A New Sieve in the Study of Prime Numbers

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Abstract

We developed a new Sieve, A-Sieve, and produced some new ways in the study of prime numbers. By using this new method, we can prove that for any natural number k, there are infinitely many pairs of primes that differ by 2k.

Contents

1. Introduction

A *prime number* (or a *prime*) is a natural number greater than 1 that has no positive divisors other than 1 and itself.

It is well known that the sieve of Eratosthenes has long been used in the study of prime numbers. Here we introduce a new Sieve, which we call ***A-Sieve.*** We start with some basic notations (for the basic Mathematical symbols, see Notes [1]).

Let $\mathbf{S_d(a) = \{a+(n-1)d,\ n \in N\}}$ be a sequence with a general term $a+(n-1)d$, $n \in N$, where N is the set of natural numbers. See the following examples:

$S_2(2)=\{2+(n-1)2,\ n \in N\}=\{2,4,6,8,\ldots\ldots\}$ is the set of all even numbers;

$S_5(5)=\{5+(n-1)5,\ n \in N\}=\{5,10,15,20,\ldots\ldots\}$ is the set of all multiples of 5;

$S_{10}(1)=\{1+(n-1)10,\ n \in N\}=\{1,11,21,31,\ldots\ldots\}$, the set of all numbers that end with 1;

$S_{10}(3)=\{3+(n-1)10,\ n \in N\}=\{3,13,23,33,\ldots\ldots\}$, the set of all numbers that end with 3;

$S_{10}(7)=\{7+(n-1)10,\ n \in N\}=\{7,17,27,37,\ldots\ldots\}$, the set of all numbers that end with 7;

$S_{10}(9)=\{9+(n-1)10,\ n \in N\}=\{9,19,29,39,\ldots\ldots\}$, the set of all numbers that end with 9.

Since all $S_{10}(1)$, $S_{10}(3)$, $S_{10}(7)$, and $S_{10}(9)$ have no divisors 2 and 5, we have:

$$S_{10}(1) \cup S_{10}(3) \cup S_{10}(7) \cup S_{10}(9) = S_1(1) - \{S_2(2) \cup S_5(5)\}$$

that means except 2 and 5, all other prime numbers end in 1, 3, 7, or 9.

2. Introducing the New Sieve: A-Sieve

We denote by **P[X]** the set of all prime numbers in the set of X. For example, P[{1,2,3,4,5,6,7,8,9,10}] = {2,3,5,7}.

Let $\mathbf{N_d(a)}$ denote the set of natural numbers associated with $S_d(a)$. Some examples are given by the following Table 1.

$N_{10}(11)$	$S_{10}(11)$	$N_{10}(13)$	$S_{10}(13)$	$N_{10}(17)$	$S_{10}(17)$	$N_{10}(19)$	$S_{10}(19)$
n	10(n-1)+11	n	10(n-1)+13	n	10(n-1)+17	n	10(n-1)+19
1	11	1	13	1	17	1	19
2	21	2	23	2	27	2	29
3	31	3	33	3	37	3	39
4	41	4	43	4	47	4	49
5	51	5	53	5	57	5	59
6	61	6	63	6	67	6	69
7	71	7	73	7	77	7	79
8	81	8	83	8	87	8	89
9	91	9	93	9	97	9	99
… …	… …	… …	… …	… …	… …	… …	… …

Table 1.

The key idea of our new **A-Sieve** is: Instead of using Sieve to the whole natural number set N, we separate N into some subsequence like $S_d(a)$ and then use the Sieve to $N_d(a)$. We take $S_{10}(11)$ and $N_{10}(11)$ as an example to demonstrate this process as follows:

For all $p \in P[N]$, where P[N] is the set of all prime numbers, since all numbers in $S_{10}(11)$ have no divisors 2 and 5, we start with p=3, the smallest prime number to mark out. Because all numbers in $S_{10}(11)$ end with 1, the multiplicands of 3 must be 7, 17, 27, 37, …. While the first number 21=3*7 in $S_{10}(11)$ is corresponding to the number 2 in $N_{10}(11)$, so we mark all numbers of $S_3(2)$ in $N_{10}(11)$.

Next, let p=7, the multiplicands of 7 must be 3, 13, 23, 33, …. Because 7*3 already marked out, the first number 91=7*13 in $S_{10}(11)$ is corresponding to the number 9 in $N_{10}(11)$, so we mark all numbers of $S_7(9)$ in $N_{10}(11)$.

The next number not yet marked out in P[N] after 3 and 7 is 11 and the multiplicands of 11 must be 11, 21, 31, 41, …. The first number 121=11*11 in $S_{10}(11)$ is corresponding to the number 12 in $N_{10}(11)$, so we mark all numbers of $S_{11}(12)$ in $N_{10}(11)$.

… …

In general, for any $p \in P[N]$ ($p \neq 2,5$), find the first p-multiple number f in $S_{10}(11)$ to get the corresponding number x in $N_{10}(11)$, then mark all numbers of $S_p(x)$ in $N_{10}(11)$. Note that some of the numbers may be marked more than once.

The following Table 2. shows some of this process (see Note [3] for the color use):

$N_{10}(11)$	$S_{10}(11)$	$N_{10}(13)$	$S_{10}(13)$	$N_{10}(17)$	$S_{10}(17)$	$N_{10}(19)$	$S_{10}(19)$
n	10(n-1)+11	n	10(n-1)+13	n	10(n-1)+17	n	10(n-1)+19
1	11	1	13	1	17	1	19
2	21	2	23	2	27	2	29
3	31	3	33	3	37	3	39
4	41	4	43	4	47	4	49
5	51	5	53	5	57	5	59
6	61	6	63	6	67	6	69
7	71	7	73	7	77	7	79
8	81	8	83	8	87	8	89
9	91	9	93	9	97	9	99
10	101	10	103	10	107	10	109
11	111	11	113	11	117	11	119
12	121	12	123	12	127	12	129
13	131	13	133	13	137	13	139
14	141	14	143	14	147	14	149
15	151	15	153	15	157	15	159
16	161	16	163	16	167	16	169
17	171	17	173	17	177	17	179
18	181	18	183	18	187	18	189
19	191	19	193	19	197	19	199
… …	… …	… …	… …	… …	… …	… …	… …

Table 2.

Denote by **$A[N_d(a)]$** the set of all the numbers remaining not marked in $N_d(a)$ after marking out all subsequence $S_p(x)$ of $N_d(a)$ for some $x \in N_d(a)$ and for all $p \in P[N]$.

Then $A[N_d(a)] = N_d(a) - \{ S_p(x),$ for some $x \in N_d(a)$ and for all $p \in P[N] \}$,

$P[S_d(a)]$ $= \{ a+(n-1)d, \quad n \in A[N_d(a)] \}$.

For example: $A[N_{10}(11)] = N_{10}(11) - \{ S_3(2) \cup S_7(9) \cup S_{11}(12) \cup \ldots \cup S_p(w) \cup \ldots \}$,

and $P[S_{10}(11)] = \{ 11+(n-1)10, n \in A[N_{10}(11)] \}$;

$A[N_{10}(13)] = N_{10}(13) - \{ S_3(3) \cup S_7(13) \cup S_{11}(14) \cup \ldots \cup S_p(x) \cup \ldots \}$;

$A[N_{10}(17)] = N_{10}(17) - \{ S_3(2) \cup S_7(7) \cup S_{11}(18) \cup \ldots \cup S_p(y) \cup \ldots \}$;

$A[N_{10}(19)] = N_{10}(19) - \{ S_3(3) \cup S_7(4) \cup S_{11}(20) \cup \ldots \cup S_p(z) \cup \ldots \}$.

After the process of A-Sieve, we have the following Table 3.

$A[N_{10}(11)]$	$P[S_{10}(11)]$	$A[N_{10}(13)]$	$P[S_{10}(13)]$	$A[N_{10}(17)]$	$P[S_{10}(17)]$	$A[N_{10}(19)]$	$P[S_{10}(19)]$
n	10(n-1)+11	n	10(n-1)+13	n	10(n-1)+17	n	10(n-1)+19
1	11	1	13	1	17	1	19
3	31	2	23	3	37	2	29
4	41	4	43	4	47	5	59
6	61	5	53	6	67	7	79
7	71	7	73	9	97	8	89
10	101	8	83	10	107	10	109
13	131	10	103	12	127	13	139
15	151	11	113	13	137	14	149
18	181	16	163	15	157	17	179
19	191	17	173	16	167	19	199
21	211	19	193	19	197	22	229
24	241	22	223	22	227	23	239
25	251	23	233	25	257	26	269
27	271	26	263	27	277	34	349
28	281	28	283	30	307	35	359
31	311	29	293	31	317	37	379
33	331	31	313	33	337	38	389
40	401	35	353	34	347	40	409
42	421	37	373	36	367	41	419
43	431	38	383	39	397	43	439
46	461	43	433	45	457	44	449
49	491	44	443	46	467	47	479
52	521	46	463	48	487	49	499
54	541	50	503	54	547	50	509
… …	… …	… …	… …	… …	… …	… …	… …

Table 3.

There are different proofs of the infinitude of primes. Since our A-Sieve used to $N_{10}(11)$, $N_{10}(13)$, $N_{10}(17)$, and $N_{10}(19)$ is quite similar to the way using sieve of Eratosthenes to N, it leads to that all $A[N_{10}(11)]$, $A[N_{10}(13)]$, $A[N_{10}(17)]$, $A[N_{10}(19)]$ (and so $P[S_{10}(11)]$, $P[S_{10}(13)]$, $P[S_{10}(17)]$, $P[S_{10}(19)]$) have similar patterns as P[N]. In other words, they behave similarly to P[N] with similar density given by the prime number theorem.

We have the following Lemma.

LEMMA 2.1. *All $A[N_{10}(11)]$, $A[N_{10}(13)]$, $A[N_{10}(17)]$, and $A[N_{10}(19)]$ are infinite.*

Proof. Suppose $A[N_{10}(11)]$ is finite with L being the largest number in the set, then there exists a prime number p such that

$$A[N_{10}(11)] = N_{10}(11) - \{S_3(2) \cup S_7(9) \cup S_{11}(12) \cup \ldots \cup S_p(w)\} \text{ for some } w \in N_{10}(11).$$

Note that $L \in A[N_{10}(11)]$ implies $L \notin \{S_3(2) \cup S_7(9) \cup S_{11}(12) \cup \ldots \cup S_p(w)\}$.

Let $d=2*5*3*7*11*\ldots*p$, the product of all prime numbers less than or equal to p,

then for all $m \in N$, $(L+ m*d) \notin \{S_3(2) \cup S_7(9) \cup S_{11}(12) \cup \ldots \cup S_p(w)\}$.

Therefore, $(L+ m*d) \in A[N_{10}(11)]$ and $(L+ m*d) > L$ for all $m \in N$.

Thus, $A[N_{10}(11)]$ must be infinite.

Similarly, $A[N_{10}(13)]$, $A[N_{10}(17)]$, and $A[N_{10}(19)]$ must be infinite. □

Lemma 1 immediately yields the following Corollary:

COROLLARY 2.2. *All $P[S_{10}(11)]$, $P[S_{10}(13)]$, $P[S_{10}(17)]$, and $P[S_{10}(19)]$ are infinite and* $P[N] = P[S_{10}(11)] \cup P[S_{10}(13)] \cup P[S_{10}(17)] \cup P[S_{10}(19)] \cup \{2,3,5,7\}$.

3. A-Sieve in the Study of Prime Pairs

Now, we consider the use of A-Sieve to the study of prime pairs. First, we introduce 8 special sequences as follows: Separate $S_{10}(11)$ into $S_{30}(31)$, $S_{30}(11)$, $S_{30}(21)$ and sieve out all 3-multiple numbers, $S_{30}(21)$, so that $S_{30}(31)$ and $S_{30}(11)$ have no divisor 3 after 2 and 5 as shown in the following Table 4.

$S_{30}(31)$	$S_{30}(11)$	$S_{30}(21)$
	11	21
31	41	51
61	71	81
91	101	111
121	131	141
151	161	171
181	191	201
… …	… …	… …

Table 4.

By the same way, after sieving out all 3-multiple numbers, $S_{30}(33)$, $S_{30}(27)$, and $S_{30}(39)$, all $S_{30}(13)$, $S_{30}(23)$, $S_{30}(7)$, $S_{30}(17)$, $S_{30}(19)$, and $S_{30}(29)$ have no divisor 3 apart from 2 and 5.

For $S_{30}(31)$, $S_{30}(11)$, $S_{30}(13)$, $S_{30}(23)$, $S_{30}(7)$, $S_{30}(17)$, $S_{30}(19)$, and $S_{30}(29)$

we use A-Sieve to the corresponding sequences:

$N_{30}(31)$, $N_{30}(11)$, $N_{30}(13)$, $N_{30}(23)$, $N_{30}(7)$, $N_{30}(17)$, $N_{30}(19)$, $N_{30}(29)$ (see Note [4]) to get

$A[N_{30}(31)]$, $A[N_{30}(11)]$, $A[N_{30}(13)]$, $A[N_{30}(23)]$, $A[N_{30}(7)]$, $A[N_{30}(17)]$, $A[N_{30}(19)]$, $A[N_{30}(29)]$

and $P[S_{30}(31)]$, $P[S_{30}(11)]$, $P[S_{30}(13)]$, $P[S_{30}(23)]$, $P[S_{30}(7)]$, $P[S_{30}(17)]$, $P[S_{30}(19)]$, $P[S_{30}(29)]$.

Similar to the Corollary 1, we have the following:

COROLLARY 3.1. *All $P[S_{30}(31)]$, $P[S_{30}(11)]$, $P[S_{30}(13)]$, $P[S_{30}(23)]$, $P[S_{30}(7)]$, $P[S_{30}(17)]$, $P[S_{30}(19)]$, $P[S_{30}(29)]$ are infinite and*

$P[S_{10}(11)] = P[S_{30}(31)] \cup P[S_{30}(11)]$, $P[S_{10}(13)] = P[S_{30}(13)] \cup P[S_{30}(23)]$,

$P[S_{10}(7)] = P[S_{30}(7)] \cup P[S_{30}(17)]$, $P[S_{10}(19)] = P[S_{30}(19)] \cup P[S_{30}(29)]$,

$P[N] = P[S_{30}(31)] \cup P[S_{30}(11)] \cup P[S_{30}(13)] \cup P[S_{30}(23)] \cup P[S_{30}(7)] \cup P[S_{30}(17)] \cup P[S_{30}(19)] \cup P[S_{30}(29)] \cup \{2,3,5\}$.

Let $\mathbf{N_d(a,b)}$ denote the set of natural numbers associated with the pair of sequences $S_d(a)$ and $S_d(b)$. (Then $A[N_d(a,b)] = A[N_d(a)] \cap A[N_d(b)]$.)

Let $\mathbf{P[S_d(a);\ S_d(b)]} = \{a+(n-1)d;\ b+(n-1)d,\ n \in A[N_d(a,b)]\}$ be the set of all pairs of prime numbers from $S_d(a)$ and $S_d(b)$ produced by $A[N_d(a,b)]$.

In order to find twin primes, we consider the following 3 pairs of sequences:

$S_{30}(11)$ and $S_{30}(13)$,

$S_{30}(17)$ and $S_{30}(19)$,

$S_{30}(29)$ and $S_{30}(31)$

as shown on the below Table 5.

$N_{30}(11,13)$	$S_{30}(11)$	$S_{30}(13)$	$N_{30}(17,19)$	$S_{30}(17)$	$S_{30}(19)$	$N_{30}(29,31)$	$S_{30}(29)$	$S_{30}(31)$
n	30(n-1)+11	30(n-1)+13	n	30(n-1)+17	30(n-1)+19	n	30(n-1)+29	30(n-1)+31
1	11	13	1	17	19	1	29	31
2	41	43	2	47	49	2	59	61
3	71	73	3	77	79	3	89	91
4	101	103	4	107	109	4	119	121
5	131	133	5	137	139	5	149	151
6	161	163	6	167	169	6	179	181
7	191	193	7	197	199	7	209	211
8	221	223	8	227	229	8	239	241
… …	… …	… …	… …	… …	… …	… …	… …	… …

Table 5.

By using the similar process of A-Sieve to $N_{10}(11)$, we use A-Sieve to $N_{30}(11,13)$ as demonstrated below:

For all $p \in P[N]$, since $S_{30}(11)$ and $S_{30}(13)$ not containing divisors 2, 5 and 3, we start with p= 7, the first number 161 = 7*23 in $S_{30}(11)$ is corresponding to the number 6 in $N_{30}(11,13)$, so we mark all numbers of $S_7(6)$ in $N_{30}(11,13)$, and the first number 133 = 7*19 in $S_{30}(13)$ is corresponding to the number 5 in $N_{30}(11,13)$, we then mark all numbers of $S_7(5)$ in $N_{30}(11,13)$;

Next, let p = 11, the first number 671 = 11*61 in $S_{30}(11)$ is corresponding to the number 23 in $N_{30}(11,13)$, so we mark all numbers of $S_{11}(23)$ in $N_{30}(11,13)$, and the first number 253 = 11*23 in $S_{30}(13)$ is corresponding to the number 9 in $N_{30}(11,13)$, so we also mark all numbers of $S_{11}(9)$ in $N_{30}(11,13)$;

We can do the same to p = 13 …

In general, for any $p \in P[N]$, find the first p-multiple number f in $S_{30}(11)$ to get the corresponding number x in $N_{30}(11,13)$, then mark all numbers of $S_p(x)$ in $N_{30}(11,13)$, also find the first p-multiple number g in $S_{30}(13)$ to get the corresponding number y in $N_{30}(11,13)$, then mark all numbers of $S_p(y)$ in $N_{30}(11,13)$. Note that some of the numbers may be marked more than once.

Therefore, we have $A[N_{30}(11,13)] = N_{30}(11,13) - \{S_7(6) \cup S_7(5) \cup S_{11}(23) \cup S_{11}(9) \cup S_{13}(8) \cup S_{13}(14) \cup \ldots. \cup S_p(x) \cup S_p(y) \cup \ldots\ldots\} = A[N_{30}(11)] \cap A[N_{30}(13)]$,

Where $A[N_{30}(11)] = N_{30}(11) - \{ S_7(6) \cup S_{11}(23) \cup S_{13}(8) \cup \ldots \cup S_p(x) \cup \ldots\}$,

and $A[N_{30}(13)] = N_{30}(13) - \{ S_7(5) \cup S_{11}(9) \cup S_{13}(14) \cup \ldots \cup S_p(y) \cup \ldots\}$.

The following Table 6 and Table 7 show some of the process:

$N_{30}(11,13)$	$S_{30}(11)$	$S_{30}(13)$	$N_{30}(17,19)$	$S_{30}(17)$	$S_{30}(19)$	$N_{30}(29,31)$	$S_{30}(29)$	$S_{30}(31)$
n	30(n-1)+11	30(n-1)+13	n	30(n-1)+17	30(n-1)+19	n	30(n-1)+29	30(n-1)+31
1	11	13	1	17	19	1	29	31
2	41	43	2	47	49	2	59	61
3	71	73	3	77	79	3	89	91
4	101	103	4	107	109	4	119	121
5	131	133	5	137	139	5	149	151
6	161	163	6	167	169	6	179	181
7	191	193	7	197	199	7	209	211
8	221	223	8	227	229	8	239	241
9	251	253	9	257	259	9	269	271
10	281	283	10	287	289	10	299	301
11	311	313	11	317	319	11	329	331
12	341	343	12	347	349	12	359	361
13	371	373	13	377	379	13	389	391
14	401	403	14	407	409	14	419	421
15	431	433	15	437	439	15	449	451
16	461	463	16	467	469	16	479	481
17	491	493	17	497	499	17	509	511
18	521	523	18	527	529	18	539	541
19	551	553	19	557	559	19	569	571
20	581	583	20	587	589	20	599	601
21	611	613	21	617	619	21	629	631
22	641	643	22	647	649	22	659	661
23	671	673	23	677	679	23	689	691
24	701	703	24	707	709	24	719	721
25	731	733	25	737	739	25	749	751
26	761	763	26	767	769	26	779	781
27	791	793	27	797	799	27	809	811
28	821	823	28	827	829	28	839	841
29	851	853	29	857	859	29	869	871
30	881	883	30	887	889	30	899	901
31	911	913	31	917	919	31	929	931
32	941	943	32	947	949	32	959	961
… …	… …	… …	… …	… …	… …	… …	… …	… …

Table 6.

A[N_{30}(11,13)]	P[S_{30}(11); S_{30}(13)]		A[N_{30}(17,19)]	P[S_{30}(17); S_{30}(19)]		A[N_{30}(29,31)]	P[S_{30}(29); S_{30}(31)]	
n	30(n-1)+11	30(n-1)+13	n	30(n-1)+17	30(n-1)+19	n	30(n-1)+29	30(n-1)+31
1	11	13	1	17	19	1	29	31
2	41	43	4	107	109	2	59	61
3	71	73	5	137	139	5	149	151
4	101	103	7	197	199	6	179	181
7	191	193	8	227	229	8	239	241
10	281	283	12	347	349	9	269	271
11	311	313	21	617	619	14	419	421
15	431	433	28	827	829	19	569	571
16	461	463	29	857	859	20	599	601
18	521	523	43	1277	1279	22	659	661
22	641	643	48	1427	1429	27	809	811
28	821	823	50	1487	1489	34	1019	1021
30	881	883	54	1607	1609	35	1049	1051
35	1031	1033	56	1667	1669	41	1229	1231
36	1061	1063	57	1697	1699	43	1289	1291
37	1091	1093	60	1787	1789	44	1319	1321
39	1151	1153	63	1877	1879	54	1619	1621
44	1301	1303	67	1997	1999	65	1949	1951
49	1451	1453	68	2027	2029	71	2129	2131
50	1481	1483	70	2087	2089	77	2309	2311
58	1721	1723	75	2237	2239	78	2339	2341
63	1871	1873	76	2267	2269	85	2549	2551
65	1931	1933	89	2657	2659	91	2729	2731
70	2081	2083	90	2687	2689	93	2789	2791
71	2111	2113	106	3167	3169	99	2969	2971
72	2141	2143	109	3257	3259	100	2999	3001
80	2381	2383	116	3467	3469	104	3119	3121
87	2591	2593	118	3527	3529	110	3299	3301
91	2711	2713	119	3557	3559	111	3329	3331
94	2801	2803	126	3767	3769	112	3359	3361
109	3251	3253	131	3917	3919	113	3389	3391
113	3371	3373	138	4127	4129	118	3539	3541
116	3461	3463	139	4157	4159	131	3929	3931
120	3581	3583	141	4217	4219	134	4019	4021
… …	… …	… …	… …	… …	… …	… …	… …	… …

Table 7.

Similar to the proof of Lemma 2.1, we prove the following:

LEMMA 3.2. *All* $A[N_{30}(11,13)]$, $A[N_{30}(17,19)]$, *and* $A[N_{30}(29,31)]$ *are infinite.*

Proof. Assume that $A[N_{30}(11,13)]$ is finite with Z being the largest number in the set, then there exists a prime number q such that

$A[N_{30}(11,13)] = N_{30}(11,13) - \{S_7(6) \cup S_7(5) \cup S_{11}(23) \cup S_{11}(9) \cup S_{13}(8) \cup S_{13}(14) \cup \ldots. \cup S_q(x) \cup S_q(y)\}$,

for some $x \in N_{30}(11)$ and some $y \in N_{30}(13)$.

Note that $Z \in A[N_{30}(11,13)]$ implies $Z \in N_{30}(11,13)$ and

$Z \notin \{S_7(6) \cup S_7(5) \cup S_{11}(23) \cup S_{11}(9) \cup S_{13}(8) \cup S_{13}(14) \cup \ldots. \cup S_q(x) \cup S_q(y)\}$.

Let $d = 2*3*5*7*11*\ldots*q$, the product of all prime numbers less or equal to q, then for all $m \in N$,

$(Z+ m*d) \notin \{S_7(6) \cup S_7(5) \cup S_{11}(23) \cup S_{11}(9) \cup S_{13}(8) \cup S_{13}(14) \cup \ldots. \cup S_q(x) \cup S_q(y)\}$ and $(Z+ m*d) \in N_{30}(11,13)$.

Therefore, $(Z + m*d) \in A[N_{30}(11,13)]$ and $(Z + m*d) > Z$ for all $m \in N$,

and thus $A[N_{30}(11,13)]$ must be infinite.

Similarly, $A[N_{30}(17,19)]$ and $A[N_{30}(29,31)]$ must be infinite. □

Now by Lemma 3.2, it is straightforward to prove the following twin primes conjecture:

THEOREM 3.3. *There are infinitely many pairs of primes that differ by 2.*

Proof. Since $P[S_{30}(11); S_{30}(13)] = \{11+(n-1)30; 13+(n-1)30, n \in A[N_{30}(11,13)] \}$,

$P[S_{30}(17); S_{30}(19)] = \{17+(n-1)30; 19+(n-1)30, n \in A[N_{30}(17,19)] \}$,

and $P[S_{30}(29); S_{30}(31)] = \{29+(n-1)30; 31+(n-1)30, n \in A[N_{30}(29,31)] \}$,

by Lemma 3.2, all $P[S_{30}(11); S_{30}(13)]$, $P[S_{30}(17); S_{30}(19)]$, and $P[S_{30}(29); S_{30}(31)]$ must be infinite, which concludes that there are infinitely many pairs of primes that differ by only 2. □

Furthermore, we have the following more general result:

THEOREM 3.4. *For any $k\in N$, there are infinitely many pairs of primes that differ by 2k.*

Proof. For any $k\in N$, let $k = 15m + h$, where $0\le m< (k/15)$ and $0\le h< 15$. Then similar to the proof of Theorem 1, we have the following corresponding infinite sets of prime pairs that differ by 2k:

h	infinite sets of prime pairs that differ by 2k, $k=15m+h$, where $0\le m<(k/15)$ and $0\le h<15$		
0 i=m-1	$P[S_{60}(31); S_{60}(61+30i)]$	$P[S_{60}(11); S_{60}(41+30i)]$	$P[S_{60}(13); S_{60}(43+30i)]$
	$P[S_{60}(23); S_{60}(53+30i)]$	$P[S_{60}(7); S_{60}(37+30i)]$	$P[S_{60}(17); S_{60}(47+30i)]$
	$P[S_{60}(19); S_{60}(49+30i)]$	$P[S_{60}(29); S_{60}(59+30i)]$	
1	$P[S_{30}(11); S_{30}(13+30m)]$	$P[S_{30}(17); S_{30}(19+30m)]$	$P[S_{30}(29); S_{30}(31+30m)]$
2	$P[S_{30}(13); S_{30}(17+30m)]$	$P[S_{30}(7); S_{30}(11+30m)]$	$P[S_{30}(19); S_{30}(23+30m)]$
3	$P[S_{30}(11); S_{30}(17+30m)]$	$P[S_{30}(13); S_{30}(19+30m)]$	$P[S_{30}(23); S_{30}(29+30m)]$
	$P[S_{30}(7); S_{30}(13+30m)]$	$P[S_{30}(17); S_{30}(23+30m)]$	$P[S_{30}(31); S_{30}(37+30m)]$
4	$P[S_{30}(11); S_{30}(19+30m)]$	$P[S_{30}(23); S_{30}(31+30m)]$	$P[S_{30}(29); S_{30}(37+30m)]$
5	$P[S_{30}(13); S_{30}(23+30m)]$	$P[S_{30}(7); S_{30}(17+30m)]$	$P[S_{30}(19); S_{30}(29+30m)]$
	$P[S_{30}(31); S_{30}(41+30m)]$		
6	$P[S_{30}(11); S_{30}(23+30m)]$	$P[S_{30}(7); S_{30}(19+30m)]$	$P[S_{30}(17); S_{30}(29+30m)]$
	$P[S_{30}(19); S_{30}(31+30m)]$	$P[S_{30}(29); S_{30}(41+30m)]$	
7	$P[S_{30}(23); S_{30}(37+30m)]$	$P[S_{30}(17); S_{30}(31+30m)]$	$P[S_{30}(29); S_{30}(43+30m)]$
8	$P[S_{30}(31); S_{30}(47+30m)]$	$P[S_{30}(13); S_{30}(29+30m)]$	$P[S_{30}(7); S_{30}(23+30m)]$
9	$P[S_{30}(11); S_{30}(29+30m)]$	$P[S_{30}(13); S_{30}(31+30m)]$	$P[S_{30}(23); S_{30}(41+30m)]$
	$P[S_{30}(19); S_{30}(37+30m)]$	$P[S_{30}(29); S_{30}(47+30m)]$	$P[S_{30}(31); S_{30}(49+30m)]$
10	$P[S_{30}(11); S_{30}(31+30m)]$	$P[S_{30}(23); S_{30}(43+30m)]$	$P[S_{30}(17); S_{30}(37+30m)]$
	$P[S_{30}(29); S_{30}(49+30m)]$		
11	$P[S_{30}(7); S_{30}(29+30m)]$	$P[S_{30}(19); S_{30}(41+30m)]$	$P[S_{30}(31); S_{30}(53+30m)]$
12	$P[S_{30}(13); S_{30}(37+30m)]$	$P[S_{30}(23); S_{30}(47+30m)]$	$P[S_{30}(7); S_{30}(31+30m)]$
	$P[S_{30}(17); S_{30}(41+30m)]$	$P[S_{30}(19); S_{30}(43+30m)]$	$P[S_{30}(29); S_{30}(53+30m)]$
13	$P[S_{30}(11); S_{30}(37+30m)]$	$P[S_{30}(23); S_{30}(49+30m)]$	$P[S_{30}(17); S_{30}(43+30m)]$
14	$P[S_{30}(13); S_{30}(41+30m)]$	$P[S_{30}(19); S_{30}(47+30m)]$	$P[S_{30}(31); S_{30}(59+30m)]$

Table 8.

For example: When k=1, then m=0 and h=1, so we have infinite sets of prime pairs $P[S_{30}(11); S_{30}(13)]$, $P[S_{30}(17); S_{30}(19)]$, and $P[S_{30}(29); S_{30}(31)]$ that differ by 2, which is the case of Theorem 3.3.

When k = 15, then m = 1, i= 0 and h = 0, we have infinite sets of prime pairs $P[S_{60}(31); S_{60}(61)]$, $P[S_{60}(11); S_{60}(41)]$, (see Table 9. below)…, $P[S_{60}(29); S_{60}(59)]$ that differ by 30.

$A[N_{60}(31,61)]$	$P[S_{60}(31); S_{60}(61)]$	
n	60(n-1)+31	60(n-1)+61
1	31	61
3	151	181
4	211	241
10	571	601
11	631	661
17	991	1021
20	1171	1201
22	1291	1321
31	1831	1861
36	2131	2161
38	2251	2281
39	2311	2341
50	2971	3001
55	3271	3301
56	3331	3361
59	3511	3541
71	4231	4261
77	4591	4621
81	4831	4861
97	5791	5821
98	5851	5881
102	6091	6121
105	6271	6301
108	6451	6481
127	7591	7621
137	8191	8221
… …	… …	… …

$A[N_{60}(11,41)]$	$P[S_{60}(11); S_{60}(41)]$	
n	60(n-1)+11	60(n-1)+41
1	11	41
2	71	101
5	251	281
8	431	461
9	491	521
16	911	941
18	1031	1061
20	1151	1181
25	1451	1481
27	1571	1601
32	1871	1901
36	2111	2141
40	2351	2381
41	2411	2441
44	2591	2621
46	2711	2741
51	3011	3041
54	3191	3221
62	3671	3701
65	3851	3881
71	4211	4241
74	4391	4421
75	4451	4481
79	4691	4721
85	5051	5081
88	5231	5261
… …	… …	… …

Table 9.

Again, notice that our A-Sieve used to $N_d(a,b)$ is similar to the use of sieve of Eratosthenes to N, it leads to that all $A[N_d(a,b)]$ have similar patterns as P[N]. Therefore, the prime pairs (p, p+2k) also behave similarly to P[N] with similar density (see Notes [5] table for k=1) and that will be our future works.

Theorem 3.4 can also be interpreted in a different way as follows:

THEOREM 3.5. *Every even number is the difference of two primes and there are infinite of such pairs of primes.*

On the other hand, we want to know if every even number greater than 2 is the sum of two primes, which is the so called Goldbach conjecture and will be explored in our future works.

4. Further Study

Our A-Sieve provides some new methods in the study of Number Theory in different aspects. Here we list some for the further study.

4.1. Prime k-tuples

Prime pairs can be generalized to ***prime k-tuples***, $(p_1,p_2,p_3,...,p_k)$, patterns in the differences between more than two prime numbers.

Let $\mathbf{N_d(a_1,a_2,\dots,a_k)}$ denote the set of natural numbers associated with k sequences $S_d(a_1)$, $S_d(a_2)$, …, and $S_d(a_k)$. Then $A[N_d(a_1,a_2,\dots,a_k)] = A[N_d(a_1)]\cap A[N_d(a_2)\cap \dots , \cap A[N_d(a_k)]$.

Let $\mathbf{P[S_d(a_1); S_d(a_2), \dots, S_d(a_k)]} = \{a_1+(n-1)d; a_2+(n-1)d; \dots ; a_k+(n-1)d, \quad n\in A[N_d(a_1,a_2,\dots,a_k)]\}$ be the set of all prime k-tuples from $S_d(a_1)$, $S_d(a_2)$, …, and $S_d(a_k)$ produced by $A[N_d(a_1,a_2,\dots,a_k)]$.

Similar to the prime pairs, if we take different combinations from $S_{30}(31)$, $S_{30}(11)$, $S_{30}(13)$, $S_{30}(23)$, $S_{30}(7)$, $S_{30}(17)$, $S_{30}(19)$, and $S_{30}(29)$, then we can study the infinitude and density of prime k-tuples.

For example: If we consider sequences $S_{30}(11)$, $S_{30}(13)$, $S_{30}(17)$ and $S_{30}(17)$, $S_{30}(19)$, $S_{30}(23)$, then by using A-Sieve to $N_{30}(11,13,17)$ and $N_{30}(17,19,23)$ we have the following (Table 10.) infinitely many prime 3-tuples (p, p+2, p+6):

$N_{30}(11,13,17)$	$P[S_{30}(11); S_{30}(13); S_{30}(17)]$			$N_{30}(17,19,23)$	$P[S_{30}(17); S_{30}(19); S_{30}(23)]$		
n	30(n-1)+11	30(n-1)+13	30(n-1)+17	n	30(n-1)+17	30(n-1)+19	30(n-1)+23
1	11	13	17	1	17	19	23
2	41	43	47	4	107	109	113
4	101	103	107	8	227	229	233
7	191	193	197	12	347	349	353
11	311	313	317	29	857	859	863
16	461	463	467	43	1277	1279	1283
22	641	643	647	48	1427	1429	1433
28	821	823	827	50	1487	1489	1493
30	881	883	887	54	1607	1609	1613
37	1091	1093	1097	67	1997	1999	2003
44	1301	1303	1307	75	2237	2239	2243
50	1481	1483	1487	76	2267	2269	2273
63	1871	1873	1877	89	2657	2659	2663
70	2081	2083	2087	90	2687	2689	2693
109	3251	3253	3257	118	3527	3529	3533
116	3461	3463	3467	131	3917	3919	3923
123	3671	3673	3677	138	4127	4129	4133
134	4001	4003	4007	151	4517	4519	4523
165	4931	4933	4937	155	4637	4639	4643
175	5231	5233	5237	160	4787	4789	4793
184	5501	5503	5507	166	4967	4969	4973
189	5651	5653	5657	183	5477	5479	5483
275	8231	8233	8237	207	6197	6199	6203
277	8291	8293	8297	228	6827	6829	6833
296	8861	8863	8867	263	7877	7879	7883
315	9431	9433	9437	270	8087	8089	8093
316	9461	9463	9467	285	8537	8539	8543
345	10331	10333	10337	348	10427	10429	10433
373	11171	11173	11177	349	10457	10459	10463
… …	… …	… …	… …	… …	… …	… …	… …

Table 10.

If we consider $N_{30}(13,17,19)$ for sequences $S_{30}(13)$, $S_{30}(17)$, $S_{30}(19)$ and $N_{30}(7,11,13)$ for $S_{30}(7)$, $S_{30}(11)$, $S_{30}(13)$, then we can get infinitely many prime 3-tuples (p, p+4, p+6).

If consider $N_{30}(11,17,19)$ for sequences $S_{30}(11)$, $S_{30}(17)$, $S_{30}(19)$ and $N_{30}(23,29,31)$ for $S_{30}(23)$, $S_{30}(29)$, $S_{30}(31)$, then we can get infinitely many prime 3-tuples (p, p+6, p+8). Thus, we have the following:

THEOREM 4.1. *There are infinitely many prime 3-tuples such as (p, p+2, p+6), (p, p+4, p+6), and (p, p+6, p+8).*

Now if we consider sequences $S_{30}(11)$, $S_{30}(13)$, $S_{30}(17)$, and $S_{30}(19)$, then by using A-Sieve to $N_{30}(11,13,17,19)$, we have the following (Table 11.) infinitely many prime 4-tuples (p, p+2, p+6, p+8):

A[$N_{30}(11,13,17,19)$]	P[$S_{30}(11)$; $S_{30}(13)$; $S_{30}(17)$; $S_{30}(19)$]			
n	30(n-1)+11	30(n-1)+13	30(n-1)+17	30(n-1)+19
1	11	13	17	19
4	101	103	107	109
7	191	193	197	199
28	821	823	827	829
50	1481	1483	1487	1489
63	1871	1873	1877	1879
70	2081	2083	2087	2089
109	3251	3253	3257	3259
116	3461	3463	3467	3469
189	5651	5653	5657	5659
315	9431	9433	9437	9439
434	13001	13003	13007	13009
522	15641	15643	15647	15649
525	15731	15733	15737	15739
536	16061	16063	16067	16069
602	18041	18043	18047	18049
631	18911	18913	18917	18919
648	19421	19423	19427	19429
701	21011	21013	21017	21019
743	22271	22273	22277	22279
844	25301	25303	25307	25309
… …	… …	… …	… …	… …

Table 11.

Hence, we have the following:

THEOREM 4.2. *There are infinitely many prime 4-tuples such as (p, p+2, p+6, p+8).*

The more general prime k-tuples will be the explored in our future works.

4.2. A-Sieve in different level

We introduced A-Sieve first start with the level of 4 sequences $S_{10}(11)$, $S_{10}(13)$, $S_{10}(17)$, and $S_{10}(19)$. Then we separate each one to get the next level of 8 sequences $S_{30}(31)$, $S_{30}(11)$, …, and $S_{30}(29)$ in order to study the prime k-tuples, especially prime pairs.

If we separate $S_{30}(11)$, into $S_{210}(11)$, $S_{210}(41)$, $S_{210}(71)$, $S_{210}(101)$, $S_{210}(131)$, $S_{210}(161)$, $S_{210}(191)$ and sieve out $S_{210}(161)$, the set of 7-multiple numbers, so that the rest 6 sequences have no divisor 7 after 2, 5, and 3 (see Table 12. below).

$S_{210}(11)$	$S_{210}(41)$	$S_{210}(71)$	$S_{210}(101)$	$S_{210}(131)$	$S_{210}(161)$	$S_{210}(191)$
11	41	71	101	131	161	191
221	251	281	311	341	371	401
431	461	491	521	551	581	611
641	671	701	731	761	791	821
851	881	911	941	971	1001	1031
1061	1091	1121	1151	1181	1211	1241
1271	1301	1331	1361	1391	1421	1451
… …	… …	… …	… …	… …	… …	… …

Table 12.

The same process to $S_{30}(1)$, $S_{30}(13)$, $S_{30}(23)$, $S_{30}(7)$, $S_{30}(17)$, $S_{30}(19)$, $S_{30}(29)$ and each derives 6 sequences not containing divisors 2,5,3, and 7 after sieving out all 7-multiple numbers.

Then, for each of these 8*6=48 of sequences $S_{210}(x)$, use A-Sieve to the corresponding sets $N_{210}(x)$ by starting on the prime number p=11.

In general, the ***i-th level of A-Sieve*** can be described as follows:

Let $p_i \in P[N]$ $(i \in N)$ be the i-th prime number in the acendent order with i except only we reverse 3and 5 such that $p_2 = 5$ and $p_3 = 3$, i.e., $p_1 = 2$, $p_2 = 5$, $p_3 = 3$, $p_4 = 7$, $p_5 = 11$, …

For $i \in N$, let $d(i) = p_1*p_2*p_3*p_4*\ldots*p_i$, the product of first i prime numbers,

and let $r(i) = (p_1-1)*(p_2-1)*(p_3-1)*(p_4-1)*\ldots*(p_i-1)$.

Then we have $d(i) = d(i-1)*p_i$ and $r(i) = r(i-1)*(p_i-1)$, where we set $r(0) = d(0) = 1$.

Now for each of $r(i-1)$ sequences $S_{d(i-1)}(x_{i-1})$, we separate it into p_i sub-sequences $S_{d(i)}(x_i)$, where $x_i = x_{i-1} + m*d(i-1)$ with $0 \le m \le (p_i-1)$, then we get $r(i-1)*p_i$ such sub-sequences. After sieving out all the set of p_i-multiple numbers, the rest $r(i-1)*(p_i-1) = r(i)$ of sub-sequences have no divisor p_1, p_2, p_3, …, and p_i. Thus we can use A-Sieve to such $N_{d(i)}(x_i)$ by starting on the next prime number $p_{i+1.}$

For example:

Start with $S_1(1)$, then $N_1(1) = S_1(1) = N$, so our A-Sieve to $N_1(1)$ is just the classical sieve of Eratosthenes, and we can put this as $i = 0$ level;

When i= 1, $d(1) = p_1 = 2$, $r(1) = (p_1-1) = 1$, $r(0) = d(0) = 1$, $x_1 = 1, 2$, then separate $S_1(1)$ into 2 subsequence $S_2(1)$ and $S_2(2)$. After sieving out $S_2(2)$, the set of even numbers, $S_2(1)$ have no divisor 2. Then we can use A-Sieve to $N_2(1)$ by starting on the next prime number $p_2 = 5$;

When i= 2, $d(2) = p_1*p_2 = 2*5 = 10$, $r(2) = (p_1-1)*(p_2-1) = 4$, $x_2 = 1, 3, 5, 7, 9$. We separate $S_2(1)$ into 5 subsequence $S_{10}(1)$, $S_{10}(3)$, $S_{10}(5)$, $S_{10}(7)$, and $S_{10}(9)$. After sieving out $S_{10}(5)$, the set of 5-multiple numbers, 4 subsequence $S_{10}(1)$, $S_{10}(3)$, $S_{10}(7)$, and $S_{10}(9)$ have no divisor 2 and 5. Then we can use A-Sieve to $N_{10}(1)$, $N_{10}(3)$, $N_{10}(7)$, and $N_{10}(9)$ by starting on the prime number next to $p_2 = 5$, i.e., $p_3 = 3$ as we did in the beginging to introduce A-Sieve;

When i= 3, $d(3) = p_1*p_2*p_3 = 2*5*3 = 30$, $r(3) = (p_1-1)*(p_2-1)*(p_3-1) = 8$, $x_3 = 1, 11, 21, 3, 13, 23, 7, 17, 27, 9, 19, 29$, then separate each $S_{10}(x_2)$ into 3 subsequence $S_{30}(x_3)$. After sieving out all 4 subsequence of the set of 3-multiple numbers, 8 subsequence $S_{30}(1)$, $S_{30}(11)$, …, and $S_{30}(29)$ have no divisor 2, 5, and 3. Then we can use A-Sieve to $N_{30}(1)$, $N_{30}(11)$, …, and $N_{30}(29)$ by starting on the prime number next to $p_3 = 3$, i.e., $p_4 = 7$ as we did in the Notes [4];

When i= 4, $d(4) = p_1*p_2*p_3*p_4 = 2*5*3*7 = 210$, $r(4) = (p_1-1)*(p_2-1)*(p_3-1)*(p_4-1) = 48$, we separate each $S_{30}(x_3)$ into 7 subsequence $S_{210}(x_4)$. After sieving out all 8 subsequence of the set of 7-multiple numbers, 48 subsequence $S_{210}(1)$, $S_{210}(31)$, …, and $S_{210}(209)$ have no divisor 2, 5, 3, and 7. Then we can use A-Sieve to $N_{210}(1)$, $N_{210}(31)$, …, and $N_{210}(209)$ by starting on the prime number next to $p_4 = 7$, i.e., $p_5 = 11$; … … (see Notes [6] table).

For each $i \in N$, at the same i-th level of A-Sieve, all $A[N_{d(i)}(x_i)]$ have the same A-Sieve pattern, except

for the starting points so all $P[S_{d(i)}(x_i)]$ should have similar size.
For example, when i= 3, the following table (Table 13.) shows the size of all $P[S_{30}(x_3)]$ up to M:

M	the number of primes in P[X] that less than M									
	P[N]	$P[S_{30}(1)]$	$P[S_{30}(11)]$	$P[S_{30}(13)]$	$P[S_{30}(23)]$	$P[S_{30}(7)]$	$P[S_{30}(17)]$	$P[S_{30}(19)]$	$P[S_{30}(29)]$	Sum
100	25	2	3	3	3	4	2	2	3	22
200	46	4	6	6	5	6	6	5	5	43
500	95	9	13	12	11	13	11	11	12	92
1000	168	18	22	20	21	24	22	18	20	165
2000	303	34	39	39	38	38	39	37	36	300
3000	430	50	53	54	55	55	56	48	56	427
5000	669	80	83	87	84	84	85	79	84	666
6000	783	92	102	101	97	98	100	93	97	780
10000	1229	152	154	154	155	155	153	150	153	1226
15000	1754	210	220	221	221	222	222	212	223	1751
20000	2264	275	289	285	283	287	283	277	282	2261
25000	2764	340	350	346	343	342	353	343	344	2761
30000	3247	402	407	406	408	407	412	395	407	3244
… …	… …	… …	… …	… …	… …	… …	… …	… …	… …	… …

Table 13.

From the table above, we can see that all $P[S_{30}(x_3)]$ have almost the same size, and the size of p[N] is the sum of the sizes of all $P[S_{30}(x_3)]$ plus 3.

$P[N] = P[S_{30}(1)] \cup P[S_{30}(11)] \cup P[S_{30}(13)] \cup P[S_{30}(23)] \cup P[S_{30}(7)] \cup P[S_{30}(17)] \cup P[S_{30}(19)] \cup P[S_{30}(29)] \cup \{2,3,5\}$, the primes are distributed evenly among all $P[S_{30}(x_3)]$.

Therefore, A-Sieve provides a way to study the distribution of P[N]. We have the following:

1) All $P[S_{d(i)}(x_i)]$ at the same i-th level have the same A-Sieve pattern and almost the same size. Furthermore, the size of p[N] is the sum of the sizes of all $P[S_{d(i)}(x_i)]$ plus i.
2) P[N] is the union of all $P[S_{d(i)}(x_i)]$ plus $\{p_1,p_2,\ldots, p_i\}$. In other word, the primes are distributed evenly among all $P[S_{d(i)}(x_i)]$.

Finally, notice that A-Sieve also provides a way to find large prime numbers. Application of the method using computers will be explored in our future works.In summary, A-Sieve provides a novel method to study or reconsider some problems in Number Theory.

Notes

[1] Basic Mathematical symbols:

$\boldsymbol{N}$ – the set of natural numbers (all numbers written in the usual decimal system),

$x \in \boldsymbol{X}$ – an element x belongs to a set $\boldsymbol{X}$,

$x \notin \boldsymbol{X}$ – an element x doesn't belong to a set $\boldsymbol{X}$,

$\boldsymbol{X} \subset \boldsymbol{Y}$ – a set $\boldsymbol{X}$ is a subset of a set $\boldsymbol{Y}$,

$\boldsymbol{X} \cup \boldsymbol{Y}$ – a union of sets $\boldsymbol{X}$ and $\boldsymbol{Y}$,

$\boldsymbol{X} \cap \boldsymbol{Y}$ – an intersection of sets $\boldsymbol{X}$ and $\boldsymbol{Y}$,

$\boldsymbol{X} - \boldsymbol{Y}$ – $\{x: \ x \in \boldsymbol{X}$ and $x \notin \boldsymbol{Y}\}$,

$\{ u_n \}$ – a sequence with a general term u_n.

[2] New Notations of this paper:

$S_d(a) = \{a+(n-1)d, n \in N\}$ – a sequence with a general term a+(n-1)d,

P[X] – the set of all prime numbers in the set of X,

$N_d(a)$ – the set of natural numbers associated with $S_d(a)$,

A-Sieve – a Sieve that used to the set $N_d(a)$ instead of $S_d(a)$,

A[N_d(a)] – the set of all the remaining numbers after **A-Sieve** to $N_d(a)$,

P[S_d(a)] = {a+(n-1)d, n ∈ A[N_d(a)]} – the set of all prime numbers in $S_d(a)$ produced by A[N_d(a)],

N_d(a,b) – the set of natural numbers associated with the pair of $S_d(a)$ and $S_d(b)$,

P[S_d(a); S_d(b)] = {a+(n-1)d; b+(n-1)d, n ∈ A[N_d(a,b)]} – the set of all pairs of prime numbers from $S_d(a)$ and $S_d(b)$ produced by A[N_d(a,b)].

$N_d(a_1,a_2,\dots,a_k)$ – the set of natural numbers associated with k sequences $S_d(a_1)$, $S_d(a_2)$, …, and $S_d(a_k)$.

P[$S_d(a_1)$; $S_d(a_2)$; …; $S_d(a_k)$]={a_1+(n-1)d; a_2+(n-1)d; … ; a_k+(n-1)d, n ∈ A[$N_d(a_1,a_2,\dots,a_k)$]}
– the set of all prime k-tuples from $S_d(a_1)$, $S_d(a_2)$, …, and $S_d(a_k)$ produced by A[$N_d(a_1,a_2,\dots,a_k)$]

[3] The color be used to the corresponding prime numbers:

3	7	11	13	17	19	23	29	31	37	41

[4] The following tables indicate some of the process of using A-Sieve to
$N_{30}(31)$, $N_{30}(11)$, $N_{30}(13)$, $N_{30}(23)$, $N_{30}(7)$, $N_{30}(17)$, $N_{30}(19)$, $N_{30}(29)$ to get A[N_{30}(31)], A[N_{30}(11)], A[N_{30}(13)], A[N_{30}(23)], A[N_{30}(7)], A[N_{30}(17)], A[N_{30}(19)], A[N_{30}(29)] and P[S_{30}(31)], P[S_{30}(11)], P[S_{30}(13)], P[S_{30}(23)], P[S_{30}(7)], P[S_{30}(17)], P[S_{30}(19)], P[S_{30}(29)].

$N_{30}(31)$	$S_{30}(31)$	$N_{30}(11)$	$S_{30}(11)$
n	30(n-1)+31	n	30(n-1)+11
		1	11
1	31	2	41
2	61	3	71
3	91	4	101
4	121	5	131
5	151	6	161
6	181	7	191
7	211	8	221
8	241	9	251
9	271	10	281
10	301	11	311
11	331	12	341
12	361	13	371
13	391	14	401
14	421	15	431
15	451	16	461
16	481	17	491
17	511	18	521
18	541	19	551
19	571	20	581
20	601	21	611
21	631	22	641
22	661	23	671
23	691	24	701
24	721	25	731
25	751	26	761
26	781	27	791
27	811	28	821
28	841	29	851
29	871	30	881
30	901	31	911
31	931	32	941
32	961	33	971
33	991	34	1001
34	1021	35	1031
35	1051	36	1061
… …	… …	… …	… …

$N_{30}(13)$	$S_{30}(13)$	$N_{30}(23)$	$S_{30}(23)$
n	30(n-1)+13	n	30(n-1)+23
1	13	1	23
2	43	2	53
3	73	3	83
4	103	4	113
5	133	5	143
6	163	6	173
7	193	7	203
8	223	8	233
9	253	9	263
10	283	10	293
11	313	11	323
12	343	12	353
13	373	13	383
14	403	14	413
15	433	15	443
16	463	16	473
17	493	17	503
18	523	18	533
19	553	19	563
20	583	20	593
21	613	21	623
22	643	22	653
23	673	23	683
24	703	24	713
25	733	25	743
26	763	26	773
27	793	27	803
28	823	28	833
29	853	29	863
30	883	30	893
31	913	31	923
32	943	32	953
33	973	33	983
34	1003	34	1013
35	1033	35	1043
36	1063	36	1073
… …	… …	… …	… …

$N_{30}(7)$	$S_{30}(7)$	$N_{30}(17)$	$S_{30}(17)$
n	30(n-1)+7	n	30(n-1)+17
1	7	1	17
2	37	2	47
3	67	3	77
4	97	4	107
5	127	5	137
6	157	6	167
7	187	7	197
8	217	8	227
9	247	9	257
10	277	10	287
11	307	11	317
12	337	12	347
13	367	13	377
14	397	14	407
15	427	15	437
16	457	16	467
17	487	17	497
18	517	18	527
19	547	19	557
20	577	20	587
21	607	21	617
22	637	22	647
23	667	23	677
24	697	24	707
25	727	25	737
26	757	26	767
27	787	27	797
28	817	28	827
29	847	29	857
30	877	30	887
31	907	31	917
32	937	32	947
33	967	33	977
34	997	34	1007
35	1027	35	1037
… …	… …	… …	… …

$N_{30}(19)$	$S_{30}(19)$	$N_{30}(29)$	$S_{30}(29)$
n	30(n-1)+19	n	30(n-1)+29
1	19	1	29
2	49	2	59
3	79	3	89
4	109	4	119
5	139	5	149
6	169	6	179
7	199	7	209
8	229	8	239
9	259	9	269
10	289	10	299
11	319	11	329
12	349	12	359
13	379	13	389
14	409	14	419
15	439	15	449
16	469	16	479
17	499	17	509
18	529	18	539
19	559	19	569
20	589	20	599
21	619	21	629
22	649	22	659
23	679	23	689
24	709	24	719
25	739	25	749
26	769	26	779
27	799	27	809
28	829	28	839
29	859	29	869
30	889	30	899
31	919	31	929
32	949	32	959
33	979	33	989
34	1009	34	1019
35	1039	35	1049
… …	… …	… …	… …

A[N_{30}(31)]	P[S_{30}(31)]	A[N_{30}(11)]	P[S_{30}(11)]	A[N_{30}(13)]	P[S_{30}(13)]	A[N_{30}(23)]	P[S_{30}(23)]
n	30(n-1)+31	n	30(n-1)+11	n	30(n-1)+13	n	30(n-1)+23
1	31	1	11	1	13	1	23
2	61	2	41	2	43	2	53
5	151	3	71	3	73	3	83
6	181	4	101	4	103	4	113
7	211	5	131	6	163	6	173
8	241	7	191	7	193	8	233
9	271	9	251	8	223	9	263
11	331	10	281	10	283	10	293
14	421	11	311	11	313	12	353
18	541	14	401	13	373	13	383
19	571	15	431	15	433	15	443
20	601	16	461	16	463	17	503
21	631	17	491	18	523	19	563
22	661	18	521	21	613	20	593
23	691	22	641	22	643	22	653
25	751	24	701	23	673	23	683
27	811	26	761	25	733	25	743
33	991	28	821	28	823	26	773
34	1021	30	881	29	853	29	863
35	1051	31	911	30	883	32	953
39	1171	32	941	35	1033	33	983
40	1201	33	971	36	1063	34	1013
41	1231	35	1031	37	1093	37	1103
43	1291	36	1061	38	1123	39	1163
44	1321	37	1091	39	1153	40	1193
46	1381	39	1151	41	1213	41	1223
49	1471	40	1181	44	1303	43	1283
51	1531	44	1301	48	1423	46	1373
54	1621	46	1361	49	1453	48	1433
58	1741	49	1451	50	1483	50	1493
60	1801	50	1481	52	1543	51	1523
61	1831	51	1511	56	1663	52	1553
62	1861	53	1571	57	1693	53	1583
65	1951	54	1601	58	1723	54	1613
67	2011	58	1721	59	1753	58	1733
… …	… …	… …	… …	… …	… …	… …	… …

A[N_{30}(7)]	P[S_{30}(7)]	A[N_{30}(17)]	P[S_{30}(17)]	A[N_{30}(19)]	P[S_{30}(19)]	A[N_{30}(29)]	P[S_{30}(29)]
n	30(n-1)+7	n	30(n-1)+17	n	30(n-1)+19	n	30(n-1)+29
1	7	1	17	1	19	1	29
2	37	2	47	3	79	2	59
3	67	4	107	4	109	3	89
4	97	5	137	5	139	5	149
5	127	6	167	7	199	6	179
6	157	7	197	8	229	8	239
10	277	8	227	12	349	9	269
11	307	9	257	13	379	12	359
12	337	11	317	14	409	13	389
13	367	12	347	15	439	14	419
14	397	16	467	17	499	15	449
16	457	19	557	21	619	16	479
17	487	20	587	24	709	17	509
19	547	21	617	25	739	19	569
20	577	22	647	26	769	20	599
21	607	23	677	28	829	22	659
25	727	27	797	29	859	24	719
26	757	28	827	31	919	27	809
27	787	29	857	34	1009	28	839
30	877	30	887	35	1039	31	929
31	907	32	947	36	1069	34	1019
32	937	33	977	38	1129	35	1049
33	967	37	1097	42	1249	37	1109
34	997	40	1187	43	1279	41	1229
37	1087	41	1217	47	1399	42	1259
38	1117	43	1277	48	1429	43	1289
42	1237	44	1307	49	1459	44	1319
44	1297	46	1367	50	1489	47	1409
45	1327	48	1427	52	1549	48	1439
49	1447	50	1487	53	1579	50	1499
53	1567	54	1607	54	1609	52	1559
54	1597	55	1637	56	1669	54	1619
55	1627	56	1667	57	1699	57	1709
56	1657	57	1697	59	1759	63	1889
59	1747	60	1787	60	1789	65	1949
… …	… …	… …	… …	… …	… …	… …	… …

[5] (p, p+2) has similar density with P[N]

M	primes< M	30M	the number of primes in P[X] that less than 30M			
	In P[N]		$P[S_{30}(11); S_{30}(13)]$	$P[S_{30}(17); S_{30}(19)]$	$P[S_{30}(29); S_{30}(31)]$	Sum
20	8	600	10	7	9	26
50	15	1500	20	12	16	48
100	25	3000	30	24	26	80
200	46	6000	51	44	46	141
300	62	9000	61	60	67	188
500	95	15000	89	88	93	270
1000	168	30000	158	156	152	466
… …	… …	… …	… …	… …	… …	… …

[6] the i-th level of A-Sieve

level i	first i primes $p_1,\ldots,p_i$	d(i) $p_1*\ldots*p_i$	r(i) $(p_1-1)*\ldots(p_i-1)$	separate r(i-1) $S_{d(i-1)}(x_{i-1})$ into r(i-1)*p_i $S_{d(i)}(x_i)$	sieve out p_i- multiples	A-Sieve to $N_{d(i)}(x_i)$	start prime p_{i+1}
0		1	1	$S_1(1)=N$		$N_1(1)=N$	2
1	2	2	1	$S_2(1)$, $S_2(2)$	$S_2(2)$	$N_2(1)$	5
2	2,5	10	4	$S_{10}(1)$, $S_{10}(3)$, $S_{10}(5)$ $S_{10}(7)$, $S_{10}(9)$	$S_{10}(5)$	$N_{10}(1)$, $N_{10}(3)$ $N_{10}(7)$, $N_{10}(9)$	3
3	2,5,3	30	8	$S_{30}(1)$, $S_{30}(11)$, $S_{30}(21)$ $S_{30}(3)$, $S_{30}(13)$, $S_{30}(23)$ $S_{30}(7)$, $S_{30}(17)$, $S_{30}(27)$ $S_{30}(9)$, $S_{30}(19)$, $S_{30}(29)$	$S_{30}(21)$ $S_{30}(3)$ $S_{30}(27)$ $S_{30}(9)$	$N_{30}(1)$, $N_{30}(11)$ $N_{30}(13)$, $N_{30}(23)$ $N_{30}(7)$, $N_{30}(17)$ $N_{30}(19)$, $N_{30}(29)$	7
4	2,5,3,7	210	48	$S_{210}(1)$, $S_{210}(31)$, … $S_{210}(209)$	$S_{210}(91)$, … $S_{210}(119)$	$N_{210}(1)$, $N_{210}(31)$, …, $N_{210}(209)$	11
5	2,5,3, 7,11	2310	480	$S_{2310}(1)$, $S_{2310}(211)$, … $S_{2310}(2309)$	$S_{2310}(2101)$, …	$N_{2310}(1)$, … $N_{2310}(2309)$	13
6	2,5,3,7, 11,13	30030	5760	$S_{30030}(1)$, $S_{30030}(2311)$, … $S_{30030}(30029)$	$S_{30030}(23101)$, …	$N_{30030}(1)$, … $N_{30030}(30029)$	17
7	2,5,3,7, 11,13,17	510510	92160	$S_{510510}(1)$, $S_{510510}(30031)$, … $S_{510510}(510509)$	$S_{510510}(60061)$ …	$N_{510510}(1)$, … $N_{510510}(510509)$	19
8	2,5,3,7, 11,13, 17,19	9699690	1658880	$S_{9699690}(1)$, $S_{9699690}(510511)$, … $S_{9699690}(9699689)$	$S_{9699690}(510511)$ …	$N_{9699690}(1)$, … $N_{9699690}(9699689)$	23
… …	… …	… …	… …	… …	… …	… …	… …