Solid Strips Configurations

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Abstract

We introduce the idea of Solid Strip Configurations which is a way of describing 3-dimensional compact manifolds alternative to Δ-complexes and CW complexes. The proposed method is just an idea which we believe deserve further formal mathematical investigation.

Key Words: compact manifolds, manifold decomposition.

1 Introduction

Compact manifolds of dimension higher then 2 are very hard to study and classify. Starting from a method in the 2D case and focusing on 3D manifolds, we propose in this paper a method to describe these manifolds which, if further developed, we believe may results very convenient.

2 Strip Configurations in 2-Dimensions

2.1 Main Definitions

A **Strip** is a 2-dimensional manifold with boundaries obtained by identifying 2 opposite edges of the 4 edges of a square. It can be done without a twist (Untwisted Strip) or with a twist (Mobius strip).

A **Strip Configuration** is a finite set of strips, crossing each other or not, such that it exist a compact 2-dimensional not self-intersecting manifold in which the set of strips can be embedded. An example of two strings that do not form a string configuration is given in Fig. 1a. Once we embed the strips on such a manifold we are allowed to move the strips on the manifold at will. If a and b are two strips then we will use the notation $a \diamond b$ for the configuration obtained by making a and b crossing 1 time.

A non path connected strip configuration can always be changed in a path connected one according to the following procedure: 1-embed the strips in a compact two dimensional manifold; 2- bring two strips from two non path connected subset of the configuration close each other without changing the configuration of the two subset(see Fig. 1b); 3- overlap the two trips so that they cross in two points (see Fig. 1b).

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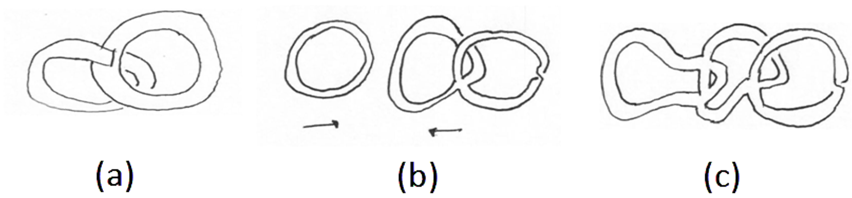

Figure 1: Definition of String Configuration

Note that the boundary of a string configuration is made of a finite number of **sub-boundaries** (i.e. non path connected parts) each of which being a circles (i.e. S^1). The **Associated (Compact) Manifold** to a strip configuration is the compact manifold obtained by making the configuration path connected (if it is not) and identifying the boundary of a disk ($\mathbf{D}^2$) to each sub-boundaries of the strip configuration. We will use the notation $\Omega(A)$ for the associated manifold to the strip configuration A.

Two strip configurations are **Homeomorphic Associated Equivalent** if their associated manifolds are homeomorphic or, which is the same, if once embedded in the associated manifold one string configuration can be changed into the other by moving the strips on the manifold and deforming the manifold by means of continuous transformations. In the process each strip shall always keep its own identity even when it crosses other strips with continuous transformations meaning that a strip cannot be cut and glued to form other strips. Two strip configurations are **Homotopy Associated Equivalent** if their associated manifolds are homotopy equivalent.

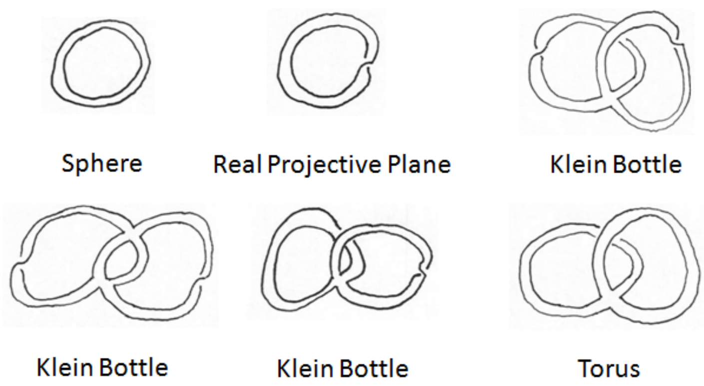

Figure 2: 1 and 2 Strip Configurations

In a strip configuration a string can be twisted n times (with $n \geq 0$) (if n is even then the string is homomorphic to an untwisted strip, if n is odd to a Mobius strip) and two strips can cross each other m times (with $m \geq 0$).

We want to give now some criteria for two strip configurations to be homeomorphic associated equivalent. Some of these criteria are not obvious and should be formally proved.

1. A non path connected strip configuration and the path connected one obtained from it using the procedure explained in the paragraph above

are equivalent.

2. An untwisted strip that does not cross any other strip can be removed from the configuration because this is equivalent to remove from the associated manifold a sphere which is sum connected to the manifold.

3. Given a strip configuration, this is equivalent to the same strip configuration where strips that are twisted an odd number of times are replaced by Mobius strips and strips that are twisted an even number of times are replaced with untwisted strips.

We note that the direct sum of 2-dimensional manifolds has a non path connected strip configuration given by the union of the two strip configurations of the two manifolds.

However, the above criteria are not enough and we want to evaluate equivalences by calculating topological invariants on the configurations. Strip configurations are very convenient from this point of view because the fundamental group of the associated manifold can be easily computed from its strip configuration using the van Kampen theorem.

To evaluate the fundamental group, the generators are given by the open maximal spanning graph obtained from the graph we get homotopyng each strips to a 1- dimensional space (i.e. we turn strips into lines) while the conditions to present the group can be evaluated on the strip configuration itself.

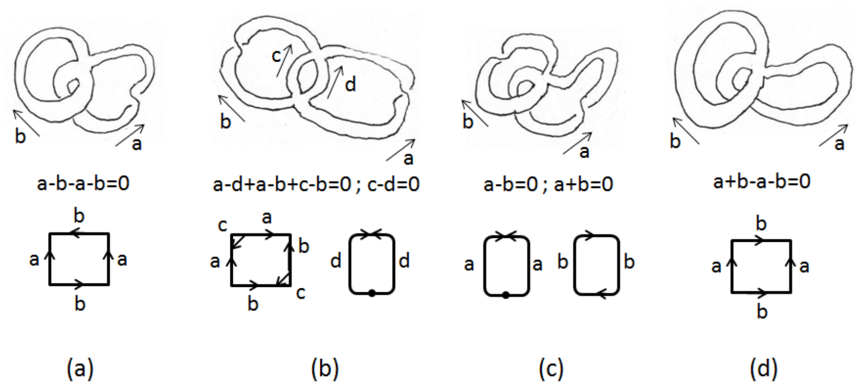

Figure 3: Strip Configurations Fundamental Groups

We will show this with some example. In figure Fig. 3 we show some strips configurations with the generators used to have the free non commutative groups. The conditions to present the fundamental group of the associated manifold are drawn in a "polygonal picture" under each configuration. These conditions are obtained starting from a point and adding the generators (group are presented with an additive operation although unusual for non commutative groups) that we encounter on the boundary going all around till we get to the same point.

For case of Fig. 3a the condition lead to the group $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_2$ which is the group of the Klein bottle. For case 3b, we have $c = d$ which, with a simple algebraic manipulations give the condition presented in the two polygon under the configuration in the figure. These lead to the group $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_2$ which is the group of the Klein bottle. For case 3.c, from the two conditions we have that $a = b$ and therefore the two conditions became $a - a = 0$ and $b + b = 0$. Once again these lead to the group $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_2$ which is the group of the Klein bottle. Condition of Fig. 3d leads to the commutative free group on two generators which is the group of the torus $\pi_1 = \mathbb{Z}^2$.

We note that for cases of Fig. 3b and 3c we need to manipulate the conditions algebraically to permute the names of edges for the polygonal representation end this because in each polygon we want to have pairs of edges with the same name.

2.2 Represented 2D Manifolds

A question we may ask is how many compact 2D manifolds we can represent with strip configurations. We have the following proposition:

Proposition 2.1: *If a 2D compact manifold has a Δ-complex representation, then it has also a solid strip representation.*

The prove of the above statement will be only sketched here. In each simplexe we have three strips joining couple of edges as shown in the Fig. 4.

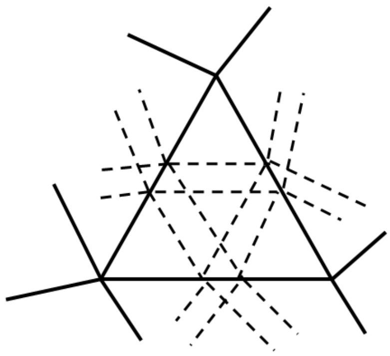

Figure 4: Strips on a Simplex

If we find all closed paths (following each strip in a given direction at each edge we have two possibilities to proceed), each possible closed path is a strip and this gives as a redundant strip configuration representation for the manifold under study. We want to find a minimal proper representation. We group the strips in classes of equivalence where two strips are equivalent if the they can be moved on the manifold till they are superimposed. For each class we choose a representative. If we obtain a non proper configuration we change the representatives or we move them on the manifold till the configuration is proper. This can always be done. We remove the strips that are not crossing at least one of the other trip an even number of times (i.e. we remove spheres sum connected to the manifold). We have now our minimal strip configuration as required.

3 Motivation for 3-Dimensions Strip Configurations

If we think for a moment to what we did in the previous paragraph we see that we represent 2D manifold starting from strip configurations or, another way to see it, we use 2D strips to probe a 2D space in a similar way homotopy theory does with loops. Given a strip configurations, this may not be embedded in $\mathbb{R}^2$ but it does exist a minimal (in away that may be made precise using the concept of associated space) 2D compact manifold where this strip configuration can be embedded.

This way to probe spaces has the advantage to see differences in some spaces that are homotopy equivalent. The most trivial example (although with boundary) is the Mobius strip which is homotopy equivalent to $\mathbf{S}^1$. However, in this space obviously a smaller Mobius Strip can be embedded while the same cannot be done in the circle.

Me may think to have a look to a strip configuration and see immediately what "strip loop" are present and tell in this way if two spaces are the same. However, the examples from Fig. 3c show that this is not so straight forward. In order to solve the problem we have build groups based on the boundaries of the strip configurations, using the the van Kampen theorem, that are eventually fundamental groups.

We will show later that we may defines some sort of 3D strips and there are at least 15 of them. Once combined in configurations, this leads to an huge amount of combinations, which may somehow be used to represent 3-manifold in a convenient way.

Obviously, before we do that, we need to show what a 3D strip is and what their configurations are. This will be done in the following sections.

4 Strip Configurations in 3-Dimensions

4.1 Main Definitions

In 2-dimensions we use 2-D strips obtained by identifying one couple of opposite edges of the two couples of edges of a square. In 3-dimensions we will use **Solids Strips** which are 3-D "strips" obtained by identifying two couples of opposite faces of the three couples of faces of a cube.

This manifolds have been studied in the paper [1] where they are named "Solid Strips".

The boundary of a solid trip is build by identifying the edges of two squares and what we get may form one or two sub-boundaries. The total homeomorphic configurations of Solid Strips are 21 (reported in Appendix A.1) but they may be further reduced to 15 Homology equivalent classes of solid strips with the same boundary and same homology groups (see [1]).

Solid strips are 3-manifolds or, another way we see it, they are **Thick Compact 2D Surfaces** where by that we mean that they are like surfaces expanded by a δL in the third dimension which, by sake of visualization for the reasoning that will follow, we may think to be small with respect to the surface itself when needed.

Being thick surfaces solid strips cross (i.e. intersect) in a solid torus, when locally embedded (i.e. just a little piece of them) in $\mathbb{R}^3$, but they may cross in 3-disks, when embedded in higher dimensions, in the same way surfaces intersect in circles and points.

We are interested in the last king of crossing where a bunch of solid strips cross each other in a finite set of 3-disks and form what we will call a **Solid Strip Configuration** in the same way 2-strips cross in 2-disks forming the 2-strip configurations described in the paragraphs above. If a and b are two solid strip then we will use the notation $a \diamond b$ for the configuration obtained by making a and b crossing 1 time.

The boundary of a solid strip configurations is formed by **Sub-Boundaries** exactly as in the 2D case. The difference here is that while in the 2D case the sub-boundaries depend on how the strip crosses, in 3D each strip has its own sub-boundary and this does not change when it crosses other strips.

In analogy with the 2D case, we will call the **Associated (Compact) Manifold** to a solid strip configurations the 3D compact manifolds that we get by filing the holes defined by its sub-boundaries (i.e. we attached manifolds to its boundaries till we get a compact space) in the "most simple" topological way where the meaning of the "most simple" will be clarified further on. In analogy with the 2D case we will use the notation $\Omega(A)$ for the associated manifold to the solid strip configuration A. We note explicitly that a non path connected strip configuration cannot be made connected using the same procedure we had for the 2D case. This is because if we make two separated strips to cross in a 3D space, locally homeomorphic to $\mathbb{R}^3$, they will cross in a torus and not in a disk. For the above reason we define the associated manifold to a non path connected strip configuration to be the connected sum of the associated manifolds to its connected components.

We need now to make more mathematically precise the two ideas of "solid strip crossing" and "filling the bubbles" of a configuration. This will be done in the following two sections.

4.2 Crossing of Solid Strips

Being thick 2D surfaces, solid strip look locally like tiles and they cross (i.e. intersect) as shown in Fig 5a where a tile A is intersecting a tile B in a 3-dimensional space. If we move all points of the tile B, apart from the point in a cube C, common to both tiles, along the 4^{th} direction (see Fig 5b), tile B will disappear from our 3D space apart from the point in the cube C where it will still intersect the tile A. Now the two tiles are again proper cubes that intersect each other only in C and the boundary of each tile intersect the boundary of C only on two opposite faces.

We can now identify the faces of the A and B, which do not intersect each other, and we can get any two solid trip a and b we like. If we do it properly two opposite faces of C will lie on the boundary of the solid strip a and other two opposite faces of C will lie on the boundary of the solid strip b. The final configuration we get is the crossing of two trips $a \diamond b$.

Although misleading, for the sake of representation we can ignore the up direction of each tiles and draw solid strips configurations in the same way we do for the 2-dimensional case where now the surface of the strip represent a volume and the lines of the boundaries represent surfaces. Each side of the strip

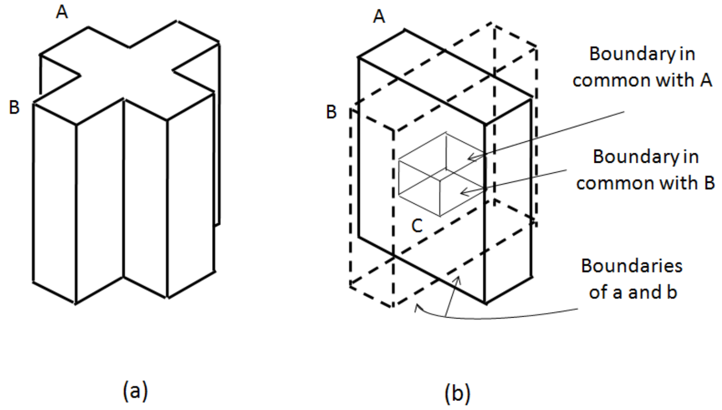

Figure 5: Crossing of two Tiles

will represent one of the squares that form the boundary of the solid strip. Strip will be represented by an untwisted or Mobius strips depending on whether their boundary has one or two sub-boundaries (see Fig. 6).

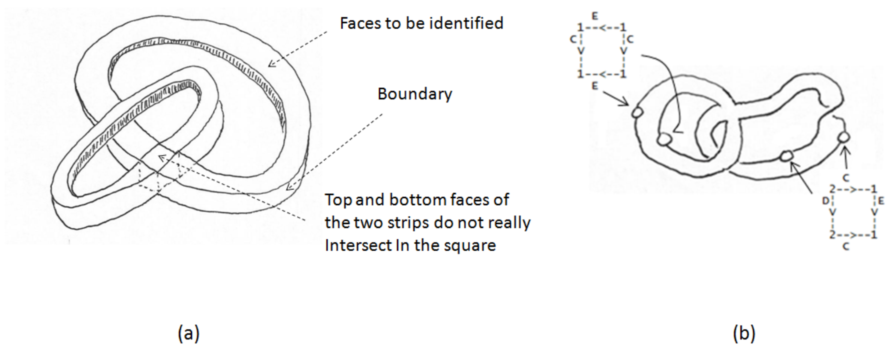

Figure 6: Crossing of two Solid Strips

Moreover we have:

Proposition 4.1: *Given two separate solid strip configurations A and B with Euler characteristic $\chi(A)$ and $\chi(B)$, and given the two solid strip $a \in A$ and $b \in B$, then if C is the composed solid strip configuration made by having a and b crossing, we have:*

$$\chi(C) = \chi(A) + \chi(B) - 1 \tag{1}$$

Proof: If we find suitable CW complexes for the two configurations A and B then, given the construction above for crossing strips, the two configuration will cross by having a cell (3-disk) in common. This will make the final configuration to have one cell less then the two separated configurations and therefore an Euler characteristics decreased by the euler characteristic of the missing cell. Since the Euler characteristic of the 3-disk is 1, this prove the proposition.

This proposition will easily allow the computation of the Euler characteristics of any strip configuration starting from the Euler characteristics of single strips which are known (see Appendix A.1).

4.3 Associated Manifold

We said above that the sub-boundaries of a solid strip configuration depend from the strips it contains and not from the way they cross. This is because each strip has its own boundary and they do not mix with each other when the strips cross. This means that once we define the associated manifold to all possible 15 strips, this will fully give a definition of associated manifold to any configuration.

Given a strip ξ, its boundary are formed by two squares which edges are identified in various way (see [1]). Given two 3D cells shaped as a pyramid with a square base, we identify the two bases of the pyramids with the two squares of the boundary of ξ. We get eight 2-simplices (twice the four sides of each pyramid) having an edges in common with one of the edges of the two squares of the boundary of ξ. We identify these eight simplices each other following the same way the edges of the boundary of ξ, to which they are attached, are identified. This will give us a compact manifold $\Omega(\xi)$ we where looking for.

Definition 4.1: *Let ξ be a solid trip. We define $\mu(\xi)$ to be:*

$$\mu(\xi) = \chi(\Omega(\xi)) - \chi(\xi) \tag{2}$$

where χ represents the Euler characteristics

We have 15 different type of solid strips and therefore all the possible value of μ we need can be given in a table. We give the flowing proposition

Proposition 4.2: *Let A be a solid trip configuration composed of solid strips a_i crossing in various way and let $\Omega(A)$ be its associated (compact) manifold. Let also $\chi(A)$ and $\chi(\Omega(A))$ be the Euler characteristics of the configuration and its associated manifold. We have:*

$$\chi(\Omega(A)) = \chi(A) + \sum_i \mu(a_i) \tag{3}$$

Given the way we have defined μ and the procedure to get the associated manifold of a strip configuration, the above proposition is trivial. Since Euler characteristics of solid trip configurations can be easily evaluated (see paragraph above), the above proposition allows to easily evaluate also also the Euler characteristics of their associated manifolds.

4.4 Represented 3D Manifolds

In the previous section we have defined the associated manifold to a Solid Strip Configuration. We note explicitly that, in the 3D case, strip configurations may represent a large class of spaces. As for the 2D case we have:

Proposition 4.3: *If a 3D compact manifold has a Δ-complex representation, then it has also a solid strip representation.*

The prove of this proposition may be sketched using a similar approach as for the 2D case. The difference in this case is that we do not have 2 strips (i.e. "thick" and "flat" lines with 2D space in it) joining two edges of 2D simplexes as in Fig. 4, but for each 3D simplex we have four thick tiles (i.e. with 3D space in it) each of which joining three 2D faces of the simplex. In this case we have thick closed surfaces rather then path. The only thing to show here is that each of these thick surfaces is homeomorphic to a solid trip. This is not done at the moment and we may do it in a further version of in this paper.

5 Further Developments

What we did in the previous paragraphs has not been presented in a rigorous mathematical form, however it has been developed following some reasonable steps. What follows is more a bunch of ideas with a less sound ground that we believe may be the right direction where to further develop the theory presented above.

We want to tell when two compact manifolds described by solid strip configurations are equivalent. We may proceed in analogy with the 2D case. We propose that we may start from the flat representation of Fig. 6.b and define a free non commutative group, in the same way we did for 2D strip configurations, where generators are derived from the maximal spanning tree of the graph associated to the flat representation. In the 2D case, generator are loops, in this case generators are half of a strip boundary. Since the two half boundaries of a strip are identical, in perfect analogy with the 2D case the two sides of a trip in the flat representation will be the same generator. As we did for the 2D case we may at this point define some conditions to present the group using the flat representation in the same way we did for the 2D case. We note explicitly that for the whole thing to make sense all the generators have to be of the same type (i.e. coming from the same class of solid strips) which is a strong limitation to what we can do. At the end we will get some groups and we will call them Λ groups. We will show this with some examples (see Fig 7).

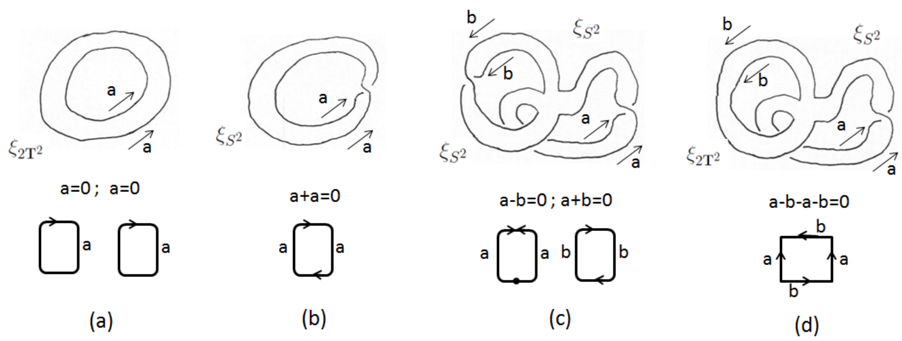

Figure 7: Examples for Groups Computation

The configuration $\Omega(\xi(g_0, a_0))$ in Fig. 7a has the only generator a. The conditions are $a = 0$; $a = 0$ which lead to the group $\Lambda_{2T^2} = 0$. We note explicitly that the double torus boundary strip is the equivalent to the untwisted strip in 2D and therefore a manifold where Λ_{2T^2} is trivial has to be equivalent to $\mathbb{S}^3$

The configurations $\Omega(\xi(g_5, a_5))$ in Fig 7b, has the only generator a. The only conditions is $a + a = 0$. We have therefore that $\Lambda_{\mathbf{S}^2} = \mathbb{Z}_2$.

The configurations $\Omega(\xi(g_5, a_5) \diamond \xi(g_5, a_5))$ in Fig 7c, has two generators a and b. The conditions are $a - b = 0$ and $a + b = 0$ (compare with example in Fig. 3c). With a simple algebraical manipulations we get easily the new conditions $a - a = 0$ and $b + b = 0$ from which we get the group $\Lambda_{\mathbf{S}^2} = \mathbb{Z} \oplus \mathbb{Z}_2$.

The configurations $\Omega(\xi(g_0, a_0) \diamond \xi(g_5, a_5))$ in Fig 7d, has two generators a and b. As for the 2D case (compare with the example in Fig. 3d) we may expect to get the same group of the previous configuration. However, generator in this configuration are not homogeneous (some from $\xi(g_0, a_0)$ and some from $\xi(g_5, a_5)$) and therefore we cannot proceed further. As a difference with the situation for the 2D case, in this case the analysis tell us that there is a strong possibility that configurations in Fig. 7c and Fig. 7d do not correspond to equivalent manifolds.

So far so good apart from the fact that we have not proved that this groups are well defined whatsoever and, most of all, that they are invariants for compact 3-manifolds.

Appendix

A.1 Solid Strip Configurations

This appendix contains the full set of solid strips equivalent class configurations. For more details and for the meaning of the $\xi(a_i, b_j)$ notation see [1].

$[\xi]$	Homology Class	ξ	$\partial\xi$	$\chi(\xi)$
1	1	$\xi(g_0, a_0)$	$\mathbf{T}^2 \sqcup \mathbf{T}^2$	0
2	2	$\xi(g_4, a_0)$, $\xi(g_0, a_4)$	$\mathbf{K} \sqcup \mathbf{K}$	0
3	3	$\xi(g_4, a_4)$	$\mathbf{RP}^2 \sqcup \mathbf{RP}^2$	1
4	4	$\xi(g_3, a_4)$, $\xi(g_2, a_4)$, $\xi(g_4, a_3)$, $\xi(g_4, a_2)$	$\mathbf{RP}^2 \vee \mathbf{RP}^2$	2
5	5	$\xi(g_3, a_3)$, $\xi(g_2, a_2)$	$\mathbf{X}_1 \vee \mathbf{X}_1$	2
6	6	$\xi(g_5, a_5)$	$\mathbf{S}^2$	1
7	7	$\xi(g_1, a_1)$	$\mathbf{T}^2$	0
8	7	$\xi(g_1, a_0)$, $\xi(g_0, a_1)$	$\mathbf{T}^2$	0
9	8	$\xi(g_5, a_0)$, $\xi(g_0, a_5)$	$\mathbf{T}^2$	0
10	9	$\xi(g_4, a_1)$, $\xi(g_1, a_4)$	$\mathbf{K}$	0
11	9	$\xi(g_5, a_1)$, $\xi(g_1, a_5)$	$\mathbf{K}$	0
12	10	$\xi(g_6, a_5)$, $\xi(g_7, a_5)$, $\xi(g_5, a_6)$, $\xi(g_5, a_7)$	$\mathbf{X}_1$	2
13	11	$\xi(g_6, a_6)$, $\xi(g_7, a_6)$, $\xi(g_6, a_7)$, $\xi(g_7, a_7)$	$\mathbf{X}_2$	1
14	12	$\xi(g_2, a_3)$, $\xi(g_3, a_2)$	$\mathbf{X}_2$	1
15	12	$\xi(g_3, a_1)$, $\xi(g_2, a_1)$, $\xi(g_1, a_3)$, $\xi(g_1, a_2)$	$\mathbf{X}_2$	1
16	12	$\xi(g_6, a_1)$, $\xi(g_7, a_1)$, $\xi(g_1, a_6)$, $\xi(g_1, a_7)$	$\mathbf{X}_2$	1
17	13	$\xi(g_3, a_0)$, $\xi(g_2, a_0)$, $\xi(g_0, a_3)$, $\xi(g_0, a_2)$	$\mathbf{Y}_1$	0
18	13	$\xi(g_6, a_0)$, $\xi(g_7, a_0)$, $\xi(g_0, a_6)$, $\xi(g_0, a_7)$	$\mathbf{Y}_1$	0
19	14	$\xi(g_6, a_4)$, $\xi(g_7, a_4)$, $\xi(g_4, a_6)$, $\xi(g_4, a_7)$	$\mathbf{Y}_1$	0
20	14	$\xi(g_5, a_3)$, $\xi(g_5, a_2)$, $\xi(g_3, a_5)$, $\xi(g_2, a_5)$	$\mathbf{Y}_1$	0
21	15	$\xi(g_6, a_3)$, $\xi(g_7, a_3)$, $\xi(g_6, a_2)$, $\xi(g_7, a_2)$, $\xi(g_3, a_6)$, $\xi(g_2, a_6)$, $\xi(g_3, a_7)$, $\xi(g_2, a_7)$	$\mathbf{Z}_1$	0
22	N/A	$\xi(g_5, a_4)$, $\xi(g_4, a_5)$	Not Feasible	N/A

Table A.1.1 : Solid Strips ξ with Strip Classes $[\xi]$, Boundaries $\partial\xi$ and the Euler Characteristics $\chi(\xi)$.

where:

- With the symbol $\sqcup$ (disjoint union) we mean two separate instances of a space which are not path connected.
- Space $\mathbf{X}_1$: is a 2-sphere where two separate points of the sphere are identified. This space has a point where the space is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.
- Space $\mathbf{X}_1 \vee \mathbf{X}_1$: is a wedge sum of two $\mathbf{X}_1$ spaces. This space has three points where the space is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.

- Space $\mathbf{X}_2$: is a 2-sphere where two couple of separate points of the sphere are identified. This space has two points where the space is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.
- Space $\mathbf{Y}_1$: is a 2-torus where two separate points of the torus are identified. This space has a point where the space is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.
- Space $\mathbf{Z}_1$: is a Klein Bottle where two separate points of the Klein Bottle are identified. This space has a point where the manifold is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.

References

[1] V. Nardozza. *Solid Strips.* http://vixra.org/abs/1910.0039 (2019)