Abstract

Goldbach’s conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states:

"Every even integer greater than 2 can be expressed as the sum of two primes ".

Manuscript content: Prove that Goldbach’s conjecture is correct.

Key words: Prime numbers, Goldbach’s conjecture, number theory.

1. Notation system

We briefly mention the symbols and theorems in number theory to apply to this manuscript.

1.1. Notation
- Symbol of positive natural number: \( N^* \)
- Symbol of prime number greater than 2: \( P^* \)
- Symbol of odd-number greater than 2: \( O^* \)

1.2. The operations express odd and prime numbers
- For every odd natural number \( O \) greater than 2, it can always be expressed as:

\[
O = 2n + 1 \quad (With: \ O \in O^*, n \in N^*)
\]  

(1)

This deduces the result: For every odd natural number \( O' \) greater than 5, it can always be expressed as:

\[
O' = 2n' + P \quad (With: \ n' \in N^*, P \in P^*, P < O')
\]  

(2)
This also deduces the result: For every prime number $P$ greater than 5, it can always be expressed as:

$$P = 2m + P' \ (With: \ m \in N^*, P' \in P^*, P' < P) \tag{3}$$

1.3. Bertrand’s postulate
Bertrand’s postulate is a theorem stating that for any integer $n > 3$, there always exists at least one prime number $p$ with

$$n < p < 2n - 2$$

A less restrictive formulation is: for every $n > 1$ there is always at least one prime $p$ such that

$$n < p < 2n$$

2. Goldbach’s conjecture
Goldbach’s conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states:

"Every even integer greater than 2 can be expressed as the sum of two primes”.

3. Proving the Goldbach’s conjecture
3.1. Consider even integer numbers $2 < N \leq 10$
- For $N = 4$, represent: $N = 2 + 2$
- For $N = 6$, represent: $N = 3 + 3$
- For $N = 8$, represent: $N = 3 + 5$
- For $N = 10$, represent: $N = 3 + 7$

3.2. Consider even integer numbers $N > 10$
Because $N$ is an even integer greater than 10, $N$ can always be expressed as the sum of two odd numbers:

$$N = O_1 + O_2 \ (With: \ O_1, O_2 \in O^*, O_1 < O_2) \tag{4}$$
Because the prime number $P_1$ is greater than 2 in the set $P^*$, it also belongs to the set $O^*$. Therefore, $N$ can always be expressed as the sum of a prime number $P_1$ and an odd number $O$:

$$N = P_1 + O \quad (With:\: P_1 \in P^*, O \in O^*, N/2 < O < N) \quad (5)$$

Based on the result of (2), we can express the odd number $O$ to:

$$O = P_1 + 2n \quad (With:\: n \in N^*, P_1 \in P^*, P_1 < O) \quad (6)$$

Finally, this is stated as follows: With every even natural number $N > 10$, there is always at least a prime number $P_1$ in the set $P^*$ such that

$$N = P_1 + (P_1 + 2n) \quad (With:\: P_1 \in P^*, n \in N^*, n < N/2) \quad (7)$$

Example: Any even natural number $N$ greater than 10, it can be expressed by $N = 3 + (3 + 2n)$.

From expression (7), we transform to produce the result:

$$n = \frac{N}{2} - P_1 \quad (With:\: P_1 \in P^*, n \in N^*, n < N/2) \quad (8)$$

In particular, $N$ has a given value, so the value of $n$ will vary with $P_1$. While $P_1 \in P^*$, we convert $P_1$ into the set $N^*$ to construct a function $f$:

$$f = \frac{N}{2} - x \quad (With:\: x \in N^*, x < N/2) \quad (9)$$

Thus, the value of $f$ contains the value of $n$ and the value of $x$ contains the value of $P_1$. This means that the values of $n$ and $P_1$ always belong the graph of the function $f$.

On the other hand, based on the expressions (5), (6), since $N/2 < P_1 + 2n < N$, this results in the value of $n$ also in the graph of the function $g$:

$$g = x - \frac{N}{2} \quad (With:\: x \in N^*, N/2 < x < N) \quad (10)$$

Thus, the value of $g$ contains the value of $n$ and the value of $x$ contains the value of $2n + P_1$. This means that the values of $n$ and $2n + P_1$ always belong the graph of the function $g$.
*Purpose of functions:* We construct two functions $f$ and $g$ to refer to the general method, then apply Bertrand’s theorem to find the value $x$ of the function $g$. In addition, it also determines the graph of the variation of $n$ and two values $P_1, 2n + P_1$ on the same coordinate system.

Graphing two functions $f$ and $g$ on the same coordinate system, we have:

![Graph showing the variation of $n$, $P_1$, $N/2$, $2n + P_1$, and $x$ on a coordinate system.](image)

*(Two graphs $f$ and $g$ are on the same coordinate system)*

**Important argument:**

- Based on the results of (5)(6)(7), we rewrite: $N = P_1 + (P_1 + 2n)$.

- Where $O = (P_1 + 2n)$ with $O \in O^*$, and $O \in (N/2, N - 1)$. This leads to if any of the prime exists $P_2 \in (N/2, N - 1)$, then it is also the value of $O \in O^*$.

- On the other hand, applying Bertrand’s theorem to the value of $x$ of the function $g = x - \frac{N}{2}$, we have: There is always at least a prime number $x = P_2$ such that $N/2 < x = P_2 < N - 1$. 

4
- Assuming $P_2 = O = P_1 + 2n$ is unique, and $N \neq P_1 + O$, deduce this will contradict the expression (7). Therefore, there is always at least a prime number $P_2 = P_1 + 2n$ such that $N = P_1 + (P_1 + 2n) = P_1 + P_2$.

Conclusion: For every even natural number $N > 10$, it can always be expressed as the sum of two primes, with $P_1, P_2 \in P^*$, and $P_2 = P_1 + 2n$.

Combining with even natural numbers $2 < N \leq 10$ has been expressed as the sum of the two primes in section 3.1, leading us to prove that the Goldbach’s conjecture is correct.

Proving end.

References