The Time Evolution Operator

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Abstract

In this article we have devised a simpler alternative solution to the operator equation for the usual time evolution operator. This is based on an interesting commutator relation which has been derived valid subject to a weak condition that two specific operators should not be simultaneously non invertible.

Introduction

The article considers an interesting commutator relation valid subject to a weak condition that two specific operators should not be simultaneously non invertible. Applying the stated relation we have devised a simpler alternative solution to the operator equation for the usual time evolution operator

Time evolution operator

Let us consider the operator function\(^{[1]}\)

\[ \hat{U}(t, t_0) = e^{i\hat{H}_0 (t-t_0)} e^{-i\hat{H}(t-t_0)} \quad (1.1) \]

We would like to transform (1.1) to our advantage is done in standard treatment to obtain a form conducive to the construction of Feynman’s diagrams.

From (1.1) we formulate the differential equation\(^{[2]}\) and solve it subject to \(\hat{U}(t_0, t_0) = 1\):

\[ i \frac{\partial \hat{U}(t, t_0)}{\partial t} = \hat{H}(t) \hat{U}(t, t_0) \quad (1.2) \]

Standard solution to (1.2) subject to \(\hat{U}(t_0, t_0) = 1\) is given by

\[ \hat{U}(t, t_0) = T \left[ \exp \left\{ -i \int_{t_0}^{t} dt' \hat{H}(t') \right\} \right] \quad (3) \]

Where by definition\(^{[3]}\),

\[ T \left[ \exp \left\{ -i \hat{H}(t) \right\} \right] = \exp \left\{ -i \int_{t_0}^{t} dt' \hat{H}(t') \right\} \]

\[ T \left[ \exp \left\{ i \hat{H}(t) \right\} \right] = \exp \left\{ i \int_{t_0}^{t} dt' \hat{H}(t') \right\} \]
\[ T \left[ \exp \left\{ -i \int_{t_0}^{t} dt' \hat{H}(t') \right\} \right] \]

\[ = I + \frac{1}{1!} \int_{t_0}^{t} dt_1 \hat{H}(t_1) + \frac{1}{2!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 T \left[ \hat{H}(t_1)\hat{H}(t_2) \right] \]

\[ + \frac{1}{3!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_{t_0}^{t} dt_3 [\hat{H}(t_1)\hat{H}(t_2)\hat{H}(t_3)] + \cdots \] (4)

The right side of (4) is conducive to the construction of Feynman’s Diagrams

**The Trial Solution and Subsequent Considerations**

We consider the following trial solution:

\[ \hat{U}(t, t_0) = \exp \left\{ -i \int_{t_0}^{t} dt' \hat{H}(t') \right\} \] (5)

Solution given by (5) satisfies: \( \hat{U}(t_0, t_0) = 1 \)

Partial differentiating the above with respect to ‘t’ we have,

\[ \frac{\partial \hat{U}(t, t_0)}{\partial t} = -i \exp \left\{ -i \int_{t_0}^{t} dt' \hat{H}(t') \right\} \hat{H}(t) \] (6)

We shall now prove that

\[ \exp \left\{ -i \int_{t_0}^{t} dt' \hat{H}(t') \right\} \hat{H}(t) = \hat{H}(t) \exp \left\{ -i \int_{t_0}^{t} dt' \hat{H}(t') \right\} \] (7)

that is

\[ \left[ \exp \left\{ -i \int_{t_0}^{t} dt' \hat{H}(t') \right\}, \hat{H}(t) \right] = 0 \] (8)

Proof of (8): We may first consider the relation

\[ \hat{A} \exp(-i\hat{A}) = \exp(-i\hat{A})\hat{A} \] (9)

which may be proved by direct expansion. Indeed

Left side of (7):

\[ \hat{A} \exp(-i\hat{A}) = \hat{A} \left[ 1 - \frac{i\hat{A}}{1!} + \frac{(i\hat{A})^2}{2!} - \frac{(i\hat{A})^3}{3!} + \cdots \right] \]
\[ \begin{aligned} A = & \left[ 1 - \frac{iA}{1!} + \frac{(iA)^2}{2!} - \frac{(iA)^3}{3!} + \cdots \right] \hat{A} \\
= & \exp(-i\hat{A}) \hat{A} \end{aligned} \]

Let

\[ \hat{A} = \int_{t_0}^t H(t')dt' \quad \text{(10)} \]

and

\[ \hat{X} = \hat{A}\exp(-i\hat{A}) = \left( \int_{t_0}^t H(t')dt' \right) \exp \left( -i \int_{t_0}^t H(t')dt' \right) \text{(11.1)} \]

By applying (9) we have

\[ \hat{X} = \exp(-i\hat{A}) \hat{A} = \exp \left( -i \int_{t_0}^t H(t')dt' \right) \left( \int_{t_0}^t H(t')dt' \right) \text{(11.2)} \]

Differentiating (11.1) with respect to time we have

\[ \frac{\partial \hat{X}}{\partial t} = \left( \int_{t_0}^t H(t')dt' \right) H(t) \exp \left( -i \int_{t_0}^t H(t')dt' \right) - i \left( \int_{t_0}^t H(t')dt' \right) \exp \left( -i \int_{t_0}^t H(t')dt' \right) H(t) \text{(12.1)} \]

Differentiating (11.2) with respect to time we have

\[ \frac{\partial \hat{X}}{\partial t} = -i \left( \int_{t_0}^t H(t')dt' \right) \exp \left( -i \int_{t_0}^t H(t')dt' \right) H(t) \]
\[ + \left( \int_{t_0}^t H(t')dt' \right) H(t) \exp \left( -i \int_{t_0}^t H(t')dt' \right) \text{(12.2)} \]

Since the left sides of (12.1) and (12.2) are identical the right sides will also be identical. This will hold if equation (8)[equivalently (7)] holds that is if we have \[ \left[ \exp \left( -i \int_{t_0}^t dt' H(t') \right), H(t) \right] = 0 \]. A relation like

\[ \left[ \exp \left( -i \int_{t_0}^t dt' \tilde{H}(t') \right), \tilde{H}(t) \right] = b(t) \neq 0 \]

will upset the expected identicalness of (10.1) and (10.2)[we may consider different forms of \( \tilde{H}(t) \)].

\[ \left[ \exp \left( -i \int_{t_0}^t dt' \tilde{H}(t') \right), \tilde{H}(t) \right] = b(t) \]

\[ \Rightarrow \exp \left( -i \int_{t_0}^t dt' \tilde{H}(t') \right) \tilde{H}(t) = b(t) + \tilde{H}(t) \exp \left( -i \int_{t_0}^t dt' \tilde{H}(t') \right) \text{(13)} \]

Using (13) with (12.1) we have,
\[
\frac{\partial \hat{x}}{\partial t} = \left( \int_{t_0}^{t} H(t')dt' \right) H(t) \exp \left( -i \int_{t_0}^{t} H(t')dt' \right) - i \left( \int_{t_0}^{t} H(t')dt' \right) \exp \left( -i \int_{t_0}^{t} H(t')dt' \right) H(t)
\]

\[
\frac{\partial \hat{x}}{\partial t} = \left( \int_{t_0}^{t} H(t')dt' \right) H(t) \exp \left( -i \int_{t_0}^{t} H(t')dt' \right) - i \left( \int_{t_0}^{t} H(t')dt' \right) \exp \left( -i \int_{t_0}^{t} H(t')dt' \right) \left[ b(t) + \hat{H}(t) \exp \left( -i \int_{t_0}^{t} dt' \hat{H}(t') \right) \right]
\]

Equating the right sides of (12.2) and (14) we obtain

\[
\left( \int_{t_0}^{t} H(t')dt' \right) H(t) \exp \left( -i \int_{t_0}^{t} H(t')dt' \right) - i \left( \int_{t_0}^{t} H(t')dt' \right) \exp \left( -i \int_{t_0}^{t} H(t')dt' \right) H(t)
\]

\[
= \left( \int_{t_0}^{t} H(t')dt' \right) H(t) \exp \left( -i \int_{t_0}^{t} H(t')dt' \right) - i \left( \int_{t_0}^{t} H(t')dt' \right) b(t)
\]

\[
- i \left( \int_{t_0}^{t} H(t')dt' \right) \hat{H}(t) \exp \left( -i \int_{t_0}^{t} dt' \hat{H}(t') \right)
\]

We have the operator equation

\[
\left( \int_{t_0}^{t} H(t')dt' \right) b(t) = 0 \quad (15)
\]

If the operator \( b(t) \) has an inverse then

\[
\left( \int_{t_0}^{t} H(t')dt' \right) b(t) [b(t)]^{-1} = 0
\]

\[
\int_{t_0}^{t} H(t')dt' = 0 \quad (16)
\]

Equation (16) cannot be entertained: we will not have any Feynman diagram as per conventional method

If the operator \( \int_{t_0}^{t} H(t')dt' \) has an inverse then

\[
\left( \int_{t_0}^{t} H(t')dt' \right)^{-1} \left( \int_{t_0}^{t} H(t')dt' \right) b(t) = 0 \Rightarrow b(t) = 0 \quad (17)
\]

If \( \int_{t_0}^{t} H(t')dt' \) and \( b(t) \) are numbers then any one will be zero \( b=0 \) would be appropriate.

\( b(t) = 0 \) seems most plausible.
[If both the operators are expressible in matrix form it might happen both are non invertible at the same time??]

Unless both A and b are non invertible we have as follows

From equations (6) and (7) we obtain:

$$\frac{\partial \hat{U}(t, t_0)}{\partial t} = -i \hat{H}(t) \exp \left\{ -i \int_{t_0}^{t} dt' \hat{H}(t') \right\}$$ (18)

Using (5) we have

$$\frac{\partial \hat{U}(t, t_0)}{\partial t} = -i H(t) \hat{U}(t, t_0)$$

$$\Rightarrow i \frac{\partial \hat{U}(t, t_0)}{\partial t} = H(t) \hat{U}(t, t_0)$$

In the above we have obtained (1.2). Our trial solution indeed satisfies (1.2)

Conclusion

As claimed an alternative solution has been considered against the existing one. This is in view of a commutator relation valid subject to a weak condition that two specific operators should not be simultaneously non invertible.

References

1. Peskin M E, Schroeder D V, Quantum Field theory, Chapter 4: Interacting fields and Feynman’s Diagrams, Section 4.2: Perturbation Expansion of Correlation Functions, p84
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