Refutation of the fan theorem

Abstract: The definition of the decidable fan theorem is evaluated as not tautologous, hence refuting it and derived conjectures such as “uniform continuity theorem with continuous moduli”. These results form a non tautologous fragment of the universal logic $VŁ4$.

We assume the method and apparatus of Meth8/VŁ4 with $\top$ as tautology, $\bot$ as falsity (contingency), and $\top$ as truthity (non-contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET

\[
\begin{array}{llllllllllllllllllll}
\text{LET} & \sim \text{ Not, } \sim ; & + \text{ Or, } \lor, \cup ; & - \text{ Not Or; } \& \text{ And, } \land, \cap, \cdot, \odot ; & \\text{ \textbackslash \ Not \ And;} \\
> \text{ Imply, greater than, } \rightarrow, \Rightarrow, \supset, \supseteq, \rightarrow ; & < \text{ Not Imply, less than, } \in, \prec, \subset, \preceq, \leq ; \\
= \text{ Equivalent, } \equiv, \iff, \leftrightarrow, \approx, \cong ; & @ \text{ Not Equivalent, } \neq, \oplus ; \\
\% \text{ possibility, for one or some, } \exists, \exists!, \Diamond, M ; & \# \text{ necessity, for every or all, } \forall, \square, \mathbb{L} ; \\
(z=z) \text{ } \top \text{ as tautology, } \top, \text{ ordinal 3} ; & (z\neq z) \text{ } \bot \text{ as contradiction, } \emptyset, \text{ Null, } \bot, \text{ zero} ; \\
(\%z>\#z) \text{ } N \text{ as non-contingency, } \Delta, \text{ ordinal 1} ; & (\%z<\#z) \text{ } C \text{ as contingency, } \nabla, \text{ ordinal 2} ; \\
(\sim( y < x)) ( x \leq y) ( x \supseteq y) ( x \forall y) ( A=B) ( A\sim B). \\
\end{array}
\]

Note for clarity, we usually distribute quantifiers onto each designated variable.

Abstract The uniform continuity theorem (UCT) states that every pointwise continuous real-valued function on the unit interval is uniformly continuous. In constructive reverse mathematics, UCT is stronger than the decidable fan theorem. In this paper, we show that when “pointwise continuous” is replaced with “having a continuous modulus”, UCT becomes equivalent to the decidable fan theorem. Here, a modulus of a real-valued function on the unit interval is a function which calculates a modulus of pointwise continuity of the given function from the input and the required accuracy of the output. Such a modulus is said to be continuous if it is pointwise continuous with respect to the representation of real numbers as regular sequences of rationals equipped with the topology of Baire space. We also show that continuous real-valued functions on the unit interval which have continuous moduli are exactly those functions induced by type one “continuous functions” described by Loeb.

1 Introduction … the continuous fan theorem (CFT) is equivalent to the uniform continuity principle: (UC) Every continuous function $f: \{0,1\} \rightarrow \mathbb{N}$ is uniformly continuous. … A natural extension of UC to real-valued functions is the uniform continuity theorem: … (UCT) Every continuous function $f: [0,1] \rightarrow \mathbb{R}$ is uniformly continuous. A natural extension of UC to real-valued functions is the uniform continuity theorem: … (UCT) Every continuous function $f: [0,1] \rightarrow \mathbb{R}$ is uniformly continuous. … In this paper, we introduce a natural strengthening of the notion of continuous function from $[0,1]$ to $\mathbb{R}$ which makes UCT equivalent to DFT.

Remark 1.0: Below we render and interpret alpha_bar as not_alpha.

5.2 Fan theorem … The decidable fan theorem reads: (DFT) Every decidable bar is uniform.

It is well known that DFT is equivalent to the decidable fan theorem for arbitrary fan … Here, a fan is a decidable subset $T \subseteq \mathbb{N}^*$ of the set of finite sequences of $\mathbb{N}$ such that
[0.] \( T \subseteq N^* \subset N \) \hspace{10cm} (5.3.0.1)

\(~(u < t) < v ; \)

\hspace{10cm} (2)

\( FFFF \ FFFF \ FFFF \ FFFF \)

\( TTTT \ TTTT \ TTTT \ TTTT \)

\( FFFF \ FFFF \ FFFF \ FFFF \) \hspace{10cm} (4)

1. \( \langle \rangle \in T \), \hspace{10cm} (5.3.1.0)

**Remark 5.3.1.1:** For the angle brackets as an enumerated list we use parentheses. In Eq. 5.3.1.0 we use \( \langle n \rangle \in T \). \hspace{10cm} (5.3.1.1)

\[
\text{LET } p, q, s, t, u, v: m, n, s, T, N^*, N.
\]

\( q < t ; \)

\hspace{10cm} (4)

\( FFFF \ FFFF \ FFFF \ FFFF \)

\( FFFF \ FFFF \ FFFF \ FFFF \)

2. \( \forall s \in N^* (s \in T \iff \exists n \in N (s \star \langle n \rangle \in T)) \), \hspace{10cm} (5.3.2.1)

\(~(s < t) \iff ((%q < v) \& ((s \& q) < t)) ; \)

\hspace{10cm} (2)

\( FFFF \ FFFF \ FFFF \ FFFF \)

\( FFFF \ FFFF \ NNNN \ NNNN \)

\( FFFF \ FFFF \ FFFF \ FFFF \) \hspace{10cm} (5.3.2.2)

3. \( \forall s \in T \exists m \in N \forall n \in N (s \star \langle n \rangle \in T \rightarrow n \leq m) \). \hspace{10cm} (5.3.3.1)

\(~(u < t) \iff ((q < t) \& (((s < u) \& (s < t) = ((%q < v) \& ((s \& q) < t))) \& (((s < t) \& (%p < v) \& ((\#q < v) \& (((s \& q) < t) \& ~(p < q)))))) ; \)

\hspace{10cm} (4)

\( FFFF \ FFFF \ FFFF \ FFFF \)

\( TTTT \ TTTT \ TTTT \ TTTT \)

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\( TTTT \ TTTT \ TTTT \ TTTT \) \hspace{10cm} (5.3.4.2)

**Remark 5.3.3.2:** We write the conjecture as Eq. 5.3.0.1 implies 5.3.1.1 and 5.3.2.1 and 5.3.3.1. \hspace{10cm} (5.3.4.1)

Eq. 5.3.4.2 as rendered is not tautologous, hence refuting the fan theorem and conjectures derived therefrom such as the “uniform continuity theorem with continuous moduli”. 