Proving the Goldbach’s conjecture

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Abstract

Goldbach’s conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states:

”Every even integer greater than 2 can be expressed as the sum of two primes”.

Manuscript content: Prove that Goldbach’s conjecture is correct.

Key words: Prime numbers, Goldbach’s conjecture, number theory.

1. Notation system

We briefly mention the symbols and theorems in number theory to apply to this manuscript.

1.1. Notation
- Symbol of positive natural number: \(N^*\)
- Symbol of prime number greater than 2: \(P^*\)
- Symbol of odd-number greater than 2: \(O^*\)

1.2. The operations express odd and prime numbers
- For every odd natural number \(O\) greater than 2, it can always be expressed as:

\[
O = 2n + 1 \quad (With: \ O \in O^*, n \in N^*)
\]  (1)

This deduces the result: For every odd natural number \(O'\) greater than 5, it can always be expressed as:

\[
O' = 2n' + P \quad (With: \ n' \in N^*, P \in P^*, P < O')
\]  (2)
This also deduces the result: For every prime number $P$ greater than 5, it can always be expressed as:

$$P = 2m + P' \quad (With: \ m \in \mathbb{N}^*, P' \in P^*, P' < P)$$  \hspace{1cm} (3)

1.3. Bertrand’s postulate

Bertrand’s postulate is a theorem stating that for any integer $n > 3$, there always exists at least one prime number $p$ with

$$n < p < 2n - 2$$

2. Goldbach’s conjecture

Goldbach’s conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states:

"Every even integer greater than 2 can be expressed as the sum of two primes”.

3. Proving the Goldbach’s conjecture

3.1. Consider even integer numbers $2 < N \leq 10$

- For $N = 4$, represent: $N = 2 + 2$
- For $N = 6$, represent: $N = 3 + 3$
- For $N = 8$, represent: $N = 3 + 5$
- For $N = 10$, represent: $N = 3 + 7$

3.2. Consider even integer numbers $N > 10$

Because $N$ is an even integer greater than 10, $N$ can always be expressed as the sum of two odd numbers:

$$N = O_1 + O_2 \quad (With: \ O_1, O_2 \in O^*, O_1 < O_2)$$  \hspace{1cm} (4)

Because the prime number $P_1$ is greater than 2 in the set $P^*$, it also belongs to the set $O^*$. Therefore, $N$ can always be expressed as the sum of a prime number $P_1$ and an odd number $O$:

$$N = P_x + O \quad (With: \ \forall P_x \in P^*, O \in O^*, P_x < N/2)$$  \hspace{1cm} (5)
Based on the result of (2), we can express the odd number $O$ to:

$$O = P_x + 2n \quad (With: \ n \in N^*, P_x \in P^*, P_x < O) \quad (6)$$

Finally, this is stated as follows: With every even natural number $N > 10$, there is always at least a prime number $P_1$ in the set $P^*$ such that

$$N = P_x + (P_x + 2n) \quad (With: \ P_x \in P^*, n \in N^*, P_x, n < N/2) \quad (7)$$

*Example:* Any even natural number $N$ greater than 10, it can be expressed by $\forall P_x < N/2$ or $N = P_x + (P_x + 2n)$.

From expression (7), we transform to produce the result:

$$n = \frac{N}{2} - P_x \quad (With: \ P_x \in P^*, n \in N^*, P_x < N/2) \quad (8)$$

In particular, $N$ has a given value, so the value of $n$ will vary with $P_x$. While $P_x \in P^*$, we convert $P_x$ into the set $N^*$ to construct a function $f$:

$$f = \frac{N}{2} - x \quad (With: \ x \in N^*, x < N/2) \quad (9)$$

Thus, the value of $f$ contains the value of $n$ and the value of $x$ contains the value of $P_x$. This means that the values of $n$ and $P_x$ always belong the graph of the function $f$.

On the other hand, based on the expressions (5) (6), since $N/2 < P_x + 2n < N$, this results in the value of $n$ also in the graph of the function $g$:

$$g = x - \frac{N}{2} \quad (With: \ x \in N^*, N/2 < x < N) \quad (10)$$

Thus, the value of $g$ contains the value of $n$ and the value of $x$ contains the value of $2n + P_x$. This means that the values of $n$ and $2n + P_x$ always belong the graph of the function $g$.

*Purpose of functions:* We construct two functions $f$ and $g$ to refer to the general method, then apply Bertrand’s theorem to find the value $x$ of the function $g$. In addition, it also determines the graph of the variation of $n$ and two values $P_x, 2n + P_x$ on the same coordinate system.
Graphing two functions $f$ and $g$ on the same coordinate system, we have:

![Graph of functions](image)

**Important argument:**

- Based on the results of (5)(6)(7), we rewrite: $N = P_x + (P_x + 2n)$.

- Where $O = (P_x + 2n)$ with $O \in O^*$, and $O \in (N/2, N - 1)$. This leads to if any of the prime exists $P_y \in (N/2, N - 1)$, then it is also the value of $O \in O^*$.

- On the other hand, applying Bertrand’s theorem to the value of $x$ of the function $g = x - \frac{N}{2}$, we have: There is always at least a prime number $x = P_y$ such that $N/2 < x = P_y < N - 1$.

- Assuming $P_y = P_x + 2m$, and $N = P_x + P_y$. This satisfies the problem.

- Assuming $P_y = P_x + 2m$, and $N \neq P_x + P_y$ with $\forall P_x < N/2$, this means:
  
  $N = O' + P_{y=\text{max}}$ (With: $O' \in O^*$)
  
  $= P_{x=\text{min}} + (2n' + P_{y=\text{max}})$ (With: $n' \in N^*$)
- In which, $P_{y=\text{max}}$ is the largest prime number of the set $(N/2, N - 1)$; and $P_{x=\text{min}}$ is the smallest prime number of the set $(2, N/2)$

- Thus, in order not to conflict with (7), the following system of equations always has solutions with $n, n' \in N^*$:

$$\begin{align*}
N &= P_{x=\text{min}} + (P_{y=\text{max}} + 2n') \\
N &= P_{x=\text{min}} + (P_{x=\text{min}} + 2n)
\end{align*}$$

$$\implies P_{y=\text{max}} + 2n' = P_{x=\text{min}} + 2n \quad (\text{With: } n, n' \in N^*) \quad (11)$$

- Consider function $K(n) = P_{x=\text{min}} + 2n$ with $n \in N^*, K(n) \in (N/2, N - 1)$:

- From the Bertrand’s theorem, we have proved: If $N > 10$, there are at least 2 primes $P_y, P_z \in (N/2, N - 1)$. If $P_z \in (N/2, N - 1)$ then this leads to a conflict with Equation (11), because then $K(n) = P_z = P_{y=\text{max}} + 2n'$ then $n' \leq 0 \neq N^*$.

- Therefore, to satisfy the problem, $P_z$ will fall into one of two cases: $P_z$ is equal to $P_z = P_{x=\text{max}} \in (N/2, N - 1)$ and $P_y \neq P_{\text{max}}$; Or $P_z = N - 1$, and (simultaneous) there exist only one prime $P_y \in (N/2, N - 1)$.

- We consider the case $P_z = N - 1$: From the Bertrand’s theorem, there always exist at least 2 primes in the set $(N/2, N - 1)$. But because the set $(N/2, N - 1)$ contains only one prime $P_y$, so the remaining prime will be equal to $N/2$ if $N/2 \notin O^*$. If $N/2 \notin O^*$ then there will be 2 primes in the set $(N/2, N - 1)$, this time the problem will fall in the case of $P_z = P_{\text{max}} \in (N/2, N - 1)$.

- Final, if $N \neq P_z + P_y$ with $\forall P_z < N/2$ then $N = P_x + P$ with $P = P_{\text{max}} \in (N/2, N - 1)$ or $P = N/2$

Conclusion: For every even natural number $N > 10$, it can always be expressed as the sum of two primes, with $P_x, P_y \in P^*$, and $P_y = P_x + 2n$.

Combining with even natural numbers $2 < N \leq 10$ has been expressed as the sum of the two primes in section 3.1, leading us to prove that the Goldbach’s
conjecture is correct.

Proving end.

References