An unconditional and conclusive argument for the truth of the continuum hypothesis is found

§1. Equivalents of the continuum hypothesis

Let \( \text{int}(S) \) and \( \partial S \) denote the interior and boundary of \( S \). \( T \supset S \) denotes the complement of \( S \) to \( T \). \( \text{card}(S) \) denotes the cardinality of \( S \). \( f(S) \) – the image of set \( S \) obtained by mapping \( f \).

Let the relation \( < \) define a partial order on the set \( S \), i.e. the relation \( < \) is transitive, and \( p < q \) implies \( p \neq q \) and \( q \not< p \). Then, let’s say that set \( S \) “is countably divisible (in oneself, i.e. when a set can be divided into parts a countable number of times and using the points of this set) in relation \( < \)”, if:

For any nonempty and at most countable sets \( M \subset S \) and \( L \subset S \) such that \( p < q \) holds for every \( p \in M \) and every \( q \in L \), there exists an element \( r \in S \) such that
\[
\forall p' \in M \forall q' \in L, p' < r < q'.
\]

A set \( S \) is called “countably limitless in relation \( < \)” (in oneself, i.e. when set cannot be limited to a countable quantity of its borders) if:

Whatever \( M \subset S, M \neq \emptyset \), \( \text{card}(M) \leq \aleph_0 \), there exist \( p \in S \) and \( q \in S \) such that
\[
\forall m \in M p < m < q \text{ is true.}
\]

Demanding (instead of “no more than countability”) the “finiteness” of the sets \( M \) and \( L \), we obtain a definition of “finite divisibility” (a set has finite divisibility) and “finite limitless” (a set cannot be limited to a finite quantity of borders). A set composed of rational numbers, for example, has finite divisibility and finite limitless.

If \( \nu \) and \( \eta \) are ordinals, then \( \nu < \eta \) is equivalent to \( \nu \in \eta \). We identify the set of all at most countable ordinals with the ordinal \( \omega_1 \). The set of all finite ordinals, i.e. natural numbers, we identify with the ordinal \( \omega = \omega_0 \).

Let \( \gamma \) be an ordinal and \( \gamma \subseteq \gamma \). “A sequence defined on \( \eta \)” is a function \( f \) defined on \( \eta \). An element of the sequence \( f \) having a “number” \( n \in \eta \) is denoted by \( f_n \) or \( f(n) \). If \( \xi \) is an ordinal, and \( \eta \) is mapped onto \( \xi \) while preserving the order of elements, then we call the sequence to “sequence of length (type) \( \xi \)”. If the elements of the sequence defined on \( \eta \) are selected in the set where the partial order relation \( < \) acts, and \( \nu \in \mu \) implies \( f_\nu > f_\mu \) (\( f_\mu > f_\nu \)), where \( f_\nu \) and \( f_\mu \) are the elements with “numbers” \( \nu \in \eta \) and \( \mu \in \eta \), then we write that the sequence “strictly decreases” (“strictly increases”).
Let \( \mathcal{U} \) be a set of all functions defined on set \( \omega_1 \) taking values 0 or 1. That is, elements \( \in \mathcal{U} \) are binary sequences of length \( \omega_1 \), which are also interpreted as “infinite binary records” and as “points”. The entries \( \in \mathcal{U} \) are ordered lexicographically. The entries “only from ones” and “only from zeros” are denoted by \( 1_\mathcal{U} \) and \( 0_\mathcal{U} \). They are the maximal and minimal element \( \in \mathcal{U} \).

If you require set \( \mathcal{U} \) to be continuous, then some points \( \in \mathcal{U} \) must be defined by two entries, while others only by one. In this sense, the orders on points and records are somewhat different. The points defined by two entries form a “set of middles” \( \mathcal{M} \subset \mathcal{U} \) (we identify these two entries in the order on the points).

The formulas \( X < Y, X > Y \) and \( X = Y \) read “\( X \) is to the left of \( Y \) (\( X \) less \( Y \))”, “\( X \) is to the right of \( Y \) (\( X \) more \( Y \))”, “point \( X \) coincides with point \( Y \)”. Comparison relationships change from the change of the sets on which they are installed. Simplifying the text, we use a single notation for some comparisons, distinguishing them by context.

Intervals \( \subset \mathcal{U} \) are divided into open, half-open, and closed. The set \( \mathcal{G} \subset \mathcal{U} \) is called “everywhere dense in \( \mathcal{U} \)” if there are points \( \in \mathcal{G} \) on any open interval \( \subset \mathcal{U} \). By means of the canonical theory, the following theorem is trivially verified.

**Theorem I.** The set \( \mathcal{M} \) is countably divisible and countably unlimited in relation \( < \), everywhere dense in \( \mathcal{U} \), \( \text{card}(\mathcal{M}) = 2^{\aleph_0} \).

The following two statements are equivalent to the Cantor hypothesis:

**Proposition I.** There exists a set \( \mathcal{Y} \), each element of which is a subset of the set \( \omega_1 \), and \( \mathcal{Y} \) is a countably divisible set and countably limitless in relation \( \subset \).

**Proposition II.** There exists a set \( \mathcal{Y} \subset \mathcal{U} \) that is everywhere dense in \( \mathcal{U} \) and such that \( \text{card}(\mathcal{Y}) = \mathfrak{N}_1 \).

These statements seem very logical. Other statements, however, are also logical.

§2. Axioms that are not compatible with the assertion that the power of the continuum is equal to the first uncountable power

Let the sector \( \mathcal{D} = \text{int}(D) \) be the intersection of an open Euclidean circle (in the meaning of “disk having an area”, that is, not in the meaning of a circular line), of unit radius, with a right angle containing its vertex \( O \) in the center of the circle. Let the arc \( C \subset \mathcal{D} \) of a circular line (circumference) contain its ends \( X \) and \( Y \) on the sides of such a right angle. Points \( X \) and \( Y \) are parts of arc \( C \).

We denote by \( C_r \) the arc \( \subset \mathcal{D} \) which ends \( X_r \) and \( Y_r \) are not included in this arc and are on the segments \( OX \) and \( OY \), respectively, and the arc \( C_r \) itself is included in a circular line of radius \( r = \text{const} < 1 \) described from center \( O \). On each arc \( C_r \) and on the arc \( C \), we introduce a linear order:

If the points \( P \) and \( Q \) are both on the arc \( C_r \), or both are on arc \( C \), then we say and write that “\( P \) is

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§2. Axioms that are not compatible with the assertion that the power of the continuum is equal to the first uncountable power
to the left of Q" ("Q is to the right of P"), P < Q (Q > P) if the angle XOQ < angle XOP.

Let a line \( \mathbf{k} \in \mathbf{H} \), if \( \mathbf{k} \) is a curve or straight line, \( \mathbf{k} \subseteq \text{int}(\mathcal{D}) \), and \( \mathbf{k} \) starts at point O and ends on the arc C (in particular, at a point on C) so that for every positive \( r < 1 \) the set \( \mathbf{k} \cap C_r \) consists of one point.

Then, for \( q \) and \( p \in \mathbf{H} \), we write \( q \ll p \) or \( p \gg q \) if for all sufficiently large \( r < 1 \) the point \( q \cap C_r \) is located to the left of the point \( p \cap C_r \), or the point \( p \cap C_r \) is located to the right of \( q \). And for \( q \) and \( p \in \mathbf{H} \) we write \( q = p \), i.e. \( q \) and \( p \) are "equivalent" when for all sufficiently large \( r < 1 \) the points \( q \cap C_r \) and \( p \cap C_r \) coincide. If it is true that \( (q = p) \lor (q \ll p) \lor (p \gg q) \), then \( q \) and \( p \) are called "comparable". The following three theorems are of the theorem of canonical (Cantor or Zermel) theory of sets:

**Theorem II.** The set \( \mathbf{H} \) is countably divisible and countably limitless in relation \( \ll \).

**Theorem III.** There exists a countably divisible and countably limitless in relation \( \ll \) set \( \mathbf{G} \subseteq \mathbf{H} \) such that every two elements of \( \mathbf{E} \subseteq \mathbf{G} \) are comparable, the cardinality of \( \mathbf{G} \) is \( 2^{\aleph_0} \), the length of the monotone sequences \( \mathbf{E} \subseteq \mathbf{G} \) does not exceed \( \omega_1 \).

**Theorem IV.** There exist a set \( \mathbf{G} \subseteq \mathbf{H} \) and a mapping \( F \) such that \( |F(\mathbf{G})| = \aleph_1 \), and if \( q \ll p \), where \( q \) and \( p \) are lines \( \in \mathbf{G} \), then \( F(q) < F(p) \).

Using countable divisibility of the set \( \mathbf{H} \), one can construct transfinite, lengths \( \omega_1 \), strictly increasing and decreasing sequences of lines \( \in \mathbf{H} \). Hence, as a hypothesis, we can consider

**Axiom I.** Let \( \mathbf{A} \) and \( \mathbf{B} \) be non-empty subsets of the set \( \mathbf{H} \), every two elements \( \in \mathbf{A} \cup \mathbf{B} \) are comparable, and \( \text{card}(\mathbf{A} \cup \mathbf{B}) = \aleph_1 \). Let each line \( \in \mathbf{A} \) end on an arc C to the left of each line \( \in \mathbf{B} \). Then there exists a line \( \mathbf{h} \in \mathbf{H} \) terminating on C to the right of each line \( \in \mathbf{A} \) and to the left of each line \( \in \mathbf{B} \).

Axiom entails violation of the continuum hypothesis. Indeed, \( \text{card}(\mathbf{H}) = 2^{\aleph_0} \), since \( \mathbf{H} \) is given by continuous lines, and the lines quantity is equal \( 2^{\aleph_0} \). On the other hand, the number of sections of the set \( \mathbf{H} \) is at least \( 2^{\aleph_0} \). According to Axiom I, each section corresponds to some line \( \mathbf{h} \in \mathbf{H} \). Therefore, \( \text{card}(\mathbf{H}) = 2^{\aleph_0} = 2^{\aleph_1} > \aleph_1 \).
**Axiom II.** For an arbitrary \( A \subset H \) such that \( \text{card}(A) = \mathcal{N}_1 \), when all elements from \( A \) are comparable to each other, there is a line \( h \in H \) that terminates on the arc \( C \) to the right of each line of the set \( A \).

Axiom II is also not compatible with the continuum hypothesis. The following three statements are equivalent to this axiom:

**Proposition III.** For arbitrary \( A \) and \( B \), when \( A \subset H \), \( \text{card}(A) = \mathcal{N}_1 \), \( B \subset H \), \( \text{card}(B) = \mathcal{N}_0 \), such that each line \( \epsilon A \) ends on \( C \) to the left of each line \( \epsilon B \), there is a line \( h \in H \) such that \( q \ll h \ll p \) for any \( q \in A \) and \( p \in B \).

**Proposition IV.** Let \( A \subset H \) be such that \( \text{card}(A) = \mathcal{N}_1 \), and there exists a line \( q \in H \) such that \( p \ll q \) holds for every \( p \in A \). Then there exists a line \( h \in H \) for which, whatever \( p \in A \), \( p \ll h \ll q \).

**Proposition V.** Let \( S \) and \( S(n, m) \) be well ordered sets, where \( n \) and \( m \) take values \( \in \omega \). Let: \( \forall n \epsilon \omega \)(\( \forall m \epsilon \omega \)) \( S(n, m+1) \supset S(n, m) \); \( \forall n \epsilon \omega \) \( S = \bigcup_{m \epsilon \omega} S(n, m) \); \( \forall n \forall m \text{card}(S) = \text{card}(S(n, m)) = \mathcal{N}_1 \). Then, there is a function \( f \) for which, for each \( P \in S \), for all sufficiently large \( n \epsilon \omega \) (depending on \( P \)), \( S(n, f(n)) \ni P \).

The axioms are expandable so that the length of the transfinite sequences under consideration can be increased. Then, from the extension of Axiom I, we can deduce:

**Proposition VI.** Whatever the ordinal \( \alpha \), \( 2^{\mathcal{N}_0} = 2^{\mathcal{N}_\alpha} > \mathcal{N}_\alpha \).

**Proposition VII.** The cardinality of the set of real numbers exceeds the cardinality of any well ordered set.

§3. Other “experimental non-Cantor sets”

The reasoning of this section is suggestive, not strict, and is not used for any further proofs. We will say that an infinite sequence is “incapable of completion” if:

- Its initial part with a finite number of elements can be determined;

- If \( M \) initial elements of a sequence are indicated, then you can always specify its \( M+1 \)-st element;

- An assumption on an existing infinite number of “all elements of the sequence” leads to a contradiction.
Are there sufficient grounds for an idea of “completion- incapable sequences”? Or perhaps this idea is far-fetched and artificial?

The Cartesian product $\mathcal{U}\times\mathcal{U}$ is represented by a “square” with $P$, $Q$, $R$, $S$ vertices. Let $QR$ be the “upper” side of this square. We represent the strictly increasing continuous functions mapping $\mathcal{U}$ by $\mathcal{U}$ lines that connect the $P$ and $R$ vertices. Let $f$ and $g$ be such functions, moreover, for all $V\in\mathcal{U}$, $f(V)\leq g(V)$. Let us prove that whatever $f$ and $g$ are, values of the functions $f$ and $g$ coincide for an uncountable set of points $\in\mathcal{U}$. Indeed, for $X_0\in\mathcal{U}$, $X_0<1\mathfrak{g}$, let $\delta_0$ be the “vertical” segment between the points $P_0=(X_0, f(X_0))$ and $Q_0=(X_0, g(X_0))$. From the point $Q_0=(X_0, g(X_0))$ to the point $P_1=(X_1, f(X_1))$, for which $g(X_0)=f(X_1)$ and $X_0<X_1\in\mathcal{U}$, we draw the “horizontal” segment $\delta_1$. From the point $P_1=(X_1, f(X_1))$ to the point $Q_1=(X_1, g(X_1))$ we draw the “vertical” segment $\delta_2$, etc. The points $P_n=(X_n, f(X_n))$ and $Q_n=(X_n, g(X_n))$ are connected by the “vertical” segment $\delta_{2n}$. With the “horizontal” segment $\delta_{2n+1}$, we connect the points $Q_n=(X_n, g(X_n))$ and $P_{n+1}=(X_{n+1}, f(X_{n+1}))$ when $g(X_n)=f(X_{n+1})$ and $X_n<X_{n+1}\in\mathcal{U}$. If there are no infinitely many distinct points $P_n$ and $Q_n$ constructed from at least one point $P_0$, then the assertion is trivial. If all the indicated points are different, then at the point $(X_0, Y_0)$, where $X_0=\lim_{\omega\in\mathfrak{a}}X_n$ and $Y_0=\lim_{\omega\in\mathfrak{a}}f(X_n)=\lim_{\omega\in\mathfrak{a}}g(X_n)$, the functions coincide: $f(X_0)=g(X_0)=Y_0$. For any countable $\mathcal{S}\subset\mathcal{U}$ that does not contain an element $1\mathfrak{g}$, there is always $\mathcal{X}^v\in\mathcal{A}$, $\mathcal{X}^v<1\mathfrak{g}$ greater than each element $\in\mathcal{S}$. When $\mathcal{S}$ is the set of constructed values of the argument, on which the functions coincide, we select this $\mathcal{X}^v$, as the “new $X_0$”, to construct a value of the argument $E$ where the functions also coincide. From here, we construct values of $\mathcal{X}^v$ and $Y^v$, just as $X^0$ and $Y^0$ were built, “an uncountable number of times”, i.e. points of functions coincidence form a set no less than $\mathfrak{t}_1$.

We transform the square $PQRS$ by continuous metamorphosis into the pentagon $PQAB$ so that all the closure points of the square remain the points of the pentagon, except for the point $R$, which turns into a segment $AB$. Let $f$ be the diagonal of the square. After transforming the square into a pentagon, let the line $f$ end at the point $E$ on the segment $AB$, and $E\neq A, E\neq B$. Then, draw a line $h$ so that $h$ ends at the point $A$ and nowhere intersects the line $f$, except the point $P$. We can assume that $h$ corresponds to a strictly increasing continuous function in the square $PQRS$. But in the latter, geometric intuition comes into conflict with logic. “By intuition”, $h$ exists “without sticking to $f$”, but “logically” – no, because, in connection with what has been proved, the line $h$ must “stick to $f$”. In other words, “the line $h$ does not exist in the canonical theory”.

Set aside the “ban”. Let us try to construct a sequence of segments $\delta_n$ this time for $f$ and the “intuitive line” $h$. Then each finite, $n$-th step of the sequence is feasible; together, however, these steps are not feasible – the sequence of segments will lead to a contradiction, if assuming sequence completion: If there is a hypothetical $h$, then each finite set of segments $\delta_n$ exists, but the countable set does not.

A similar state of things holds for the “diagonal of the rectangle” $\mathcal{U}\times\mathcal{B}$, where $\mathcal{B}$ is the continuum of binary sequences of length $\omega_0$. This kind of questions can be asked precisely for the cantor’s non-cofinal ordinals $\omega_0$ and $\omega_1$. 

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In the “world of sets”, where the “countable axiom of choice” is violated, the “incomplete infinity” is also visible. Indeed, for a countable set of sets, one can always make a finite number of choices and continue selection. A countable set of choices is then impossible, since it is equivalent to having a complete selection function.

For a line from Axiom I, would every point sequence directed along the line to arc C be a non-completion?

The “idea of not being able to complete the set” can be expanded, if no more than countable and other ordinals are used instead of natural numbers. The most extreme case gives the most famous “set that cannot be completed” - a series of all ordinals.

Adding or eliminating “numbers” (from a set of real numbers), which are the “limits” of these hypothetical “incomplete sequences,” according to the canonical theory, will not add or exclude any numbers, because “incomplete sequences simply do not exist”. But it is precisely this kind of “additions” and “exceptions” that can be the reason why different powers can be attributed to the continuum.

Is it possible that “next to canonical sets” are “non-sets”, which are “sets” in another natural subject space? Suppose that in steps n, subjects P_n and Q_n are compared, and the sequence P_n is “capable of completion”, and the sequence Q_n is “incapable”. Then we “change the completion in the opposite direction”, attributing the “incompleteness” to the sequence P_n and to the “completion” – the sequence of objects Q_n, and only because the comparison itself cannot be “completed”. “The relativity of the ability to complete” can mean, and that in one world, sets are capable of completion, but in another, as alien to such a world, no.

Ignoring a mathematical habit, we ask one more question: Can a countable set include an uncountable subset? Let β be a subset of the natural series ω, and a elements of the set β are selected from the ω by property ψ. How does it guarantee “counting” β? We try to indicate the only M ∈ β corresponding to the number N for each N ∈ ω according to the schema of replacement. But then, every N-th time it is necessary to compose a formula φ_M(N), which depends on N and M “not according to the canonical rule”, i.e. N is not a free variable of a particular formula Φ, for which “by a given N there is a unique M for which Φ(N, M) is true”. In fact, with each new N, the formula φ_M(N) is compiled, which is different from the previously devised formulas φ_L(M) for L < N, because the element M, in addition to referring to the property ψ, will need to be defined as “the minimum value greater than the indicated numbers in the previously constructed formulas”. Application of the countable axiom of choice seems difficult, because it is not known whether β is countable? The induction scheme is called into question for similar reasons. There probably also exists some simple proof of countability of β. Nevertheless, imaginary sets are interesting as they can be attributed to “counting” by other natural criteria – on other grounds calling their power “countable”.

“Habitual countable sets”, while remaining infinite, cease to be “countable” in relation to “sets of the new world”, acquiring a power of “greater”, “lesser”, or possibly “not comparable with countable”. Then, the “lack of ability to complete the set” finds a simple explanation in the “absence of a one-to-one correspondence between sets”, i.e. the cardinalities of compared sets “similar to countable” turn out to be different.
§4. Transfinite sequence theorem from which the continuum hypothesis is derived

Let $<$ be a partial order relation on the set $M$, $U \subseteq M$ and $V \subseteq M$. $U < V$ means that for every $X \in U$ and $Y \in V$, $X < Y$ is true. Thus, the relation $<$ depends on $\prec$, it is constructed with respect to $\prec$. By a “section” of a set $W \subseteq M$ (for a given order relation) we mean an ordered pair, where elements $U$ and $V$ are such that $U < V$ is true, and for any $P \in W$ is true $P \in U$, either $P \in V$, or $P$ is not compared with every element $E U V$. The sets $U$ and $V$ are called “defining the section”.

Let the segments $A_{\mu}$ be elements of a transfinite sequence, $\mu \in \omega_1$. Suppose that each of the segments $A_{\mu}$ can be identified with an Euclid segment (through continuous and one-to-one mapping), and $A_{\alpha}$ is “parallel” to $A_{\beta}$, whatever $\alpha$ and $\beta$ are. Suppose that if $\alpha \in \beta$, then $A_{\alpha}$ is conditionally “lower” than $A_{\beta}$, and these segments do not intersect in pairs. On each segment $A_{\mu}$, let us introduce a linear order of points in which, for arbitrary points $P$ and $Q$ of the segment $A_{\mu}$, $P \{ Q, Q \} P$, or $P \Rightarrow Q$ is always true (we read: “$P$ to the left of $Q$”, “$Q$ to the right of $P$”, “$P$ and $Q$ are the same”).

We identify points with singleton sets. An element of the set $H$ is every point sequence $p$ defined on the set $\omega_1$ and such that the $\mu$-th element of the sequence $p(\mu)$ is located on the interval $A_{\mu}$. For two sequences $q$ and $p$ taken from $H$, we denote $q \ll p (p \gg q)$ if $q(\mu) \ll p(\mu)$ for all sufficiently large $\mu \in \omega_1$. If $q(\mu) = p(\mu)$ is true for all sufficiently large $\mu \in \omega_1$, then $p$ and $q$ are “equivalent,” and we write $q \equiv p$. When $q \ll p$ or $q \gg p$, $q$ and $p$ are called “comparable”.

For some $G \subseteq H$, we can always consider that the “limit” of each $q \in G$ is at an “infinitely distant point”, and such infinitely distant points form an “infinitely distant continuous line” $C$. Then we can assume $C = H$ for the following reason:

**Theorem V.** There exists a $G \subseteq H$ such that every middle $\in M$ is a limit of the sequence $\in G$.
If $q \ll p$, $q \in G$ and $p \in G$, then for the limits of these sequences $\lim(q) \in M$ and $\lim(p) \in M$,

it turns out that $\lim(q) < \lim(p)$ on $C$.

Let us prove at this stage not Theorem V itself, but only that Theorem V implies the continuum hypothesis. Let a countable set $W$ be finitely divisible and finitely limitless in oneself in respect $\ll$, and $W \subseteq G$. The relation $\ll'$ is constructed with respect to $\ll$. Let $U$ and $V$ define a section $\delta$ in $W$, $U \ll' V$. Replacing the sequences with equivalent ones (if it’s necessary), for all $p \in W$ and $q \in W$ such that $p \ll q$, for all $v \in \omega_1$ we achieve the condition $p(v) \ll q(v)$. Then, for any section $\delta$ due to countable divisibility of $G$, there are (depending on the section $\delta$) $a \in G$ and $b \in G$ and the ordinal $\mu \in \omega_1$ so that for all $v \in \omega_1$, $v \geq \mu$, for every $p \in U$ and $q \in V$ it turns out that $p(v) \ll a(v) \ll b(v) \ll q(v)$. Thus, for whatever $\delta$, there is a “strip” – a union of all intervals with ends at points $a(v)$ and $b(v)$ such that $v \geq \mu$. And let the interval ends $a(v)$ and $b(v)$ are excluded from each $v$-th interval located on $A_v$ in such a strip. Two different sections then correspond to two
different, non-intersecting strips. On each segment $A_p$, we define an everywhere dense countable point set $\Sigma_p \subset A_p$. The union of all sets $\Sigma_p$ is denoted by $\Sigma$. We set the point from $\Sigma$ chosen in the strip corresponding to the section $\delta$ in $W$. Different sections then correspond to different points from the set $\Sigma$. The cardinality of the set of sections is $2^\aleph_0$, the cardinality of $\Sigma$ is equal to the cardinal number $\aleph_1$. Q.E.D.

§5. Logic techniques employed in the proof

Let $\mathcal{F}$ be a fixed ultrafilter on the set $\Omega$, $\mathcal{F}$ does not contain finite sets, and $T$ be the canonical set theory. The latter means that $T$ contains axioms equivalent to the axioms of $\text{ZFC}$, but has the constants $\omega$ and $\omega_1$, and other ordinary constants. Using the theory $T$, we define theory $S$ by the theory of ultraproducts, which has been given a form suitable for further proofs. The subjects of the theory $S$, called “$S$-subjects”, are arbitrary functions defined on the set $\Omega$, which values are subjects of the theory $T$ called “$T$-subjects”. The subjects of the theory $T$ are ordinary sets, also called “$T$-sets”. Then, $S$-subjects - functions with values of $T$-sets - will be called “$S$-sets”. Variables and constants which values are $T$-subjects will be called “$T$-variables” and “$T$-constants”. Variables and constants defined among $S$-subjects will be called “$S$-variables” and “$S$-constants”. Let’s write that “$V \in T$”, “$\Phi \in T$”, “$\Phi \in S$”, etc. for subjects, formulas and other elements of theories.

Let $\alpha$ be an atomic formula $\in T$ depending on $N \in \omega$ free variables $t_1, \ldots, t_N$ taking values of $T$-subjects, and on $M \in \omega$ $T$-constants $c_1, \ldots, c_M$. I.e., the formula $\alpha$ has the form $\alpha(t_1, \ldots, t_N; c_1, \ldots, c_M)$. We interpret a formula $\alpha^\delta$ as the atomic formula $\in S$, which corresponds “by signs” to the formula $\alpha$ and has the form $\alpha^\delta(t_1^\delta, \ldots, t_N^\delta; c_1^\delta, \ldots, c_M^\delta)$, if $\alpha^\delta$ is actually

$$(\exists \sigma \in \mathcal{F}) (\forall \nu \in \sigma) \alpha(t_1^\delta(\nu), \ldots, t_N^\delta(\nu); c_1^\delta(\nu), \ldots, c_M^\delta(\nu)),$$

where the variables $t_1^\delta, \ldots, t_N^\delta$ take the values of $S$-subjects, and $c_m^\delta(\nu) = c_m$ is true for each $m$-th constant $c_m$, whatever $\nu \in \Omega$. That is, the symbols $t_j^\delta$ and $c_j^\delta$ denote functions, $t_j^\delta(\nu)$ and $c_j^\delta(\nu)$ are the values of the functions for $\nu \in \Omega$. Let the variables $t_j^\delta$ and $t_j$ and the constants $c_j^\delta$ and $c_j$ also have “sign correspondence” with each other.

Let a formula $\varphi^S$ belong to the theory $S$ if there is a formula $\varphi \in T$ such that $\varphi^S$ is obtained: by replacing the atomic subformulas of the formula $\varphi$ with the corresponding atomic formulas $\in S$; and when replacing the symbols of $T$-variables in the quantifiers with the corresponding symbols of $S$-variables so that the quantifiers in the formula $\varphi^S$ are bounded by $S$-sets. The formulas $\varphi^S$ and $\varphi$ also “correspond to each other by signs”. If $\varphi$ is an axiom of the theory $T$, then $\varphi^S$ is an axiom of the theory $S$.

We also often specify what subjects we mean by calling them “$S$-numbers”, “$S$-lines”, “$T$-functions”, etc. Suppose that for a subject $X \in T$ the unary predicate $P \in T$ is true, and $P(X)$ means “$X$ is a number”. Then $X$ is called the “$T$-number”. And then, any subject $Y \in S$, for which the
predicate $\mathcal{P}(Y)$ is true, is called an “$S$-number”. In the same way we name subjects according to other characteristics, when “$X$ is an ordinal”, etc.

From the theory of ultraproducts is known, in fact, found by Los [1] (trivially following from the theory of Los):

**Theorem VI.** A formula $\varphi \in \mathcal{T}$ is true if and only if the formula $\varphi^S$ is true. A formula $\varphi^S$ is provable by means of $S$ if and only if $\varphi$ is provable by means of $T$. If $T$ is a canonical set theory, then by means of the theory $T$ it is provable that $\varphi$ is true if and only if $\varphi^S$ is true.

Let us prove that the following is true

**Theorem VII.** When $T$ is a canonical set theory, a formula $\varphi \in \mathcal{T}$ is provable by means of $T$, if a formula $\varphi^S$ is provable by means of $T$.

Evidence. Suppose the formula $\mathcal{T}\varphi$ is true. Then, by Theorem VI, $\mathcal{T}\varphi^S$ is true. Under hypothesis, we prove $\varphi^S$ by means of $T$, and the assumption leads to a contradiction. Therefore, $\varphi$ is true. This is a proof of the formula $\varphi$ by $T$.

We will use Theorem VII as a logical technique, proving $S$-theorems by means of the canonical sets theory $T$ and extracting from $T$ the truth of the corresponding $T$-theorems. The theory $T$ essentially interprets itself as a theory $S$. By Theorem VII, the theory $T$—as a metatheory—does not prove more $S$-facts than $S$, in the proofs of $S$-formulas.

Note that whenever the existence of some $S$-sets is proved by means of the theory $T$, one can apply axioms or theorems of the theory $S$ to the found $S$-sets, as if they were ordinary axioms and theorems applied to ordinary sets. We consider that the speaker of the theory $S$ does not “know” what he is talking about $S$-sets. He believes that he speaks of ordinary Cantor, Zermel canonical sets, and applies the axioms and conclusions of the usual theory to them, i.e. draws conclusions without decoding atomic $S$-formulas. And only those “who know the theory $T$ know” that “who speaks the language $S$ speaks about some functions”.

The usual membership relation “$\in$” is called “$T$-membership”, denoting “$\in^T$”. And “$T$-membership” corresponds by means of signs to the “$S$-affiliation”. “$S$-affiliation” is a relation denoted by “$\in^S$”, when the following is true:

$$q \in^S b \iff \exists \sigma \in \mathcal{B} \left( \forall v \in \sigma \right) q(v) \in^T b(v).$$

The equality relation “$=$” is called “$T$-equivalence”, denoted as “$=^T$”. The relation “$=^S$”, corresponding to the relation “$=^T$”, is called “$S$-equivalence”. For the relation “$=^S$”, the statements $\in^S$ corresponding to the axioms of equality $\in^T$ are true. Inequality relations are denoted by $>^S$, $<^S$, $>^T$ and $<^T$, respectively, where $>^T$ and $<^T$ denote the relations $>$ and $<$. We use the a similar notation for the relations $\cup^S$, $\cap^S$, $\cup^T$ and $\cap^T$, the operators $\cap^S$, $\cup^S$, $\cap^S$, etc.
§6. Features of the transformation of the theory $T$ into the theory $S$

It is useful to see how objects and statements of the theory $T$ are transformed into objects and statements of the theory $S$ in some cases.

The set $\omega$ is transformed into a function $\omega^S$, each value of which, for each $v \in \Omega$, coincides with $\omega = \omega^S(v)$. A "natural number of the theory $S$" will be every function $\nu$ defined everywhere on $\Omega$ with values in $\omega^S(v)$, or $S$-equivalent to such a function. Since $\Omega \in \mathcal{F}$, for the function $\nu$ defined everywhere in $\Omega$, we obtain $(\forall v \in \Omega) \nu(v) \in \omega^S(v)$, i.e. $\nu \in \omega^S\omega^S$.

What does the statement of theory $S$ that "the set $U$ is countable"? In relations $\mathcal{E}T$, it means that for some $\alpha \in \mathcal{F}$, for all those $v$ that belong to $\alpha$, there is a bijection mapping $U(v)$ onto $\omega^S(v)$, i.e. on $\omega$. In other words, "$U$ is countable" according to theory $S$, if "$U(v)$ is countable for every $v \in \alpha$" according to theory $T$.

It is necessary not to confuse the $T$-set of $S$-numbers $\epsilon^S\omega^S$ and the $S$-set of $\omega^S$. Indeed, when $\Omega$ is infinite, the uncountable $T$-set $\rho$ of these "natural $S$-numbers" $\nu$ for which $\nu \in \epsilon\rho \iff \nu \in \epsilon^S\omega^S$ does not coincide with the function $\omega^S$.

An $S$-finite $S$-set $U$ (up to an $S$-equivalent function) is a function, where value $U(v)$ is a finite set for each $v \in \Omega$. If a number of elements $U(U(v))$ grows "fast enough" with a change in $v$, the theory $T$ deduces that the $T$-set of elements $\epsilon^S U$ has a continuum power.

The statement $\epsilon S$ that "the set $U$ has the first uncountable cardinality" in the decoding of the theory $T$ means that there is $\alpha \in \mathcal{F}$ so that for all $v \in \alpha$, there is a bijection between the sets $U(v)$ and $\omega_1 = \omega^S_1(v)$. As "$S$-ordinal" $\epsilon^S\omega^S_1$ we define function with values in $\omega_1$ defined on $\Omega$. If the $S$-formula is provable with respect to $\omega^S_1$ by means of theory $T$, and in particular by means of theory $S$, then by Theorem VII the sign-corresponding $T$-formula is provable in $T$ for $\omega_1$.

Let the set of real numbers $\mathcal{R}$ be a constant of the theory $T$. Then, "a real number of the theory $S$", i.e. an "$S$-real number" is a function $r$ with values $r(v) \in \mathcal{R}$ (with accuracy to $S$-equivalent function), defined on argument $v \in \Omega$. When the set $\Omega$ is infinite, the "$S$-real numbers" make up the "hyper continuum" $\mathcal{R}$ according to the theory $T$, that is, $\mathcal{R}$ is the $T$-set of all $S$-real numbers $\epsilon^T\mathcal{R}$. But $\mathcal{R}$ is not the subject of theory $S$. The function $\mathcal{R}\omega^S$, to which the $S$-real numbers $S$-belong does not match with $\mathcal{R}$. $\mathcal{R}$ is countably separable set and countably limitless in respect $<^S$, and all elements of $\epsilon^T \mathcal{R}$ are comparable.

Let $r$ be the "real number of the theory $S$". We construct an $S$-countable $S$-sequence of $S$-real numbers that "converges" to $r$ in the interpretation of $S$. Indeed, let $S$-numbers $r_M$ be numbered by ordinary numbers $M \in \omega$, according to the theory $T$, and for each $v \in \Omega$, $r_M(v) > r_{M+1}(v)$, and $r(v) = \lim_{M \to \omega} r_M(v)$. The $S$-real number $q$ enters the $S$-sequence under the "number" $N \in \mathcal{S}\omega^S$ if $q(v) = r_N(v)$. For each $S$-real $t >^S r$, for all sufficiently large $N \in \mathcal{S}\omega^S$, when $q$ is "numbered" by such $N$, it turns out $t >^S q >^S r$. 

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§7. Proof of the continuum hypothesis

Let the theory $S$ be such that $\Omega = \omega$, and the ultrafilter $\mathcal{F}$ does not contain finite sets. The subjects of the theory $S$ are sequences, functions defined on $\omega$. Let $q \in \mathcal{H}$. Then the line $q$, as a constant, is transformed into a sequence $q^S$ such that when changing argument values $n \in \omega$ the value $q^S(n)$ does not change, $q^S(n) = q$. For a variable line $b$, it turns out that $b \in ^S \mathcal{H}^S$, if for each $n \in \omega$ there is a line $q_n \in \mathcal{H}$, depending on $n$, for which $q_n = b(n)$. Or, $b \in ^S \mathcal{H}^S$ if $b$ is $S$-equivalent to one of the $S$-lines $\in ^S \mathcal{H}^S$ defined everywhere on $\omega$.

Let $\mathcal{G}$ be the set of lines $\subset \mathcal{H}$ from Theorem IV. Then $S$-lines $\in ^S \mathcal{G}^S$ are each-to-each comparable. Indeed, an $S$-element of an $S$-set $\mathcal{G}^S$ is each sequence of lines $b(n)$ such that $b(n) \in \mathcal{G}$ is true for each $n \in \omega$, or such an element is sequence $S$-equivalent to the specified sequence. Let $q \in ^S \mathcal{G}^S$ and $b \in ^S \mathcal{G}^S$. Since the values from $\mathcal{G}$ are each-to-each comparable, the disjunction $q(n) \ll b(n) \lor q(n) \gg b(n)$ is true for each $n \in \omega$. Hence, for one of the terms of the disjunction, there exists $\beta \in \mathcal{F}$ such that for each element $\in \beta$ this term is true. Suppose, for example, that for each $n \in \beta$ there exists a real $\rho(n) < 1$ depending on $n \in \omega$ such that for all positive real numbers $r > \rho(n)$, $r \subset \subset 1^S$, and for some $\alpha \in \mathcal{F}$ depending on $r$ and $\rho$, $q(n) \cap C_r(n) < b(n) \cap C_r(n)$, i.e. we can assume $\alpha \cap \gamma \cap C_r \subset \subset \beta \cap \gamma \cap C_r$; and similarly in other cases.

A theorem, which corresponds by signs to Theorem IV, is true for the sets $\mathcal{G}^S$ and $\mathcal{M}^S$. Let the lines $q_i \in \mathcal{G}$ be numbered by natural numbers $l$ (which is known only in the metatheory $T$). Then, if the $S$-line $b$ is such that $b(n) = q_i(l(n))$, then $S$-line $b$ $S$-belongs to the $S$-set $\mathcal{G}^S$. By Theorem IV, there are middles $X_i \in \mathcal{M}$ such that $F(q_i) = X_i$. Then the $S$-line $b$ is $S$-corresponding to the $S$-middile $Y \in ^S \mathcal{M}^S$ such that $Y(n) = X_i(l(n))$.

Using the metatheory $T$, we choose a $T$-sequence of $S$-real numbers, where length of $T$-sequence is the cardinal $\eta$, and its elements are $r_\alpha \in \eta$, so that:

For every $\alpha \in \eta$ and $\beta \in \eta$ such that $\alpha \neq \beta$, for every $n \in \omega$ and $m \in \omega$, $r_\alpha(n) \neq r_\beta(m)$ is true.

If $\alpha \in \beta \in \eta$, then $r_\alpha \subset \subset r_\beta \subset \subset 1^S$. 

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Whatever the $S$-real number $X <^S 1^S$, there is $v \in \eta$ such that $X <^S r_v$.

The sequence of $S$-numbers $r_v$ can always be chosen for some $\eta$ having cardinality less than the continuum. Indeed, one can always find a strictly increasing (with respect to $<^S$) sequence of $S$-numbers that its elements are $S$-numbers $X_v$, $y$ is a sequence length and cardinal with the cardinality less than or equal to the cardinality $\mathfrak{R}$, and there is no $S$-number $X$ such that for each $X_v$ it turns out that $X_v <^S X$. Suppose that for some $\alpha \in \mathfrak{y}$ a set of $S$-numbers is already selected, the selected numbers are marked $r_v$, $X_v <^S r_v$ and $\nu < \alpha$. By the definition of $\mathfrak{y}$, there exists an $S$-number $X_\beta >^S X_\alpha$ such that for each of the chosen $r_v$ it turns out $r_v <^S X_\beta$. Then, for each $n \in \omega$, the value $r_\alpha(n)$ is chosen on the open interval between $X_\beta(n)$ and 1 so that it does not coincide with any $r_\nu(m)$ value chosen earlier. A suitable value of $r_\alpha(n)$ is found, because in the previous steps, the number of values less than the continuum power was selected from the interval. Then we define $\eta = \gamma$. The difference in the values of $r_v$ makes further arguments more obvious, although one can dispense with such differences.

Only one of the cases can be fulfilled: 1) $\eta = \omega_1$, or 2) $\eta \geq \omega_2$. We first consider the first case (Only the assumption $\eta = \omega_1$ does not yet directly exclude that the power of the continuum can be greater than the first uncountable power).

Let the $S$-arc $C^\nu$ be such that, for each $n \in \omega$, $C^\nu(n)$ takes the value of the arc $C_{\rho(v,n)}(\nu,n)$ $= r_v(n)$. The arc $C^\nu$ is interpreted as segment $J^\nu$ by the metatheory $T$, by means of the mapping (image) $J$ so that:

For any $v \in \eta$, the segment $J^\nu = J[C^\nu(n)]$ does not change from the change $n \in \omega$.

If $\alpha \in \eta$ and $\beta \in \eta$, and $\alpha \neq \beta$, then the sets $J[C^\alpha(n)]$ and $J[C^\beta(n)]$ do not intersect.

If $\alpha \in \beta \in \eta$, then the segment $J[C^\beta(n)]$ is located “above” the segment $J[C^\alpha(n)]$, and these segments are “parallel”. If $P$ and $Q$ are points of the arc $C^\nu(n)$, and $P < Q$ on $C^\nu(n)$, then $J[P] \cap J[Q]$ on the segment $J[C^\nu(n)]$.

Let an $S$-element of an $S$-set $\Theta$ (not to be confused with the set from the example of §6) is every variable segment $K^\mu$ whose values for each $n \in \omega$ are $K^\mu(n) = J[\nu(n)] = J[C^\nu(n)]$, where $\mu$ is an arbitrary function defined everywhere on $\Omega = \omega$ with values $\in \eta = \omega_1$ (in the first case under consideration). I.e. $\mu$ - $S$-ordinal, which is the “number” for function-segment $K^\mu$. In fact, the segments $K^\mu$ are the segments $J^\mu$ “numbered” by $S$-ordinals.

For each $b \in S^S$, we define $S$-set $L_b$ such that for each $n \in \omega$, $L_b(n) = b(n) \cap (U_v C^v(n))$, where $v$ runs through all ordinals $\in \eta = \omega_1$. Thus, for some $S$-set $M_b$ we get $M_b(n) = J[L_b(n)]$. $S$-sets $M_b$ are $S$-elements of some $S$-set $M_\Theta$. Note that the set $M_b(n)$ can change even if the line $b(n)$
does not change, when change $n \in \omega$. From this definition we find that there exists an $S$-bijection between $M_b$ and $G$, when the $S$-line $b \in G$ $S$-corresponds to the $S$-set $M_b$, $f$ $S$-maps the $S$-set $M_b$ to the same $S$-middle $M^S$, which $S$-corresponds $S$-line $b$ by $F_S$.

Let $b \ll p$, $b \in G$, $p \in G$. Then for all sufficiently large $T$-ordinals $\xi \in \eta = \omega_1$, by construction, it turns out that $J|b(n)\cap C(n)| \prec J|p(n)\cap C(n)|$. When the ordinal $\xi$ is substituted for $\tau(n)$, and $\tau$ is an arbitrary $S$-ordinal, the last inequality may not hold. But there is an $S$-uncountable $S$-set $\Pi bp \subseteq S \omega_1$ for which:

There exists an $S$-ordinal $\tau \in S \Pi bp, \tau > S \tau$, there exists $\sigma \in F$ (depending on $\xi$) such that $J|b(n)\cap C(n)| \prec J|p(n)\cap C(n)|$, whatever $n \in \sigma$ (This $S$-property is interpreted by an “$S$-theoretician” who thinks that he describes “ordinary sets” by the theory $S$ in the following phrase: “For all sufficiently large ordinals $\xi$ belonging to $\Pi bp$, the point of the set $M_b$ is located to the left of the point of the set $M_p$ on the segment $K_2^\Psi$”).

For each $S$-ordinal $\xi \in S \omega_1$, which is obtained from the $T$-ordinal $\xi \in \omega_1$, so that for any $n \in \omega$, the formula $\xi(n) = \xi$ is true, it turns out that $\xi \in S \Pi bp$. (This is a $T$-property that says that all $S$-ordinals that do not change $T$-values when changing the number $n \in \omega$ are $S$-belong to the $S$-set $\Pi bp$. Only a “$T$-theoretician” knows about this property).

Indeed, for $b$ and $p$, whatever the $S$-ordinal $y$ is, there exists an $S$-ordinal $\xi > S y$ that does not change values when $n \in \omega$ changes so that the required $S$-property holds for $\xi$ (i.e. $J|b(n)\cap C(n)| \prec J|p(n)\cap C(n)|$, for any $n \in \omega$ belonging to some set $\sigma \in F$). We call the required property “feature”. For an “$S$-theoretician”, certainly, the $T$-property of the $S$-ordinal $\xi$ (do not change $T$-values) will not be mentioned. I.e. for the theory $S$, the metatheory $T$ establishes only the fact that the $S$-ordinal $\xi$ is $S$-superior to the $S$-ordinal $y$, and “the point of the set $M_b$ is located to the left of the point of the set $M_p$ when points of the sets $M_b$ and $M_p$ are located on the segment $K^\Psi_2$”. From this we find the $S$-unique for $b$ and $p$ (up to the $S$-equivalent $S$-set), the $S$-uncountable $S$-set $\Psi bp \subseteq S \omega_1$, $S$-set $\Psi bp$ obtained from $\omega_1$ $S$-the schema of specification ($S$-axiom $S$-schema of separation). Then each $S$-ordinal $\in S \Psi bp$ has this feature. Each sufficiently large $S$-ordinal that does not change the values, when the argument $n$ changes, has a marked feature, therefore, such an $S$-ordinal is $S$-owned by $\Psi bp$. We could declare $\Psi bp = \Pi bp$ if all $S$-ordinals that do not change $T$-values when changing $n \in \omega$ are $S$-belong to $\Psi bp$. There may be a $T$-countable $T$-set of such $S$-ordinals, however, that are not included in $\Psi bp$. Therefore, we define $\Pi bp \in S \Psi bp \subseteq S \omega_1$, where $\Psi bp$ is the $S$-set of all $S$-ordinals $\mu$, for which there is at least one $S$-ordinal $v \in S \Psi bp$ such that $\mu > S v$.

Now note that there is an $S$-set $\Psi$, that $S$-contains only $S$-sets $\Psi bp$, i.e. defined by the above method for $S$-lines $b \in G$, $p \in G$ such that $b \ll S p$. Indeed, denote “by $S$-elements of the
first kind" $S$-sets $\mathcal{H}_{bp}$, for which the $S$-lines $b$ and $p$ do not change their values when $n \in \omega$ is changed, that is, for each $n \in \omega$, $b(n) = b(n+1)$ and $p(n) = p(n+1)$. The $T$-property of $S$-elements “to be elements of the first kind” is known only to a $T$-theoretician. Then, $\mathcal{H}_{bp}(n)$ is the $T$-set of those $T$-ordinals $\xi \in \omega_1$, for which a point from the set $\mathcal{M}_b(n)$ is located to the left of the point from the set $\mathcal{M}_p(n)$ on the segment $F(n)$. If at least one of the $S$-lines $b$ and $p$ changes its values when changing $n \in \omega$, then for whatever $l \in \omega$, $\mathcal{H}_{bp}(l) = \mathcal{H}_{dp}(l)$, where $q$ and $d$ is an $S$-elements the first kind for some $q$ and $d$ depending on $l$. Let (already for fixed $g$ and $h$), by definition, for each $n \in \omega$, property $\mathcal{H}_{gh}(n) \in V(n)$ is true (and hence $\mathcal{H}_{gh} \in \sigma^M$ is true) – for any $g \in \mathcal{G}$ and $h \in \mathcal{F}$, which do not change values when values of $n \in \omega$ change. Then, for any $b \in \mathcal{G}$ and $p \in \mathcal{F}$, when $b \ll^S p$, the condition $\mathcal{H}_{bp} \in \sigma^M$ is true.

Let an element $\Pi_{bp}$ belong to an $S$-set $E$ if $\mathcal{H}_{bp} \in \sigma^M$. Then, every a $S$-ordinal $\xi$ which does not change when changing $n \in \omega$, this $S$-ordinal will be $S$-belonging to every $\Pi \in \sigma^E$. Therefore, there is an $S$-uncountable $S$-intersection of all $S$-sets $S$-belonging to $E$. We denote the last $S$-intersection by $Y$.

Let $\Pi \subseteq \omega_1^S$, $M \subseteq \mathcal{M}$, and by $\mathcal{M} \mu_{\Pi}$ we denote the $S$-intersection of the $S$-set $M$ with the $S$-union of all such $S$-segments $K^\mu \in \mathcal{G}$ for which $\mu \in \mathcal{P} \Pi$ is true.

We argue as an $S$-theoretician, i.e. without decoding the atomic formulas $E$, and using the $S$-set $Y \subseteq \omega_1^S$ proved in $T$. Then we find that for any $b$ and $p$, when $b \ll^S p$, $b \in \mathcal{G}$, $p \in \mathcal{F}$, for all sufficiently large $\xi \in \mathcal{Y}$ (depending on $b$ and $p$) “the point of the set $M_b \{ \xi \}$ is located on the segment $K_\xi$ to the left of the point from the set $M_p \{ \xi \}$. Thus, for each sufficiently large $\xi \in \mathcal{Y}$ there is $\sigma \in \mathcal{F}$ so that for all $n \in \sigma$ the point $M_b \{ \xi \}(n) \setminus K_\xi(n)$ lies to the left of $M_p \{ \xi \}(n) \setminus K_\xi(n)$.

An $S$-theoretician believes that he speaks of the “usual point sets $M_b \{ \xi \}$”. Since $\mathcal{Y}$ is an “uncountable set of usual ordinals” (as an $S$-theoretician expresses), then, by means of an “ordinary mapping”, let the $S$-theoretician identify the “set” of $\mathcal{Y}$ with the $\omega_1$, the “segments” $K_\xi$ $S$-theoretician identify with the segments $\Delta_\xi$, and the “sets” $M_b \{ \xi \}: S$-theoretician identify with the sequences $E \mathcal{G}$ (it is clear that “actually” we are talking about $S$-sets and $S$-relations). Strictly speaking, in the latter case, an $S$-theoretician translates the “point set” $M_b \{ \xi \}$ into the “points transfinite sequence” $M_b'$; assigning the “number” $\xi$ to each “point” of $M_b \{ \xi \}$. I.e., a number of that “segment” $\Delta_\xi$ on which the “point” is located (considering – in interpretation of an $S$-theoretician – “the renaming of the ordinals of the set $\mathcal{Y}$ by the names of the ordinals from $\omega_1$”). The set that an $S$-theoretician identifies with $\mathcal{G}$ “in reality” is some S-set $\mathcal{G} \mu$. Each S-element $B$, $S$-belonging to the S-set $\mathcal{G} \mu$, as an S-transfinite S-sequence, is obtained from some $A \in \mathcal{M} \mu$ so that $B = A'$; i.e. by assigning S-numbers to S-points, which $S$-belong to $A \mathcal{G} \mathcal{Y}$. If $b^S$ is an $S$-relation corresponding to the relation $\ll$ in $\mathcal{S}$, then $M_b' \ll^S M_p'$ when $b \ll^S p$. Using the S-bijection between $\mathcal{M} \mu$ and $\mathcal{M}^S$, we find the S-isomorphism between $\mathcal{G} \mu$ and $\mathcal{M}^S$. From the S-theorem, corresponding to Theorem $V$, we derive the validity of the Cantor hypothesis.
Suppose $\eta = \omega_2$. Then all reasoning is repeated almost verbatim. The segments $\Delta_\mu$ are then elements of a transfinite sequence of length $\omega_2$. Strictly speaking, this is an $S$-sequence, which an $S$-theoretician considers “ordinary.” Follow his phrases though. Each segment $\Delta_\mu$ is again Euclidean. Let each interval $b \cap \Delta_\mu$ be an element of the set of intervals $\Xi_\mu$, where $b = b(\delta)$ is the strip corresponding to the section $\delta$ defined in the derivation of the continuum hypothesis from Theorem $V$, and $\delta \in \mathcal{W}$, where $\mathcal{W}$ is the predefined set of sections of cardinality $\aleph_1$. Then there is a sufficiently large $\tau \in \omega_2$ such that, whatever $\delta \in \mathcal{W}$, strip $b(\delta)$ intersects the segment $\Delta_\tau$. Because the strips do not intersect, the segment $\Delta_\tau$ includes each interval from such an uncountable set $\Xi_\tau$ that every two intervals $\in \Xi_\tau$ do not intersect. The latter is excluded by the canonical theory. Thus, we find that the case $\eta = \omega_2$ is not satisfied. We similarly exclude the case $\eta > \omega_2$. Q.E.D.

Pay attention to those “non-cantor sets” that were given in §2 and §3. There is no doubt about their existence. Hence, and in connection with the found evidence, there is no doubt that the “state of things in the set theory” is not reducible to a “logical independence of statements”. A proof of a mathematical fact, apparently, does not always exclude an “opposite idea”, which is true in a different sense. And it is not a fact that “incompatible worlds of sets” cannot be combined.

References