partial v}{\partial \varphi},$$

(72) $$\nabla^2 v = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial v}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 v}{\partial \varphi^2}.$$

The incompressibility condition (equation of continuity) is

(73) $$\nabla \cdot u = \frac{1}{r^2} \frac{\partial (r^2 u_r)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial (u_\theta \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial u_\varphi}{\partial \varphi} = 0.$$

You can also consult in internet for a brief comparison the links https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes_equations and https://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates. In the first link the terms $\frac{1}{3} \nu \frac{\partial}{\partial r} \nabla \cdot u$, $\frac{1}{3} \nu \frac{1}{r} \frac{\partial}{\partial \theta} \nabla \cdot u$ and $\frac{1}{3} \nu \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} \nabla \cdot u$ are added respectively in right side of each one of equations (68) for viscous compressible fluid.

Using the substitutions

(74) $$\begin{cases} x_1, x \mapsto r \\ x_2, y \mapsto \theta \\ x_3, z \mapsto \varphi \end{cases}$$

and, respectively,

(75) $$\begin{cases} u_1, \alpha_1, A_1, B_1, A_{1j} \mapsto u_r, \alpha_r, A_r, B_r, A_{rj} \\ u_2, \alpha_2, A_2, B_2, A_{2j} \mapsto u_\theta, \alpha_\theta, A_\theta, B_\theta, A_{\theta j} \\ u_3, \alpha_3, A_3, B_3, A_{3j} \mapsto u_\varphi, \alpha_\varphi, A_\varphi, B_\varphi, A_{\varphi j} \end{cases}$$

in section 3, equations (17) to (22) and (27) to (31) for rectangular coordinates, it is possible obtain similar relations for spherical coordinates, such that

(76) $$\begin{cases} r = A_r(t)\, r_0 + B_r(t) \\ \theta = A_\theta(t)\, \theta_0 + B_\theta(t) \\ \varphi = A_\varphi(t)\, \varphi_0 + B_\varphi(t) \end{cases}$$

supposing the time functions are smooth, with the initial conditions

(77) $$\begin{cases} A_r(0) = A_\theta(0) = A_\varphi(0) = 1 \\ B_r(0) = B_\theta(0) = B_\varphi(0) = 0 \end{cases}$$

Differentiating in relation to time we have the velocity components in Lagrangian description

(78) $$\begin{cases} \dot{r} = \frac{d}{dt} r = u_r^L = \alpha_r = A_r'(t)\, r_0 + B_r'(t) \\ \dot{\theta} = \frac{d}{dt}\theta = u_\theta^L = \alpha_\theta = A_\theta'(t)\, \theta_0 + B_\theta'(t) \\ \dot{\varphi} = \frac{d}{dt}\varphi = u_\varphi^L = \alpha_\varphi = A_\varphi'(t)\, \varphi_0 + B_\varphi'(t) \end{cases}$$

and using the initial conditions parameters obtained from (76)

(79) $$\begin{cases} r_0 = \frac{r - B_r(t)}{A_r(t)} \\ \theta_0 = \frac{\theta - B_\theta(t)}{A_\theta(t)} \\ \varphi_0 = \frac{\varphi - B_\varphi(t)}{A_\varphi(t)} \end{cases}$$

in the Lagrangian velocity components (78) we have the Eulerian velocity components

(80) $$\begin{cases} u_r = u_r^E = u_r^L|_E = \frac{A_r'(t)}{A_r(t)} r - \frac{A_r'(t) B_r(t)}{A_r(t)} + B_r'(t) \\ u_\theta = u_\theta^E = u_\theta^L|_E = \frac{A_\theta'(t)}{A_\theta(t)} \theta - \frac{A_\theta'(t) B_\theta(t)}{A_\theta(t)} + B_\theta'(t) \\ u_\varphi = u_\varphi^E = u_\varphi^L|_E = \frac{A_\varphi'(t)}{A_\varphi(t)} \varphi - \frac{A_\varphi'(t) B_\varphi(t)}{A_\varphi(t)} + B_\varphi'(t) \end{cases}$$

Differentiating (78) in relation to time we have the Lagrangian acceleration components

(81) $$\begin{cases} \ddot{r} = \frac{d}{dt}\dot{r} = \frac{D}{Dt} u_r^L = \frac{D}{Dt}\alpha_r = A_r''(t)\, r_0 + B_r''(t) \\ \ddot{\theta} = \frac{d}{dt}\dot{\theta} = \frac{D}{Dt} u_\theta^L = \frac{D}{Dt}\alpha_\theta = A_\theta''(t)\, \theta_0 + B_\theta''(t) \\ \ddot{\varphi} = \frac{d}{dt}\dot{\varphi} = \frac{D}{Dt} u_\varphi^L = \frac{D}{Dt}\alpha_\varphi = A_\varphi''(t)\, \varphi_0 + B_\varphi''(t) \end{cases}$$

and using again the initial conditions parameters (79) now in (81) we have the Eulerian acceleration components

(82) $$\begin{cases} \frac{Du_r^E}{Dt} = \frac{Du_r^L}{Dt}|_E = \frac{A_r''(t)}{A_r(t)} r - \frac{A_r''(t) B_r(t)}{A_r(t)} + B_r''(t) \\ \frac{Du_\theta^E}{Dt} = \frac{Du_\theta^L}{Dt}|_E = \frac{A_\theta''(t)}{A_\theta(t)} \theta - \frac{A_\theta''(t) B_\theta(t)}{A_\theta(t)} + B_\theta''(t) \\ \frac{Du_\varphi^E}{Dt} = \frac{Du_\varphi^L}{Dt}|_E = \frac{A_\varphi''(t)}{A_\varphi(t)} \varphi - \frac{A_\varphi''(t) B_\varphi(t)}{A_\varphi(t)} + B_\varphi''(t) \end{cases}$$

Being true that numerically we have

$$(83)\quad \begin{cases} \dfrac{Du_r}{Dt} = \dfrac{\partial u_r}{\partial t} + (u^L \cdot \nabla)u_r - \dfrac{u_\theta^L u_\theta + u_\varphi^L u_\varphi}{r} \\ \dfrac{Du_\theta}{Dt} = \dfrac{\partial u_\theta}{\partial t} + (u^L \cdot \nabla)u_\theta + \dfrac{u_\theta^L u_r}{r} - \dfrac{u_\varphi^L u_\varphi \cot\theta}{r} \\ \dfrac{Du_\varphi}{Dt} = \dfrac{\partial u_\varphi}{\partial t} + (u^L \cdot \nabla)u_\varphi + \dfrac{u_\varphi^L u_r}{r} + \dfrac{u_\varphi^L u_\theta \cot\theta}{r} \end{cases}$$

when all functions are converted into time functions, where

$$(84)\quad \begin{cases} (u^L \cdot \nabla)u_r = u_r^L \dfrac{\partial u_r}{\partial r} \\ (u^L \cdot \nabla)u_\theta = \dfrac{u_\theta^L}{r} \dfrac{\partial u_\theta}{\partial \theta} \\ (u^L \cdot \nabla)u_\varphi = \dfrac{u_\varphi^L}{r\sin\theta} \dfrac{\partial u_\varphi}{\partial \varphi} \end{cases}$$

for

$$(85)\quad \frac{\partial u_r}{\partial \theta} = \frac{\partial u_r}{\partial \varphi} = \frac{\partial u_\theta}{\partial r} = \frac{\partial u_\theta}{\partial \varphi} = \frac{\partial u_\varphi}{\partial r} = \frac{\partial u_\varphi}{\partial \theta} = 0,$$

according (80), we write the Navier-Stokes equations (68) in spherical coordinates as

$$(86)\quad \begin{cases} \dfrac{\partial p}{\partial r} + \dfrac{Du_r}{Dt} = \nu\left(\nabla^2 u_r - \dfrac{2}{r^2\sin^2\theta}\dfrac{\partial(u_\theta \sin\theta)}{\partial\theta} - \dfrac{2}{r^2\sin\theta}\dfrac{\partial u_\varphi}{\partial\varphi} - \dfrac{2u_r}{r^2}\right) + f_r \\ \dfrac{1}{r}\dfrac{\partial p}{\partial\theta} + \dfrac{Du_\theta}{Dt} = \nu\left(\nabla^2 u_\theta - \dfrac{2\cos\theta}{r^2\sin^2\theta}\dfrac{\partial u_\varphi}{\partial\varphi} - \dfrac{u_\theta}{r^2\sin^2\theta}\right) + f_\theta \\ \dfrac{1}{r\sin\theta}\dfrac{\partial p}{\partial\varphi} + \dfrac{Du_\varphi}{Dt} = \nu\left(\nabla^2 u_\varphi - \dfrac{u_\varphi}{r^2\sin^2\theta}\right) + f_\varphi \end{cases}$$

From (72) and (80) we have

$$(87)\quad \begin{cases} \nabla^2 u_r = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u_r}{\partial r}\right) = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{A_r'(t)}{A_r(t)}\right) = \dfrac{2}{r}\dfrac{A_r'(t)}{A_r(t)} \\ \nabla^2 u_\theta = \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial u_\theta}{\partial\theta}\right) = \dfrac{1}{r^2}\dfrac{A_\theta'(t)}{A_\theta(t)}\cot\theta \\ \nabla^2 u_\varphi = \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 u_\varphi}{\partial\varphi^2} = 0 \end{cases}$$

and then in this case the Navier-Stokes equations are

$$(88)\quad \begin{cases} \dfrac{\partial p}{\partial r} + \dfrac{Du_r}{Dt} = \nu\left(\dfrac{2}{r}\dfrac{A_r'(t)}{A_r(t)} - \dfrac{2}{r^2\sin^2\theta}\dfrac{\partial(u_\theta\sin\theta)}{\partial\theta} - \dfrac{2}{r^2\sin\theta}\dfrac{A_\varphi'(t)}{A_\varphi(t)} - \dfrac{2u_r}{r^2}\right) + f_r \\ \dfrac{1}{r}\dfrac{\partial p}{\partial\theta} + \dfrac{Du_\theta}{Dt} = \nu\left(\dfrac{1}{r^2}\dfrac{A_\theta'(t)}{A_\theta(t)}\cot\theta - \dfrac{2\cos\theta}{r^2\sin^2\theta}\dfrac{A_\varphi'(t)}{A_\varphi(t)} - \dfrac{u_\theta}{r^2\sin^2\theta}\right) + f_\theta \\ \dfrac{1}{r\sin\theta}\dfrac{\partial p}{\partial\varphi} + \dfrac{Du_\varphi}{Dt} = \nu\left(-\dfrac{u_\varphi}{r^2\sin^2\theta}\right) + f_\varphi \end{cases}$$

Defining

(89) $$\begin{cases} S_r = -\frac{Du_r}{Dt} + \nu\left(\frac{2}{r}\frac{A_r^{'}(t)}{A_r(t)} - \frac{2}{r^2 \sin^2\theta}\frac{\partial(u_\theta \sin\theta)}{\partial\theta} - \frac{2}{r^2 \sin\theta}\frac{A_\varphi^{'}(t)}{A_\varphi(t)} - \frac{2u_r}{r^2}\right) + f_r \\ S_\theta = -r\frac{Du_\theta}{Dt} - \nu\frac{1}{r}\left(\frac{A_\theta^{'}(t)}{A_\theta(t)}\cot\theta - \frac{2\cos\theta}{\sin^2\theta}\frac{A_\varphi^{'}(t)}{A_\varphi(t)} - \frac{u_\theta}{\sin^2\theta}\right) + rf_\theta \\ S_\varphi = -r\sin\theta\frac{Du_\varphi}{Dt} - \nu\frac{u_\varphi}{r\sin\theta} + r\sin\theta f_\varphi \end{cases}$$

to have some solution to the system (88) it is necessary that

(90)
$$\begin{cases} \frac{\partial S_r}{\partial\theta} = \frac{\partial S_\theta}{\partial r} \Rightarrow -\nu\frac{2}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin^2\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} + \frac{1}{\sin\theta}\frac{A'_\varphi(t)}{A_\varphi(t)}\right) + \frac{\partial f_r}{\partial\theta} = -\frac{Du_\theta}{Dt} + \nu\frac{1}{r^2}\sigma + f_\theta + r\frac{\partial f_\theta}{\partial r} \\ \frac{\partial S_r}{\partial\varphi} = \frac{\partial S_\varphi}{\partial r} \Rightarrow \frac{\partial f_r}{\partial\varphi} = -\sin\theta\frac{Du_\varphi}{Dt} + \nu\frac{u_\varphi}{r^2\sin\theta} + \sin\theta\left(f_\varphi + r\frac{\partial f_\varphi}{\partial r}\right) \\ \frac{\partial S_\theta}{\partial\varphi} = \frac{\partial S_\varphi}{\partial\theta} \Rightarrow r\frac{\partial f_\theta}{\partial\varphi} = -r\cos\theta\frac{Du_\varphi}{Dt} + \nu\frac{u_\varphi}{r}\frac{\cos\theta}{\sin^2\theta} + r\left(\cos\theta f_\varphi + \sin\theta\frac{\partial f_\varphi}{\partial\theta}\right) \end{cases}$$

where

(91) $$\sigma = \left(\frac{A'_\theta(t)}{A_\theta(t)}\cot\theta - \frac{2\cos\theta}{\sin^2\theta}\frac{A'_\varphi(t)}{A_\varphi(t)} - \frac{u_\theta}{\sin^2\theta}\right),$$

so there is not always a solution to the Euler and Navier-Stokes equations in spherical coordinates, according to the above system, as too occurs in the cases of rectangular and cylindrical coordinates.

When there is some solution for the system (88), given the velocity and external force, a solution for pressure is then as in the rectangular and cylindrical coordinates cases,

(92) $$p = \int_{A_0}^{A} S \cdot dl + q(t),$$

where$A = (r, \theta, \varphi)$, $A_0 = (r_0, \theta_0, \varphi_0)$, $\left(\frac{\partial p}{\partial r}, \frac{\partial p}{\partial\theta}, \frac{\partial p}{\partial\varphi}\right) = (S_r, S_\theta, S_\varphi) = S$ and the line differential element is $dl = (dr,\ r\,d\theta,\ r\sin\theta\,d\varphi)$. It is also possible the use of the vector S transformed as time function only, the Lagrangian description. The value of line integral is independent of path. $q(t)$ is any smooth and limited time function, a physically reasonable time function.

6 – Conclusion

Essentially the present conclusion has already been obtained in [1], using only rectangular coordinates, and here we are only updating it with the three coordinate systems we have seen.

From equation (16) in rectangular coordinates,

$$(93)\qquad \frac{\partial p}{\partial x_i}+\frac{D\alpha_i}{Dt}=\nu\nabla^2 u_i+\frac{1}{3}\nu\frac{\partial}{\partial x_i}(\nabla\cdot u)+f_i,$$

we realize that if $\nu=0$ and f is not conservative then there is no solution for Euler equations, as well as if u is conservative and f is not conservative there is no solution for Navier-Stokes equations, which now it is very clear to see and it is complementing [5]. More specifically, if u^0, the initial velocity, is conservative (irrotational or potential flow) and f is not conservative then there is no solution for Navier-Stokes equations, because it is impossible to obtain the pressure. This then solve [6] for the cases (C) and (D), the breakdown of solutions, for both u^0 and f belonging to Schwartz Space in case (C), and smooth functions with period 1 in the three orthogonal directions e_1, e_2, e_3 in case (D). As u^0 need obey to the incompressibility condition, $\nabla\cdot u^0=0$, with $\nabla\times u^0=0$ and $u^0=\nabla\varphi^0$, where φ^0 is the potential of u^0, we have $\nabla^2 u^0=0$ and $\nabla^2\varphi^0=0$, i.e., u^0 and φ^0 are harmonic functions, unlimited functions except the constants, including zero. As u^0 need be limited, we choose $u^0=0$ for case (C) (where it is necessary that $\int_{\mathbb{R}^3}|u^0|^2 dx\, dy\, dz$ is finite) and any constant for case (D), of spatially periodic solutions. In case (D) the external force need belonging to Schwartz Space with relation to time.

The conditions (64) for cylindrical coordinates and (90) for spherical coordinates also show that there is not always a solution to the Euler and Navier-Stokes equations, with even more difficult equations to be obeyed.

Note that the application of a non conservative force in fluid is naturally possible and there will always be some movement, even starting from rest. So that this is not a paradoxical situation it seems certain that the pressure in this case cannot be scalar, but rather vector, and thus the equation returns to solution in all cases (assuming all derivatives are possible, etc.). It is as indicated in (2), or substituting p by p_i in (16) and $p_r, p_\varphi, p_z, p_\theta$ in the others correspondent coordinates.

According to what we saw in this article, solve the Navier-Stokes equations can be synonymous to solve the Euler equations, at least in rectangular coordinates, and we can take advantage of this facility. For the time being, specifically in rectangular coordinates case, section 3, I do not know any reason for having to a more complicated solution than the those described here, when the use of $\nabla^2 u\neq 0$ is necessary, except if the compromise with the motion of particles is forgotten or we intend to describe a spatially periodic solution in Fourier series or the pressure is given and is not $\nabla p = f$ or yet, the worst, the velocity is not smooth (C^∞) and there are boundary conditions. Nevertheless, even in the most complicated cases, the movement of particles can be transformed into functions exclusively of time and parameters as initial position and others. Perhaps naval or aeronautical engineers have other motives, but with a greater rigor, involving temperature and the collision of particles (between them and on rigid surfaces), other equations must be constructed.

So that there are no contradictions between the three reference systems, if in rectangular coordinates the solution is independent of the coefficient of viscosity, then the same movement in the cylindrical and spherical coordinates will also be, taking into account the additional terms that appear after $\nabla^2 u_r, \nabla^2 u_\varphi$ and $\nabla^2 u_\theta$ in (42) and (68) and the terms related to the partial derivatives of $\nabla \cdot u$ (equation of continuity).

We deduce all these equations thinking about a generic particle of fluid, hence with a single initial position, as (x_0, y_0, z_0) in rectangular coordinates. We can generalize all these results also for several particles and its respective initial positions, $(x_{0m}, y_{0m}, z_{0m}), (r_{0m}, \varphi_{0m}, z_{0m}), (r_{0m}, \theta_{0m}, \varphi_{0m})$, using for example an identification index for the various functions and variables, as $A_{im}, B_{im}, A_{rm}, A_{\theta m}, A_{\varphi m}, A_{zm}, p_m$, etc., $1 \le m \le n$, with n the total number of particles (ideally $n \to \infty$) and m indicating the specific particle. An interesting study is to calculate the time instants of the various collisions between particles, and the consequences of these collisions. A statistical treatment seems to be the most appropriate.

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