Abstract
With the vector form of $\zeta$, RH’s validity is direct.

1 Introduction

...

2 Proof

Let:

$$V(c) = (v_1(c), v_2(c), v_3(c), \ldots) \text{ where } v_n(c) = n^{-c} \text{ for } c \in \mathbb{C}$$

$\sigma = a + it$

$\circ$ be Hadamard product and $\bullet$ be dot product

$COS = (\cos(t \ln(1)), \cos(t \ln(2)), \cos(t \ln(3)), \ldots)$

$\delta$ be any real value

and take note of the identity:

$$n^\sigma = n^a \cos(t \ln(n)) + i \sin(t \ln(n))$$

If $\sigma + \delta$ and $\sigma$ are both $\zeta$ roots, then $\zeta(\sigma) = \sum_{n=1}^{\infty} n^a (\cos(t \ln(n)) + i \sin(t \ln(n))) = (V(a) \circ COS) \bullet V(0) + i((V(a) \circ \ldots) \bullet V(0)) = 0$. But for a complex number to be zero, both the real and imaginary components will have to be simultaneously zero. Therefore:

$$\sum_{n=1}^{\infty} n^a (\cos(t \ln(n)) = 0 \quad \text{(and } \sum_{n=1}^{\infty} n^a \sin(t \ln(n)) = 0)$$

$$\sum_{n=1}^{\infty} n^\sigma n^\delta (\cos(t \ln(n)) = 0 \quad \text{(and } \sum_{n=1}^{\infty} n^\sigma n^\delta \sin(t \ln(n))) = 0)$$

It can be observed that the following vectors

$$V(a) \circ COS$$
$$V(a) \circ V(\delta) \circ COS$$
$$V(\beta)$$

are linearly independent, so not coplanar unless $\delta = 0$. (Q. E. D.)

---

1. We ignore the symmetry and prove, equivalently, that there cannot be more than one root among complex numbers with the same non-zero imaginary part $it$.

2. $V(\beta) \bullet V(0) = (V(a + \delta) - V(a)) \bullet V(0)$ for some $\beta$, observing that $V(x - y) \neq V(x) - V(y)$.

3. If the assumption (of two distinct symmetric roots) holds, the dot products of (1), (2) and (3) with $V(0)$ will have to all result in 0, so (1), (2) and (3) will have to be all orthogonal to $V(0)$, implying they need be coplanar.