Non-Abelian Class Field Theory and Langlands Program

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We review classical class field theory in the section 1. The result is that there exists a class field K/k over an ideal group $H_{\mathfrak{m}}$ in k and it is an abelian extension. We will consider our example such that $k = \mathbb{Q}$.

We may say that the classical class field theory is *abelian class field theory*. Our next problem is that "how about *non−abelian class field theory*?", namely "how about the case of non-abelian extensions?". However, we have known that non-abelian theory can't be constructed based on ideal groups H. We have to pursue our plan from a different direction. This is Langlands program. We will reform the abelian class field theory according to Langlands program in the section 2. Similarly we will consider our example such that $k = \mathbb{Q}$.

We will describe the local Langlands program in the section 3. Let k be a finite extension of $\mathbb{Q}_p$. We will call it a local field. Let $\mathrm{Irr}(GLn(k))$ be the set of irreducible smooth representations of $GLn(k)$ and let $\mathcal{G}_{n,\ell}(k)$ be the set of ℓ-adic representations of Weil group W_k. The local Langlands program is to show the existence of the isomorphism

$$\mathrm{rec}_k\colon \mathrm{Irr}(GLn(k)) \xrightarrow{\simeq} \mathcal{G}_{n,\ell}(k).$$

We try to define rec_k which is consistent with Zelevinsky's classification. Therefore, our aim is to show

$$\mathrm{rec}_k\colon \mathrm{Irr}^{\mathrm{sc}}(GLn(k)) \xrightarrow{\simeq} \mathcal{G}_{n,\ell}{}^{\mathrm{Irr}}(k).$$

Here, $\mathrm{Irr}^{\mathrm{sc}}(GLn(k))$ is the subset of $\mathrm{Irr}(GLn(k))$ which consists of irreducible supercuspidal representations and $\mathcal{G}_{n,\ell}{}^{\mathrm{Irr}}(k)$ is the subset of $\mathcal{G}_{n,\ell}(k)$ which consists of ℓ-adic irreducible representations of W_k. In the section 4, we give the definition of rec_k. In order to define rec_k, we will use $H^i{}_{\mathrm{LT}}$ that is the ℓ-adic étale cohomology of "Lubin-Tate tower".

Basically we will take the strategy to show the local Langlands program from the "local-global compatibility". Namely, we need the global arguments to show the local Langlands correspondence. Currently there is no purely local proof of the local Langlands correspondence. The global arguments are used at the following points. On the one hand, it is very hard to understand the action of the inertia group on the cohomology H^i_{LT}. We will think of an ℓ-adic n-dimensional representation (ρ, V) where V is an n-dimensional $\bar{\mathbb{Q}}_\ell$-vector space. Then there exists an open subgroup I_1 of the inertia group I_k such that, for all $\sigma \in I_1$, $\rho(\sigma)$ is unipotent. However it is very hard to show the same thing of cohomology groups. On the other hand, we need the Jacquet-Langlands correspondence. In the section 5, we shall think of these problems. It must be possible for us to solve these problems in a local manner. Strauch shows Jacquet-Langlands correspondence in a purely local manner.

We will think of the global Langlands correspondence in the section 6. A local field k is the completion of a number field $K/\mathbb{Q}$, i.e., there exists a place v and $k = K_v$. Put $\mathrm{rec}_K = \prod_v \mathrm{rec}_{K_v}$. It realizes the global Langlands correspondence. In order to show that local Langlands correspondence we need a totally real or a CM-field L. Thus we see that rec_K is obtained via such a field L. It must become our problem to show the global Langlands correspondence independently of the field L. We will see that the local Langlands correspondence is shown purely locally in the section 5. This means that the global Langlands correspondence is obtained independently of the field L.

Let k be an algebraic number field. Denote the places of k by $\mathfrak{p}_v$ and put

$$\mathfrak{m} = \prod_{v=1}^{u} \mathfrak{p}_v{}^{e_v}.$$

A k-modulus is such a formal product $\mathfrak{m} = \mathfrak{m}_0\,\mathfrak{m}_\infty$, where $\mathfrak{m}_0$ (the finite part) consists of $\mathfrak{p}_v$ $(v \neq \infty)$ and $\mathfrak{m}_\infty$ consists of infinite places. A fractional ideal of k is called relatively prime to $\mathfrak{m}$ when it is relatively prime to $\mathfrak{m}_0$. Set

$$S_{\mathfrak{m}} = \{(\alpha_0) \mid \alpha_0 \equiv 1 \ (\text{mod. } \mathfrak{m})\}$$

and

$$A_{\mathfrak{m}} = \{\text{all fractional ideal of } k \text{ which are relatively prime to } \mathfrak{m}\}.$$

We can define ideal classes as the elements of $A_{\mathfrak{m}}/S_{\mathfrak{m}}$. When

$$S_{\mathfrak{m}} \subset H \subset A_{\mathfrak{m}},$$

we will call H "ideal group modulo $\mathfrak{m}$" and denote it by $H_{\mathfrak{m}}$. We can also define ideal classes as the elements of $A_{\mathfrak{m}}/H_{\mathfrak{m}}$, and the index $h = (A_{\mathfrak{m}} : H_{\mathfrak{m}})$ is its class number.

Let K/k be a Galois field of the degree n,

$$\mathrm{N}_{\mathfrak{m}}(K/k) =_{\text{def}} \{\mathfrak{a} \text{ in } k \mid \mathfrak{a} = N_{K/k}(\mathfrak{A}) \text{ for a fractional ideal } \mathfrak{A} \text{ in } K, \\ \mathfrak{a} \text{ is relatively prime to } \mathfrak{m}\}$$

and

$$H_{\mathfrak{m}}(K/k) =_{\text{def}} S_{\mathfrak{m}}\,\mathrm{N}_{\mathfrak{m}}(K/k).$$

It turns out that

$$h = (A_{\mathfrak{m}} : H_{\mathfrak{m}}(K/k)) \leq n.$$

Thus, when a Galois field K/k of the degree n is given then an ideal group $H_{\mathfrak{m}}(K/k)$ in k whose index $h \leq n$ is obtained. If $\mathfrak{p} \in H_{\mathfrak{m}}(K/k)$ then $\mathfrak{p} = N_{K/k}(\mathfrak{P})$. So, we can say that $\mathfrak{p} = \mathfrak{P}_1\mathfrak{P}_2 \cdots \mathfrak{P}_n$ where $\mathfrak{P}_1 = \mathfrak{P}$, $\mathfrak{P}_i \neq \mathfrak{P}_j$ and $\mathfrak{P}_i = \mathfrak{P}^{\sigma}$ $\sigma \in \mathrm{Gal}(K/k)$.

Definition 1.1. Suppose that a Galois field K/k of the degree n corresponds to an ideal group $H_{\mathfrak{m}}$ in k of the index h. When $h = n$ then K is called *a class field over* $H_{\mathfrak{m}}$.

According to this definition, when K/k is a class field then it must be reasonable to say that the ideal group $H_{\mathfrak{m}}(K/k)$ is determined associated with a given field K/k.

We will think of the case $k = \mathbb{Q}$. Let ζ_N be a primitive Nth root of unity and consider the Nth cyclotomatic field $\mathbb{Q}(\zeta_N)/\mathbb{Q}$. Its degree is $n = \varphi(N)$, where φ is Euler's totient function. The field of rational numbers has the unique p_∞, so a $\mathbb{Q}$-modulus $\tilde{N} = Np_\infty$ is given. We will see that $\mathbb{Q}(\zeta_N)/\mathbb{Q}$ corresponds to the ideal group

$$H_{\tilde{N}}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) = S_{\tilde{N}} = \{(a_0) \mid a_0 \equiv 1 \pmod{\tilde{N}}\}$$

and that

$$\mathbb{Q}(\zeta_N)/\mathbb{Q} \text{ is a class field over } S_{\tilde{N}}.$$

Here, $\mathbb{Q}(\zeta_N)/\mathbb{Q}$ is an abelian extension. In general, the following is satisfied.

Theorem 1.1. There exists a class field K/k over an ideal group $H_{\mathfrak{m}}$ in k and it is an abelian extension.

We know that the maximal abelian extension $\mathbb{Q}^{\mathrm{ab}}$ of $\mathbb{Q}$ is given by the union of all cyclotomatic fields $\mathbb{Q}(\zeta_N)/\mathbb{Q}$ and that it is also a class field. Now, it turns out that

$$\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^*.$$

So

$$\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \cong \hat{\mathbb{Z}}^*, \quad \hat{\mathbb{Z}} = \varprojlim(\mathbb{Z}/N\mathbb{Z}).$$

We may say that the class field theory of the case $k = \mathbb{Q}$ boils down to the above formulae.

2.

We may say that the above theory is *abelian class field theory*. Our next problem is that "how about *non−abelian class field theory*?", namely "how about the case of non-abelian extensions?". However, according to **Theorem 1.1**, we will know that non-abelian theory can't be constructed based on ideal groups H. We have to pursue our plan from a different direction. This is Langlands program. We will use the following notations.

K/k A Galois extension over a number field k. Denote prime ideals in k and K by v and w respectively.

$\overline{k}$ The algebraic closure of k.

Fr_w The Frobenius elements corresponding to v. Here suppose that K/k is unramified at v. The prime ideal splits as a product of prime ideals of K. Pick one of them and denote it by w. It holds that

$$x^{N_k(v)} \equiv x^{\mathrm{Fr}_w} \ (\mathrm{mod.}\ w) \qquad x \in \mathcal{O}_K$$

Frob_v The geometric Frobenius element. Denote the conjugacy class of Fr_w by Fr_v. Then $\mathrm{Frob}_v =_{\mathrm{def}} \mathrm{Fr}_v{}^{-1}$.

$\mathbb{A}_k$ The ring of adeles. Let $\mathbb{A} = \mathbb{A}^\infty \times \mathbb{R}$ where $\mathbb{A}^\infty = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Then

$$\mathbb{A}_k = \mathbb{A} \otimes_{\mathbb{Q}} k.$$

Here, $k_\infty = \mathbb{R} \otimes_{\mathbb{Q}} k$.

According to Langlands program, our problem becomes to establish the correspondence between n-dimensional representations

$$\rho \colon \mathrm{Gal}(\overline{k}/k) \longrightarrow GLn$$

and

irreducible automorphic representations π of $GLn(k) \backslash GLn(\mathbb{A}_k)$.

From the Galois group side, we will think of $\rho(\mathrm{Frob}_v)$. From the automorphic representation side, we will think of the infinite tensor product $\pi = \bigotimes_v \pi_v \otimes \pi_\infty$. Our task is to establish the correspondence between $\rho(\mathrm{Frob}_v) = (\mathrm{v}_1, \cdots, \mathrm{v}_n)$ and the parameter (Satake parameter) $(\mathrm{z}_1, \cdots, \mathrm{z}_n)$ of π_v.

Take the Langlands program's view, the abelian class field theory becomes the case of GL_1. Consider our example such that $k = \mathbb{Q}$.

Let $K/\mathbb{Q}$ be a finite Galois extension. We see that the element of $\mathrm{Gal}(K/\mathbb{Q})$ is extended to the element of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For example, the Frobenius element Fr_w of $\mathrm{Gal}(K/\mathbb{Q})$ is extended to $\mathrm{Fr}_{\overline{w}}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Thus a one-dimensional representation

$$\tilde{\rho}: \mathrm{Gal}(K/\mathbb{Q}) \longrightarrow \mathbb{C}^*$$

can be naturally extended to a one-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

$$\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathbb{C}^*.$$

It is also satisfied that a representation

$$\tilde{\chi}: (\mathbb{Z}/N\mathbb{Z})^* \longrightarrow \mathbb{C}^*, \text{ for some } N$$

can be naturally extended to a one-dimensional representation of $\hat{\mathbb{Z}}^*$

$$\chi\colon \hat{\mathbb{Z}}^* \longrightarrow \mathbb{C}^*.$$

We will call such a representations as $\tilde{\rho}$ or $\tilde{\chi}$ "a finite image representation".

Let's start on a one-dimensional representation of $\hat{\mathbb{Z}}^*$. It turns out that there is no finite image one-dimensional representation of $\hat{\mathbb{Z}}^*$ except $\tilde{\chi}: (\mathbb{Z}/N\mathbb{Z})^* \longrightarrow \mathbb{C}^*$ for some N since $\hat{\mathbb{Z}} = \varprojlim(\mathbb{Z}/N\mathbb{Z})$. Suppose that there is no finite image one-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ except $\tilde{\rho}: \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) \longrightarrow \mathbb{C}^*$. Here, μ_N is the group of N^{th}-roots-of-unity. We have shown that

$$\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^*.$$

Since $\mathrm{Frob}_p \in \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ $(p \nmid N)$ corresponds to $(p \bmod N)^{-1} \in (\mathbb{Z}/N\mathbb{Z})^*$, we can obtain a certain map which sends Frob_p to $(p \bmod N)$. It enables us to introduce a one-dimensional representation $\tilde{\rho}$ of $\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ like

$$\tilde{\rho}(\mathrm{Frob}_p) = \tilde{\chi}((p \bmod N)).$$

Then it holds that

$$\tilde{\rho} \xleftrightarrow{1:1} \tilde{\chi}.$$

We can say that

$$\left\{\begin{array}{l}\text{finite image 1-dimensional} \\ \text{representation of } \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{l}\text{finite image 1-dimensional} \\ \text{representation of } \hat{\mathbb{Z}}^*\end{array}\right\}$$

$$= \{ \text{ character of } (\mathbb{Z}/N\mathbb{Z})^* \mid N \in \mathbb{Z} \}.$$

The abelian class field theory of $\mathbb{Q}$ boils down to this correspondence. We shall confirm the above supposition. It is clear that

$$\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})^{\mathrm{ab}} \longrightarrow \mathbb{C}^*$$

since $\mathbb{C}^*$ is abelian. Namely, ρ has to be an extension of $\tilde{\rho}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})^{\mathrm{ab}} \longrightarrow \mathbb{C}^*$. Here $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})^{\mathrm{ab}} = \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$. We know that $\mathbb{Q}^{\mathrm{ab}}$ of $\mathbb{Q}$ is given by the union of all cyclotomatic fields $\mathbb{Q}(\mu_N)/\mathbb{Q}$. Thus there is no finite image one-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ except $\tilde{\rho}\colon \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) \longrightarrow \mathbb{C}^*$.

This formulation is somewhat primitive. A sophisticated one is given by using the ring of adeles $\mathbb{A}$. On automorphic representations side, we will expand a character (1-dimensional representation) of $\hat{\mathbb{Z}}^*$ into a character of $\mathbb{Q}^*\backslash\mathbb{A}^*$. This character is called *a Hecke character*

$$\pi\colon \mathbb{Q}^*\backslash\mathbb{A}^* \longrightarrow \mathbb{C}^*.$$

Here $\mathbb{A}^* = (\mathbb{A}^\infty)^* \times \mathbb{R}^*$ and $(\mathbb{A}^\infty)^* = \mathbb{Q}_{>0}{}^* \times \hat{\mathbb{Z}}^*$. It turns out that $\mathbb{Q}^*\backslash\mathbb{A}^* \cong \mathbb{R}_{>0}{}^*\,\hat{\mathbb{Z}}^*$ since $\mathbb{Q}^*\mathbb{R}_{>0}{}^*\backslash\mathbb{A}^* \cong \hat{\mathbb{Z}}^*$. When a Hecke character $\pi\colon \mathbb{A}^* \longrightarrow \mathbb{C}^*$ is given, let $\pi_p\colon \mathbb{Q}_p{}^* \longrightarrow \mathbb{C}^*$ be the restriction of π to $\mathbb{Q}_p{}^* \subset \mathbb{A}^*$. Then a Hecke character gives rise to the infinite product:

$$\pi(x) = \prod_{\ell} \pi_\ell(x_\ell) \times \pi_\infty(x_\infty)$$

where $x = (\cdots, x_\ell, \cdots) \times (x_\infty) \in \mathbb{Q}^*\backslash\mathbb{A}^*$ ($\hat{\mathbb{Z}}^* \cong \prod_{\ell} \mathbb{Z}_\ell{}^*$). When $\pi_\infty(x_\infty) = 1$ then π is a character of $\hat{\mathbb{Z}}^*$. We have seen that such a character is a character of $(\mathbb{Z}/N\mathbb{Z})^*$. Moreover, the absolute value $|\cdot|\colon \mathbb{Q}^*\backslash\mathbb{A}^* \longrightarrow \mathbb{R}_{>0}{}^*$ is also a Hecke character. On Galois representations side, we will prepare an ℓ-adic representation

$$\rho_\ell\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \overline{\mathbb{Q}}_\ell{}^*.$$

Take the Hecke character π which coincides with a character of $\hat{\mathbb{Z}}^*$. Now, there is a natural homomorphism;

$$\begin{array}{ccccc} \mathbb{Q}_p{}^* & \longrightarrow & (\mathbb{A}^\infty)^* & \longrightarrow & \mathbb{Q}_{>0}{}^*\backslash(\mathbb{A}^\infty)^* \cong \hat{\mathbb{Z}}^* \\ \Cup & & \Cup & & \Cup \\ p^{-1} & \longmapsto & (\cdots,1,p^{-1},1,\cdots) & \longmapsto & (\cdots,p,\ 1,\ p,\cdots) \end{array} .$$

Recall $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \cong \hat{\mathbb{Z}}^*$. We can identify $(\cdots, p,\ 1,\ p, \cdots)$ with $\mathrm{Fr}_p = \mathrm{Frob}_p{}^{-1}$. Thus we obtain a certain map

$$\begin{array}{ccc} \mathrm{Art}_{\mathbb{Q}_p}\colon \mathbb{Q}_p{}^* & \longrightarrow & \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \\ \cup & & \cup \\ p & \longmapsto & \mathrm{Frob}_p \end{array} .$$

Put $\mathrm{Art}_{\mathbb{Q}} = \prod_p \mathrm{Art}_{\mathbb{Q}_p}$ and ι: $\overline{\mathbb{Q}}_\ell{}^* \cong \mathbb{C}^*$. We can introduce an ℓ-adic representation ρ_ℓ: $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \overline{\mathbb{Q}}_\ell{}^*$ as follows;

$$\rho_\ell = \iota^{-1} \circ \pi \circ \mathrm{Art}_{\mathbb{Q}}{}^{-1}.$$

Then

$$\rho_\ell(\mathrm{Frob}_p) = \iota^{-1} \circ \pi_p(p).$$

We have seen that $|\cdot|$ is also a Hecke character. On ℓ–adic representations side,

$$\chi_\ell\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \cong \hat{\mathbb{Z}}^* \longrightarrow \mathbb{Z}_\ell{}^* \subset \overline{\mathbb{Q}}_\ell{}^*$$

coincides with such a character as $|\cdot|$. Namely, we can obtain the one-to-one correspondence between χ_ℓ and $|\cdot|$.

We will summarize the above discussion. Consider a Hecke character π as an algebraic Hecke character of weight $-m$ where π_∞: $z \longmapsto z^m$ $(z \in \mathbb{R}_{>0}{}^*)$. Thus a character of $\hat{\mathbb{Z}}^*$ gives rise to an algebraic Heceke character of weight 0. Put ι: $\overline{\mathbb{Q}}_\ell{}^* \cong \mathbb{C}^*$. We can obtain an ℓ-adic Hecke character π_ι

$$\pi_\iota\colon \mathbb{Q}^* \backslash \mathbb{A}^* \longrightarrow \overline{\mathbb{Q}}_\ell{}^*, \quad x \longmapsto \iota^{-1}(\pi(x)/x_\infty{}^m) \cdot x_\ell{}^m.$$

On the other hand, we can obtain an ℓ-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

$$\rho_\ell\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \cong \hat{\mathbb{Z}}^* \xrightarrow{\pi_\iota} \overline{\mathbb{Q}}_\ell{}^*.$$

We will call it "an algebraic ℓ-adic character".

Theorem 2.1. Let π: $\mathbb{Q}^* \backslash \mathbb{A}^* \longrightarrow \mathbb{C}^*$ be an algebraic Hecke character and ρ_ℓ: $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \overline{\mathbb{Q}}_\ell{}^*$ be an algebraic ℓ-adic character. It is satisfied that

$$\pi_p(p) = \iota \circ \rho_\ell(\mathrm{Frob}_p)$$

for almost all prime numbers p.

Therefore, there exists a one-to-one correspondence between an ℓ-adic representation ρ_ℓ: $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \overline{\mathbb{Q}}_\ell{}^*$ and a Hecke character of $\mathbb{Q}^* \backslash \mathbb{A}^*$.

In general, there exists a one-to-one correspondence between an ℓ-adic representation ρ_ℓ: $\mathrm{Gal}(\bar{k}/k) \longrightarrow \bar{\mathbb{Q}}_\ell^*$ and a Hecke character of $k^*\backslash\mathbb{A}_k^*$. Including this general one, we will reform. First, we shall give some comment about Weil group. Let K be a local field i.e. a finite extension of $\mathbb{Q}_p$, let $\mathbb{F}_q$ be the residue field of K and $\mathrm{Frob}_q \in \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ be the geometric Frobenius element. There is a surjective map;

$$f: \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q).$$

Put $\mathrm{U}_i = \{f^{-1}(\mathrm{Frob}_q{}^i)\}$. Then

$$W_K = W(\bar{K}/K) =_{\mathrm{def}} \coprod_{i\in\mathbb{Z}} \mathrm{U}_i.$$

Theorem 2.2.

(Local theory) Let K be a local field with uniformizer v. W_K is the Weil group of K and $W_K{}^{\mathrm{ab}} = W(\bar{K}/K)^{\mathrm{ab}} = W(K^{\mathrm{ab}}/K)$. There exists a unique isomorphism

$$\mathrm{Art}_K: K^* \xrightarrow{\simeq} W_K{}^{\mathrm{ab}} \subset \mathrm{Gal}(K^{\mathrm{ab}}/K).$$

(Global theory) Let k be an algebraic number field. Denote its completion at v by k_v. Put

$$\mathrm{Art}_v = \mathrm{Art}_{k_v} \text{ and } W_v = W_{k_v}.$$

When $v|\infty$ then Art_v: $k_v{}^*/k_v{}^{*0} \xrightarrow{\simeq} W_v$ where $k_v{}^{*0} = \mathbb{R}_{>0}{}^*$ or $\mathbb{C}^*$. There exists a group homomorphism

$$\mathrm{Art}_k: \mathbb{A}_k{}^* \longrightarrow \mathrm{Gal}(\bar{k}/k)^{\mathrm{ab}}$$

which satisfies $\mathrm{Art}_k|_{k_v{}^*} = \mathrm{Art}_v$. It gives rise to

$$\mathrm{Art}_k: \overline{k^* k_\infty{}^{*0}}\backslash\mathbb{A}_k{}^* \xrightarrow{\simeq} \mathrm{Gal}(\bar{k}/k)^{\mathrm{ab}}.$$

We will review the **Theorem 2.2 (Local theory)** in the case $K = \mathbb{Q}_p$. Put

$$\mathbb{Q}_p{}^{\mathrm{cyc}} = \mathbb{Q}_p{}^{\mathrm{ur}} \cdot \mathbb{Q}_p{}^{\mathrm{ram}}; \quad \mathbb{Q}_p{}^{\mathrm{ur}} = \cup_{p\nmid N} \mathbb{Q}_p(\mu_N),\ \mathbb{Q}_p{}^{\mathrm{ram}} = \cup_m \mathbb{Q}_p(\mu_{p^m}).$$

- $\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^f-1})/\mathbb{Q}_p) \cong \mathrm{Gal}(\mathbb{F}_{p^f}/\mathbb{F}_p)$ $\quad f \geq 1$. Thus
$$\mathrm{Gal}(\mathbb{Q}_p{}^{\mathrm{ur}}/\mathbb{Q}_p) \cong \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \cong \hat{\mathbb{Z}}.$$
- $$\mathrm{Gal}(\mathbb{Q}_p{}^{\mathrm{ram}}/\mathbb{Q}_p) \cong \mathbb{Z}_p{}^*.$$

Then,

$$\mathrm{Gal}(\mathbb{Q}_p{}^{\mathrm{cyc}}/\mathbb{Q}_p) \cong \mathrm{Gal}(\mathbb{Q}_p{}^{\mathrm{ram}}/\mathbb{Q}_p) \times \mathrm{Gal}(\mathbb{Q}_p{}^{\mathrm{ur}}/\mathbb{Q}_p).$$

Since $\mathbb{Q}_p{}^{\mathrm{cyc}} = \mathbb{Q}_p{}^{\mathrm{ab}}$, we obtain

$$\begin{array}{ccccc} \mathrm{Art}_{\mathbb{Q}_p}\colon \mathbb{Q}_p{}^* & \xrightarrow{\simeq} & W_{\mathbb{Q}_p}{}^{\mathrm{ab}} & \subset & \mathrm{Gal}(\mathbb{Q}_p{}^{\mathrm{ab}}/\mathbb{Q}_p) \\ & & \cong \mathbb{Z}_p{}^* \times \mathbb{Z} & & \cong \mathbb{Z}_p{}^* \times \hat{\mathbb{Z}} \end{array} .$$

In the case of $\mathbb{R}$

$$\mathrm{Art}_{\mathbb{R}}\colon \mathbb{R}^* \longrightarrow W_{\mathbb{R}} = \mathrm{Gal}(\mathbb{C}/\mathbb{R}).$$

It turns out that

$$W_{\mathbb{R}} = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{R}^*/\mathbb{R}_{>0}{}^*.$$

We will also review the **Theorem 2.2 (Global theory)** in the case $k = \mathbb{Q}$. Put

$$\mathbb{Q}^{\mathrm{cyc}} = \mathbb{Q}_{\mathrm{cyc}}{}^{p\text{-ur}} \cdot \mathbb{Q}_{\mathrm{cyc}}{}^{p\text{-ram}}; \quad \mathbb{Q}_{\mathrm{cyc}}{}^{p\text{-ur}} = \cup_{p \nmid N} \mathbb{Q}(\mu_N),\ \mathbb{Q}_{\mathrm{cyc}}{}^{p\text{-ram}} = \cup_m \mathbb{Q}(\mu_{p^m}).$$

Since $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$,

$$\mathbb{Q}^{\mathrm{cyc}} \hookrightarrow \mathbb{Q}_p{}^{\mathrm{cyc}} \text{ and } \mathrm{Gal}(\mathbb{Q}_p{}^{\mathrm{cyc}}/\mathbb{Q}_p) \hookrightarrow \mathrm{Gal}(\mathbb{Q}^{\mathrm{cyc}}/\mathbb{Q}).$$

We can say that $\mathbb{Q}^{\mathrm{cyc}} = \mathbb{Q}^{\mathrm{ab}}$. Let $\mathbb{R} = \mathbb{Q}_\infty$. We have

$$\mathrm{Art}_{\mathbb{Q}_v}\colon \mathbb{Q}_v{}^* \longrightarrow \mathrm{Gal}(\mathbb{Q}_v{}^{\mathrm{ab}}/\mathbb{Q}_v) \subset \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}); \quad v = p, \infty.$$

Taking the product of the maps $\mathrm{Art}_{\mathbb{Q}_v}$, we can define a surjective map;

$$\Pi_v \mathrm{Art}_{\mathbb{Q}_v}\colon \mathbb{A}^* \longrightarrow \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}).$$

Put $\mathrm{Art}_{\mathbb{Q}} = \Pi_v \mathrm{Art}_{\mathbb{Q}_v}$. Since $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) = \varprojlim \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$, it is necessary for us to confirm the validity of the following map;

$$\mathrm{Art}_{\mathbb{Q}}\colon \mathbb{A}^* \longrightarrow \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}); \ \ N \geq 1.$$

Recall $x = (\ \cdots, x_p, \cdots\ ) \in \mathbb{A}^*$ if and only if $x_p \in \mathbb{Z}_p{}^*$ for all but finitely many primes p. So we see that $\mathrm{Art}_{\mathbb{Q}_p}(x_p) = \mathrm{id} \in \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ for all but finite many x_p. Therefore, we can say that $\mathrm{Art}_{\mathbb{Q}}(x) \in \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ is given by the product of finite many elements $\mathrm{Art}_{\mathbb{Q}_p}(x_p) \in \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$. Moreover, taking $x_p \in \mathbb{Q}_p{}^*$ into account, the above map is consistent with $\varprojlim$. The kernel of the map $\mathrm{Art}_{\mathbb{Q}}$ is $\mathbb{Q}^*\mathbb{R}_{>0}{}^*$.

Theorem 2.3. Let $\pi\colon k^*\backslash\mathbb{A}_k{}^* \longrightarrow \mathbb{C}^*$ be an algebraic Hecke character and $\rho_\ell\colon \mathrm{Gal}(\overline{k}/k) \longrightarrow \overline{\mathbb{Q}}_\ell{}^*$ be an algebraic ℓ-adic character. It is satisfied that

$$\pi_v = \iota \circ \rho_{\ell v} \circ \mathrm{Art}_v$$

for all $v \nmid \ell$ ($v \neq \infty$). Here $\rho_{\ell v} = \rho_\ell|_{W_v}$.

3.

Let k be a finite extension of $\mathbb{Q}_p$. We will call it a local field. Think of the L parameter of $GLn(k)$

$$\textit{homomorphism } \phi\colon W_k \times SL_2(\mathbb{C}) \longrightarrow GLn(\mathbb{C}).$$

We will think of the equivalence class of a given L parameter. Let $\Phi(GLn(k))$ be the set of all the equivalence classes. When $\phi \in \Phi(GLn(k))$ then ϕ is decomposed as a representation of $W_k \times SL_2(\mathbb{C})$ into direct sums as follows;

$$\phi = \bigoplus_{i=1}^{k} \phi_i \boxtimes \mathrm{Sym}^{m_i-1}\mathbf{Std}$$

where $\phi_1, \cdots, \phi_k$ are irreducible smooth representations of W_k, $\mathrm{Sym}^{m_i-1}\mathbf{Std}$ is the unique irreducible m_i-dimensional algebraic representation of $SL_2(\mathbb{C})$ and $m_1, \cdots, m_k$ are positive integers such that $n = \sum_{i=1}^{k} m_i \dim_{\mathbb{C}} \phi_i$. When $\phi \in \Phi(GLn(k))$ is irreducible as a representation of $W_k \times SL_2(\mathbb{C})$ then we call it "a discrete L parameter". In general, a given L parameter is the direct sum of discrete L parameters.

The n-dimensional Weil-Deligne representation (r, V, N) is defied as follows;

- (r, V) is an n-dimensional smooth representation of W_k,
- $N \in \mathrm{End}_{\mathbb{C}}(V)$

$$N r(\sigma) = q^{d(\sigma)} r(\sigma) N \ \ (\sigma \in W_k).$$

There is a surjective map $\mathrm{Gal}(\overline{k}/k) \longrightarrow \mathrm{Gal}(\overline{\kappa}/\kappa)$ $(\kappa = \mathcal{O}_k/v \cong \mathbb{F}_q)$. Let I_k (the inertia group) be the kernel of the map. Fix $\phi \in W_k$, which satisfies $\mathrm{Im}(\phi) = \mathrm{Frob}_v$. We call it a "Frobenius lift". Then $\sigma = \phi^{d(\sigma)}\theta \in W_k$; $\theta \in I_k$, $d(\sigma) \in \mathbb{Z}$.

If a Weil-Deligne representation (r, V, N) satisfies the following three conditions;

- (r, V) is semi-simple,
- $r(\phi)$ is a semi-simple linear map for a lift ϕ,
- $r(\phi)$ is a semi-simple linear map for any lift ϕ,

then we call (r, V, N) a "Frobenius semi-simple Weil-Deligne representation". It is known that a Frobenius semi-simple Weil-Deligne representation is obtained from a given Weil-Deligne representation (r, V, N). Let $r(\phi) = \mathrm{T}\,\mathrm{U}$ where T is a semi-simple matrix and U is an unipotent matrix. Define

$$r^{\mathrm{ss}}(\sigma) = r^{\mathrm{ss}}(\phi^{d(\sigma)}\theta) = \mathrm{T}^{d(\sigma)} r(\theta).$$

Then (r^{ss}, V, N) is a Frobenius semi-simple Weil-Deligne representation. Denote it by $(r, V, N)^{\text{Frob-ss}}$. On the other hand, we denote $(r^{ss}, V, 0)$ by $(r, V, N)^{ss}$. It turns out that

$$\Phi(GLn(k)) \ni \phi \quad \overset{1:1}{\longleftrightarrow} \quad (r, V, N) \in \{\text{Frobenius semi-simple Weil-Deligne representations}\}/\sim \qquad \text{— (3.1).}$$

We shall denote the Weil-Deligne representation corresponding to $\mathbf{1} \boxtimes \mathrm{Sym}^{n-1}\mathbf{Std} \in \Phi(GLn(k))$ by $\mathbf{Sp}_n$.

We will think of an ℓ-adic ($\ell \neq p = \mathrm{char}\,\kappa$, κ is the residue field of k) n-dimensional representation of W_k

$$\rho\colon W_k \longrightarrow GL(V)$$

where V is an n-dimensional $\overline{\mathbb{Q}}_\ell$-vector space.

Let (r, V) be a smooth n-dimensional representation of W_k over $\mathbb{C}$. Fix $\iota\colon \overline{\mathbb{Q}}_\ell \xrightarrow{\simeq} \mathbb{C}$. By the isomorphism ι, we can obtain an ℓ-adic n-dimensional representation of W_k. In general, an ℓ-adic representation isn't always smooth, so we can't always say that all ℓ-adic representations of W_k are obtained by this method. However, if $\ell \neq p$, we can show that all ℓ-adic n-dimensional representations are obtained from n-dimensional Weil-Deligne representations. We call such an ℓ-adic representation as relates to a Frobenius semi-simple Weil-Deligne representation a "Frobenius semi-simple ℓ-adic representation". We think of the equivalence class of the Frobenius semi-simple ℓ-adic n-dimensional representations of W_k. Denote the class by $\mathcal{G}_{n,\ell}(k)$. According to the above (3.1), we can say that

$$\Phi(GLn(k)) \ni \phi \quad \overset{1:1}{\longleftrightarrow} \quad \rho \in \mathcal{G}_{n,\ell}(k).$$

for $p \neq \ell$.

We will think of irreducible smooth representations of $GLn(k)$. As usual, think of the equivalence class of an irreducible smooth representation, and let $\mathrm{Irr}(GLn(k))$ be the set of all equivalence classes. We have seen that

$$\Phi(GLn(k)) \xrightarrow{\simeq} \mathcal{G}_{n,\ell}(k).$$

When

$$\mathrm{Irr}(GLn(k)) \longrightarrow \Phi(GLn(k)); \quad \pi \longrightarrow \phi_\pi$$

is given then we can obtain

$$\mathrm{rec}_k\colon \mathrm{Irr}(GLn(k)) \longrightarrow \Phi(GLn(k)) \xrightarrow{\simeq} \mathcal{G}_{n,\ell}(k).$$

Our aim is to show that rec_k is a bijection, i.e., $\mathrm{rec}_k(\pi) \xleftrightarrow{1:1} \rho_\pi$. We call it "local Langlands correspondence". We will start with summarizing the theory of types. Put $n = n_1 + \cdots + n_t$. For any irreducible smooth representation π of $GLn(k)$, it will be a subquotient of the normalized parabolic induction of the irreducible representation $\pi_1 \boxtimes \cdots \boxtimes \pi_t$ of $GLn_1(k) \times \cdots \times GLn_t(k)$. We will denote such π by the Langlands quotient $\pi_1 \boxplus \cdots \boxplus \pi_t$. What we have to do is

(i) to define rec_k for all $n \geq 1$;
(ii) to show that $\mathrm{rec}_k(\pi) = \mathrm{rec}_k(\pi_1) \oplus \cdots \oplus \mathrm{rec}_k(\pi_t)$ for $\pi = \pi_1 \boxplus \cdots \boxplus \pi_t$;
(iii) to show that rec_k is a bijection.

Let $\mathrm{Irr}^{\mathrm{sc}}(GLn(k))$ be the subset of $\mathrm{Irr}(GLn(k))$ which consists of irreducible supercuspidal representations and $\mathcal{G}_{n,\ell}{}^{\mathrm{Irr}}(k)$ be the subset of $\mathcal{G}_{n,\ell}(k)$ which consists of ℓ-adic irreducible representations of W_k. We try to show

$$\mathrm{rec}_k\colon \mathrm{Irr}^{\mathrm{sc}}(GLn(k)) \xrightarrow{\simeq} \mathcal{G}_{n,\ell}{}^{\mathrm{Irr}}(k) \qquad — (3.2).$$

Let π be an irreducible smooth representation of $GLn(k)$. It turns out that π is an irreducible representation with supercuspidal support, i.e., $\pi = \pi_1 \boxplus \cdots \boxplus \pi_t$ where π_i is supercuspidal. Consider the above property (ii). If we can show (3.2) then we can extend it to $\mathrm{rec}_k\colon \mathrm{Irr}(GLn(k)) \xrightarrow{\simeq} \mathcal{G}_{n,\ell}(k)$.

We try to show

$$\mathrm{rec}_{n,k}\colon \mathrm{Irr}^{\mathrm{sc}}(GLn(k)) \xrightarrow{\simeq} \mathcal{G}_{n,\ell}{}^{\mathrm{Irr}}(k).$$

We shall look back according to M. Harris's view.

Think of maps for certain global fields L:

$$\mathrm{rec}_{n,L}\colon \mathcal{A}^{good}(n,\, L) \hookrightarrow \mathcal{G}(n,\, L).$$

Here,

- L is supposed to have a place w such that $L_w \xrightarrow{\simeq} k$,
- $\mathcal{A}^{good}(n,\, L)$ is a class of cuspidal automorphic representations of $GLn(\mathbb{A}_L)$ chosen to fit some circumstances,
- $\mathcal{G}(n,\, L)$ can be taken to be the set of equivalence classes of compatible families of n-dimensional semi-simple ℓ-adic representation of $\mathrm{Gal}(\bar{L}\,/L)$.

Fix an automorphic representation $\Pi = \otimes_v \Pi_v$ of $GLn(\mathbb{A}_L)$. The ℓ-adic representation $\mathrm{rec}_{n,L}(\Pi)$ of $\mathrm{Gal}(\bar{L}\,/L)$, when it exists, should have the property that

$$\mathrm{rec}_{n,L}(\Pi)\,|_{W_{L_v}} = \mathrm{rec}_{n,L_v}(\Pi_v)$$

for almost all v such that $\Pi_v \in \mathcal{A}^{unr}(n,\, L_v)$. Here, $\mathcal{A}^{unr}(n,\, L_v)$ is the class of unramified representations of $GLn(L_v)$. Denote the unramified subset of $\mathcal{G}(n,\, L_v)$ by $\mathcal{G}^{unr}(n,\, L_v)$. We can define a bijection, a special case of Satake parametrization,

$$\mathrm{rec}_{n,L_v}\colon \mathcal{A}^{unr}(n,\, L_v) \xrightarrow{\simeq} \mathcal{G}^{unr}(n,\, L_v).$$

Let w be a finite place of L where $L_w \cong k$. We will think of the Weil-Deligne representation which corresponds to a given ℓ-adic representation of $\mathrm{Gal}(\bar{L}_v\,/L_v)$. Let $(\mathrm{rec}_{n,L_w})^{\mathrm{ss}}$ be the semisimplification of rec_{n,L_w} and set

$$(\mathrm{rec}_{n,k})^{\mathrm{ss}} = (\mathrm{rec}_{n,L_w})^{\mathrm{ss}}.$$

We will show that

P4.1. for any $\pi \in \mathrm{Irr}^{\mathrm{sc}}(GLn(k))$ there exists $\Pi \in \mathcal{A}^{good}(n,\, L)$, for some L, with $\Pi_w \cong \pi$;

P4.2. for $\Pi \in \mathcal{A}^{good}(n,\, L)$, $\Pi' \in \mathcal{A}^{good}(n',\, L)$, the completed L-function

$$\Lambda(s,\, \mathrm{rec}_{n,L}(\Pi) \otimes \mathrm{rec}_{n',L}(\Pi'))$$

satisfies the functional equation

$$\Lambda(s,\, \mathrm{rec}_{n,L}(\Pi) \otimes \mathrm{rec}_{n',L}(\Pi'))$$

$$= \varepsilon(s,\ \mathrm{rec}_{n,L}(\Pi)\otimes\mathrm{rec}_{n',L}(\Pi'))\,\Lambda(1-s,\ \check{\mathrm{rec}}_{n,L}(\Pi)\otimes\check{\mathrm{rec}}_{n',L}(\Pi'));$$

$\varepsilon(s,\ \mathrm{rec}_{n,L}(\Pi)\otimes\mathrm{rec}_{n',L}(\Pi')) = \prod_v \varepsilon_v(s,\ \mathrm{rec}_{n,L}(\Pi)\otimes\mathrm{rec}_{n',L}(\Pi'),\ \Psi_v)$ is the product of local Deligne-Langlands ε factors, $\check{}$ denotes contragredient and the local additive characters Ψ_v are assumed to be the local components of a continuous character of $\mathbb{A}_L/L$.

Our task is that

- the construction of a class $\mathcal{A}^{good}(n,\ L)$ where the ℓ-adic representation $\mathrm{rec}_{n,L}(\Pi)$ for $\Pi\in\mathcal{A}^{good}(n,\ L)$ exists,
- to make the class $\mathcal{A}^{good}(n,\ L)$ large enough to satisfy **P4.1** and **P4.2**.

Let's begin our work with looking back ℓ-adic representations. We shall take example by elliptic curves. Let $E/\mathbb{Q}$ be an elliptic curve. Denote the kernel of multiplication map by ℓ^m by $E[\ell^m]$. Set

$$T_\ell(E) = \lim_{\leftarrow} E[\ell^m], \quad V_\ell(E) = T_\ell(E)\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell.$$

Since $\mathrm{G} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $V_\ell(E)$ and it leads $\mathrm{Aut}(V_\ell(E)) \cong GL_2(\mathbb{Q}_\ell)$, we can identify the act of G on $V_\ell(E)$ with the act of $GL_2(\mathbb{Q}_\ell)$ on $V_\ell(E)$ as the act of $\mathrm{Aut}(V_\ell(E))$. It means that a homomophism

$$\rho\colon \mathrm{G} \longrightarrow \mathrm{Aut}(V_\ell(E)) \cong GL_2(\mathbb{Q}_\ell)$$

is given. There exists a commutative module $\mathrm{End}(E)$, which is called endomorphism ring of E. Set

$$J = \mathrm{End}(E)\otimes_{\mathbb{Z}}\mathbb{Q}_\ell.$$

J acts on $V_\ell(E)$. We may say that $J^* = \mathrm{Aut}(E)\otimes_{\mathbb{Z}}\mathbb{Q}_\ell$ since $\mathrm{Aut}(E)$ is formed by the invertible elements of $\mathrm{End}(E)$. We will understand that $j\in J^*$ realizes $\mathrm{Aut}(V_\ell(E))$ although the element of J does not always realize $\mathrm{Aut}(V_\ell(E))$. We see that $V_\ell(E)$ has actions of important groups $GL_2(\mathbb{Q}_\ell)$ and J. In order to think of our problem, we take a strategy to find the space like $V_\ell(E)$ which has actions of important groups GL_2 and J.

Let L be an algebraic field and let B be a central simple algebra over L of dimension n^2. There exists a certain division ring D containing L and

$$B \cong M_m(D).$$

We call a field F a splitting field for B if

$$B\otimes_L F \cong M_n(F).$$

Set $G = \mathrm{Res}_{L/\mathbb{Q}} B^*$, which is an algebraic group over $\mathbb{Q}$. Here

$$G(\mathbb{Q}) = B^* \cong GLm(D).$$

It turns out that

$$G(\mathbb{Q}_p) = (B \otimes \mathbb{Q}_p)^* = (\prod_{v|p} B \otimes_L L_v)^*.$$

If L_v is a splitting field for B then $B \otimes_L L_v \cong M_n(L_v)$. Otherwise we call B ramified at v. So when B is ramified at v then $(B \otimes_L L_v)^*$ is not isomorphic to $GLn(L_v)$. Think of

$$G(\mathbb{A}) = (B \otimes \mathbb{A})^* = \prod_p G(\mathbb{Q}_p).$$

This group $B \otimes \mathbb{A}$ corresponds to J appearing in the above example. Denote a class of automorphic representations of $G(\mathbb{A})$ chosen to fit proper conditions by $\mathcal{A}^G(n, L)$. We can construct $\mathcal{A}^{good}(n, L)$ via $\mathcal{A}^G(n, L)$. Let $\boldsymbol{\Pi}$ be automorphic representations of $G(\mathbb{A})$. Set

$(\mathrm{CH})_B$: $\boldsymbol{\Pi}$ is a cuspidal representation of $G(\mathbb{A})$ satisfying certain conditions.

When $\boldsymbol{\Pi}$ is the automorphic representation of $G(\mathbb{A})$ which satisfies $(\mathrm{CH})_B$ then we can obtain an ℓ-adic n-dimensional representation $\mathcal{R}_{\ell}'(\boldsymbol{\Pi})$ of $\mathrm{Gal}(\bar{L}/L)$. On the other hand, let Π be automorphic representations of $GLn(\mathbb{A}_L)$. Set

(CH): Π is a cuspidal representation satisfying certain conditions,

$\mathrm{DS}(\Pi)$: the set of finite places v where Π_v becomes a discrete series of representations.

Let $\mathrm{ram}(B)$ be the set of places where B is ramified. We have Jacquet-Langlands correspondence

$$\begin{array}{cccc} \mathrm{JL:} & \{\text{cuspidal representations of } G(\mathbb{A})\} & \xleftrightarrow{1-1} & \{\text{cuspidal representations of } GLn(\mathbb{A}_L)\}' \\ & \cup\!\!| & & \cup\!\!| \\ & \boldsymbol{\Pi} & \longleftrightarrow & \Pi \\ & & & \text{which satisfies } \mathrm{ram}(B) \subset \mathrm{DS}(\Pi) \end{array}$$

Set the ℓ-adic representation $\mathrm{rec}_{n,L}(\Pi)$ of $\mathrm{Gal}(\bar{L}/L)$ as follows;

$$\mathrm{rec}_{n,L}(\Pi) = \mathcal{R}_{\ell}'(\mathrm{LJ}(\Pi)).$$

Here $\mathrm{LJ} = \mathrm{JL}^{-1}$.

More precisely, M. Harris and R. Taylor prove that $\mathrm{rec}_k = \mathrm{rec}_{L_w}$ can be calculated explicitly in the vanishing cycles of certain formal deformation spaces. In order to define rec_k, we will use H^i_{LT} that is the ℓ-adic étale cohomology of "Lubin-Tate tower".

Let R be a commutative ring. An n-dimensional commutative formal group $\mathscr{G}$ over R is a power series

$$F(Y, Z) = (F_i(Y_1, \cdots, Y_n; Z_1, \cdots, Z_n))_{1 \le i \le n} \in \mathcal{A} = R[\![X_1, \cdots, X_n]\!]$$

satisfying the properties

- $F(X, Y) = X + Y + \text{higher order terms}$
- $F(F(X, Y), Z) = F(X, F(Y, Z))$
- $F(X, Y) = F(Y, X)$.

Write $F(X, Y) = X +_F Y$ and we call it the formal group law of $\mathscr{G}$. Let the multiplication by p be $[p]$: $X \longrightarrow X +_F \cdots +_F X$ (p times). It is clear that $[p]$ is a homomorphism $\mathscr{G}$ to $\mathscr{G}$. We see that $\mathscr{G}$ is made into an additive group, so it has the additive identity. Denote it by $\mathcal{O}$. The $\mathscr{G}$ is said to be divisible if $[p]$ is an isogeny, i.e., $[p]$ has the finite kernel. Here $\#\mathrm{Ker}\,[p] = p^h$. This h is called the height of $\mathscr{G}$. We assume R complete, noetherian, local, with residue field κ of characteristic $p > 0$. Let $\mathscr{G}$ be an n-dimensional commutative formal group over R. Suppose that $\mathscr{G}$ is divisible. We can form an associated p-divisible group of height h over R, which is an inductive system

$$\mathscr{G}(p) = (\mathscr{G}(p)_u, i_u) \quad u \ge 0.$$

(i) $\mathscr{G}(p)_u = \mathrm{Spec} A_u$; where $A_u = \mathcal{A}/J_u$ and $J_u = [p^u](I)\mathcal{A}$ is the ideal in $\mathcal{A}$ generated by the elements $[p^u](X_i)$, $1 \le i \le n$. (Here, $I = J_0$ denotes the ideal generated by the variables X_i). It turns out that $\varprojlim (A_u)$ is isomorphic to $\mathcal{A}$.

(ii) For each $u \ge 0$

$$0 \longrightarrow \mathscr{G}(p)_u \xrightarrow{i_u} \mathscr{G}(p)_{u+1} \xrightarrow{[p^u]} \mathscr{G}(p)_{u+1} \longrightarrow 0$$

is exact.

It follows that $\mathscr{G}(p) = \varinjlim \mathscr{G}(p)_u$.

Proposition 4.1. Let R be a complete noetherian local ring whose residue field κ is of characteristic $p > 0$. Then the association $\mathscr{G} \longmapsto \mathscr{G}(p)$ is an equivalence between the category of divisible commutative formal Lie groups over R and the category of connected p-divisible groups over R.

Let $k/\mathbb{Q}_p$ be a finite extension of $\mathbb{Q}_p$ with uniformizer v and let $\kappa = \mathcal{O}_k/\mathfrak{p} \cong \mathbb{F}_q$ $(\mathfrak{p} = (v))$. Let R be an $\mathcal{O}_k$-algebra with structure map $\iota\colon \mathcal{O}_k \longrightarrow R$, i.e., R has the structure of $\mathcal{O}_k$-module via the map $\iota\colon \mathcal{O}_k \longrightarrow R$. A formal $\mathcal{O}_k$-module $\mathscr{G}$ over R is a formal divisible group $\mathscr{G}$ over R together with a family of power series $[a]$ for $a \in \mathcal{O}_k$ which together represent a homomorphism $\iota\colon \mathcal{O}_k \longrightarrow \mathrm{End}(\mathscr{G})$. Here it is required that $[a](X) = \iota(a)X + O(X^2)$. Let $\mathcal{O}_{\breve{k}}$ be the completion of the maximal unramified extension of $\mathcal{O}_k$ and $\breve{k}$ be the field of fractions of $\mathcal{O}_{\breve{k}}$. The residue field of $\breve{k}$ is $\bar{\kappa}$ which is an algebraic closure of κ.

[Remark] In other words,

$$\breve{k} = k \otimes_{W(\kappa)} W(\bar{\kappa}).$$

Let $W(\kappa)$ be the ring of Witt vectors over κ. It is the unramified extension of degree n $(q = p^n)$ of the ring of p-adic integers. The ring of integers of $\breve{k}$ is $\mathcal{O}_{\breve{k}}$. It holds that $\mathcal{O}_{\breve{k}} = \mathcal{O}_k \otimes_{W(\kappa)} W(\bar{\kappa})$.

We will think of moduli spaces of formal groups. Fix $h \geq 1$. There is a unique one-dimensional formal p-divisible $\mathcal{O}_k$-module $\mathscr{G}_{h\,0}$ over $\bar{\kappa}$ of height h (the degree of the multiplication by v is q^h) up to isomorphism. Let $J = \mathrm{End}_{\mathcal{O}_k}(\mathscr{G}_{h\,0}) \otimes_{\mathcal{O}_k} k$. J is a central division algebra over k of invariant $1/h$. Let R be a complete local noetherian $\mathcal{O}_{\breve{k}}$-algebra with the structure map $\iota\colon \mathcal{O}_{\breve{k}} \longrightarrow R$ which induces isomorphisms between $\bar{\kappa} \xrightarrow{\simeq} R/\mathfrak{m}_R$ ($\mathfrak{m}_R$ is the maxima ideal of R). Choose a lift $\mathscr{G}$ of $\mathscr{G}_{h\,0}$, which is a one-dimensional formal p-divisible $\mathcal{O}_k$-module over R. Assign to R the set of pairs $(\mathscr{G}, \rho)$, where $\rho\colon \mathscr{G}_{h\,0} \longrightarrow \mathscr{G} \otimes_R \bar{\kappa}$ is an isomorphism. Let $\mathcal{C}$ be the category of such $\mathcal{O}_{\breve{k}}$-algebras R. Let $\mathcal{M}_0$ be the functor

$$\mathcal{C} \longrightarrow \mathbf{Sets}\,; \quad R \longmapsto \{(\mathscr{G}, \rho)\}/\sim .$$

Fix $h \geq 1$ and $n \geq 1$. There exists up to isomorphism a unique one-dimensional formal p-divisible $\mathcal{O}_k$-module over $\bar{\kappa}$ of height h with Drinfeld $\mathfrak{p}^n$-structure α^{triv} over $\bar{\kappa}$. We will denote it by $\mathscr{G}_{h\,n} = (\mathscr{G}_{h\,0}, \alpha^{triv})$. Assign to R the set of pairs $(\mathscr{G}, \rho, \alpha)$ consisting of a formal p-divisible $\mathcal{O}_k$-module $\mathscr{G}$ over R, of a Drinfeld $\mathfrak{p}^n$-structure α of $\mathscr{G}$ over R and $\rho\colon \mathscr{G}_{h\,n} \xrightarrow{\simeq} (\mathscr{G} \otimes_R \bar{\kappa}, \alpha \text{ over } \bar{\kappa})$. Let $\mathcal{M}_n$ be the functor

$$\mathcal{C} \longrightarrow \mathbf{Sets}\,; \quad R \longmapsto \{(\mathscr{G}, \rho, \alpha)\}/\sim .$$

Drinfeld defined a tower of rings

$$\mathcal{M}_0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \cdots .$$

$\mathcal{M}_n$ is a covering of $\mathcal{M}_0$ with Galois group $GL_h(\mathcal{O}_k/\mathfrak{p}^n)$. The functor $\mathcal{M}_0$ is represented by a $(h-1)$-dimensional regular local $\mathcal{O}_{\breve{k}}$-algebra $R_{h\,0}$, which is a complete local noetherian ring. $R_{h\,0}$ is isomorphic to the power series ring $\mathcal{O}_{\breve{k}}[\![X_1, \cdots, X_{h-1}]\!]$. For every integer $n \geq 1$, the deformation functor $\mathcal{M}_n$ is represented by a h-dimensional regular local $\mathcal{O}_{\breve{k}}$-algebra R_{hn}. It turns out that

$$\mathcal{M}_n \cong \mathrm{Spf}(R_{hn}), \quad n \geq 0.$$

Let M_0 be the rigid generic fiber of $\mathcal{M}_0$. Correspondingly, we obtain M_n and it is an étale covering of M_0. We will have a projective system $\{M_n\}_{n \geq 0}$, and call it "Lubin-Tate tower". We will produce the space M which is the inverse limit of the M_n. M has right actions of three important groups.

- $G = GL_h(k)$. An element $g \in G$ sends a triple $(\mathscr{G}, \rho, \alpha)$ to $(\mathscr{G}, \rho, \alpha \circ g)$.
- J^*. An element $j \in J^*$ acts on $\mathscr{G}_{h0}$, then we will see that

$$\rho \circ j\colon (\mathscr{G}_{h0}, \alpha^{triv}) \xrightarrow{j} (\mathscr{G}_{h0}, \alpha^{triv}) \xrightarrow{\simeq} (\mathscr{G} \otimes_R \bar{\kappa}, \alpha \text{ } over \text{ } \bar{\kappa}).$$

Thus we can say that an element $j \in J^*$ sends a triple $(\mathscr{G}, \rho, \alpha)$ to $(\mathscr{G}, \rho \circ j, \alpha)$.

- W_k (the Weil group). Let $\sigma \in W_k$ then σ induces $\mathrm{Frob}_v^{d(\sigma)}$ $(d(\sigma) \in \mathbb{Z})$ on $\bar{\kappa}/\kappa$ $(= \overline{\mathbb{F}}_q/\mathbb{F}_q)$. The pair $(\mathscr{G}, \alpha)$ consists of a formal p-divisible $\mathcal{O}_k$-module over R and of a Drinfeld $\mathfrak{p}^n$-structure α of $\mathscr{G}$ over R. Thus, applying σ to them, we can obtain $\mathscr{G}^\sigma$ and the level structure α^σ on $\mathscr{G}^\sigma$. Recall $\rho\colon \mathscr{G}_{hn} \xrightarrow{\simeq} (\mathscr{G} \otimes_R \bar{\kappa}, \alpha \text{ } over \text{ } \bar{\kappa})$. Since $\mathscr{G}_{hn}$ is a formal group over $\bar{\kappa}$, ρ^σ will be the map $\mathscr{G}_{hn}{}^{(q^{d(\sigma)})} \xrightarrow{\simeq} (\mathscr{G}^\sigma \otimes_R \bar{\kappa}, \alpha^\sigma \text{ } over \text{ } \bar{\kappa})$. Here $\mathscr{G}_{hn} \longrightarrow \mathscr{G}_{hn}{}^{(q^{d(\sigma)})}$ is the $q^{d(\sigma)}$-power Frobenius map. We will denote it by $F^{d(\sigma)}$. We can obtain

$$\rho^\sigma \circ F^{d(\sigma)}\colon \mathscr{G}_{hn} \longrightarrow \mathscr{G}_{hn}{}^{(q^{d(\sigma)})} \xrightarrow{\simeq} (\mathscr{G}^\sigma \otimes_R \bar{\kappa}, \alpha^\sigma \text{ } over \text{ } \bar{\kappa}).$$

We can say that the element σ sends a triple $(\mathscr{G}, \rho, \alpha)$ to $(\mathscr{G}^\sigma, \rho^\sigma \circ F^{d(\sigma)}, \alpha^\sigma)$.

Set $H^i{}_{\mathrm{LT}}$ that is the ℓ-adic étale cohomology of "Lubin-Tate tower";

$$H^i{}_{\mathrm{LT}} = \varinjlim H_c{}^i(M_n \otimes_{\breve{k}} \bar{k}^\wedge, \overline{\mathbb{Q}}_\ell).$$

Here $\bar{k}^\wedge$ is the completion of an algebraic closure of k. The three groups $GL_h(k)$, J^* and W_k act on $H^i{}_{\mathrm{LT}}$. It gives a representation of $GL_h(k) \times J^* \times W_k$:

$$\tau(g \times j \times \sigma) = \pi(g) \otimes \vartheta(j) \otimes \rho(\sigma); \quad g \in GL_h(k), j \in J^*, \sigma \in W_k.$$

Here

$$\mathrm{Hom}_{J^*}(H^i{}_{\mathrm{LT}}, \mathrm{LJ}(\pi)) \cong (H^i{}_{\mathrm{LT}}{}^* \otimes \mathrm{LJ}(\pi))^{J^*},$$

where $H^i{}_{\mathrm{LT}}{}^*$ is the dual space of $H^i{}_{\mathrm{LT}}$. We will denote the dual by "$*$".

[remark]

Let $\mathrm{Irr}^{(2)}(G)$ be the set of equivalence classes of irreducible admissible essentially square integrable representation of G. There exists a bijection, which is called Jacquet-Langlands correspondence,

$$\mathrm{JL} : \mathrm{Irr}^{(2)}(J^*) \longleftrightarrow \mathrm{Irr}^{(2)}(GL_h(k)).$$

Here $\mathrm{LJ} = \mathrm{JL}^{-1}$.

Therefore, $\mathrm{Hom}_{J^*}(H^i_{\mathrm{LT}}, \mathrm{LJ}(\pi))$ where $\pi \in \mathrm{Irr}^{\mathrm{sc}}(GL_h(k))$ gives a representation:

$$\begin{aligned}(\tau^*(g \times j \times \sigma) \otimes \mathrm{LJ}(\pi))^{J^*} &= (\pi^*(g) \otimes \vartheta^*(j) \otimes \rho^*(\sigma) \otimes \mathrm{LJ}(\pi)(j))^{J^*} \\ &= \pi^*(g) \otimes \rho^*(\sigma).\end{aligned}$$

If we use the contragredient of $\mathrm{Hom}_{J^*}(H^i_{\mathrm{LT}}, \mathrm{LJ}(\pi))$ then it gives a representation $\pi(g) \otimes \rho(\sigma)$. We can define rec_k.

Theorem 4.2. For $\pi \in \mathrm{Irr}^{\mathrm{sc}}(GL_h(k))$,

$$\begin{aligned}\pi \otimes \mathrm{rec}_k(\pi)(\tfrac{1-h}{2}) &= \pm\,[(\mathrm{Hom}_{J^*}(H_c(M), \mathrm{LJ}(\pi)))^\vee] \\ &=_{\mathrm{def}} \pm \sum_i (-1)^i (\mathrm{Hom}_{J^*}(H^i_{\mathrm{LT}}, \mathrm{LJ}(\pi)))^\vee.\end{aligned}$$

In general $\mathrm{rec}_k(\pi)$ isn't always irreducible for $\pi \in \mathrm{Irr}(GL_h(k))$. Since $\pi \in \mathrm{Irr}^{\mathrm{sc}}(GL_h(k))$, $\mathrm{rec}_k(\pi)$ is irreducible. So $\mathrm{rec}_k(\pi)$ is twisted. Denote the one-dimensional representation of W_k

$$W_k \longrightarrow \mathbb{C}^*;\ \ \sigma \longmapsto q^{-(1-h)/2\cdot d(\sigma)}$$

by $\mathbb{C}(\frac{1-h}{2})$. Fix ι: $\overline{\mathbb{Q}}_\ell \xrightarrow{\simeq} \mathbb{C}$, so we will identify $\mathbb{C}$ with $\overline{\mathbb{Q}}_\ell$. Then $\mathrm{rec}_k(\pi)(\frac{1-h}{2}) = \mathrm{rec}_k(\pi) \otimes \mathbb{C}(\frac{1-h}{2})$.

We can show that the rec_k has the expected properties.

We will give a overview of the validity of the theorem 4.2. It is related to what is called "local-global arguments". We shall carry out this according to M. Strauch's view (see Introduction in [18]). We have produced a space M, which is the inverse limit of the M_n. He points out "whereas the spaces M_n are defined purely locally, the analysis of the inductive limit above is carried out in by embedding the local situation into a global one". He says "it is very hard to understand the action of the inertia group on H^{n-1}". Why do we need global arguments? Because it is very hard to understand the action of I_k on cohomology groups. We will think of an ℓ-adic n-dimensional representation (ρ, V) of W_k where V is an n-dimensional $\overline{\mathbb{Q}}_\ell$-vector space. Then there exists an open subgroup I_1 of I_k such that, for all $\sigma \in I_1$, $\rho(\sigma)$ is unipotent. However it is very hard to show the same thing of cohomology groups. It must be caused by the fact that modules are to rings what vector spaces are to fields. Our aim is to show the statement: there exists an open subgroup I_1 of I_k such that, for all $\sigma \in I_1$ and all $i \in \mathbb{Z}$, σ acts unipotently on $H^i{}_{\mathrm{LT}}$.

We will think of this problem in more general setting. Consider a regular, proper and flat scheme X over (S, s, η). We shall think that there exists a morphism $f: X \longrightarrow \mathrm{Spec}(S)$. Here s is a closed point and η is a generic point. Let $\overline{\eta}$ be the fixed algebraic closure of η and let $X_{\overline{\eta}} = X \times_S \overline{\eta}$ $(= X \times_{\mathrm{Spec}(S)} \mathrm{Spec}(\overline{\eta}))$. We will show the following proposition.

Proposition 5.1. There exists an open subgroup I_1 of the inertia group I such that, for all $\sigma \in I_1$ and all $i \in \mathbb{Z}$, σ acts unipotently on $H_c{}^i(X_{\overline{\eta}}, \Lambda)$.

Its proof is very difficult. We shall think of a special case. Assume that S is a smooth curve over C, i.e., there exists a morphism h: $S \longrightarrow \mathrm{C}$. Here

$$\mathrm{C} = \mathrm{Ker}(\ (n_1,\ \ldots,\ n_r):\ \mathbb{Z}^r \longrightarrow \mathbb{Z}\ \ ),$$

and $\gcd(n_i) = dp^m$, with $(d, p) = 1$ ($p = \mathrm{char}\, s$). Denote a X such that there exists a morphism h: $S \longrightarrow \mathrm{C}$ by X/C. Now the tame inertia group I_{t};

$$I_{\mathrm{t}} \ \cong\ \varprojlim_{(d,p)=1}\ (\mathbb{Z}/d\mathbb{Z})(1) \cong \prod_{\ell \neq p} \mathbb{Z}_\ell(1).$$

Thus

$$I\ \twoheadrightarrow\ I_{\mathrm{t}}\ \twoheadrightarrow\ (\mathbb{Z}/d\mathbb{Z})(1).$$

Therefore the inertia group I acts on $(\mathbb{Z}/d\mathbb{Z})(1)$. Here

$$(\mathbb{Z}/d\mathbb{Z})(1) \cong \mu_d \cong \text{a cyclic group of order } d.$$

Especially, $(\mathbb{Z}/(q^h-1)\mathbb{Z})(1) \cong \mathbb{F}_{q^h}{}^*$ $(q = p^n)$. The inertia group I acts on C via its action on $(\mathbb{Z}/d\mathbb{Z})(1)$. Grothendieck shows the following for X/C.

Proposition 5.2. There exists an open subgroup I_1 of the inertia group I such that, for all $\sigma \in I_1$ and all $i \in \mathbb{Z}$, $(\sigma-1)^{i+1} = 0$ on $H_c{}^i(X_{\bar{\eta}}, \Lambda)$ (resp. $H^i(X_{\bar{\eta}}, \Lambda)$).

T. Yoshida gives a purely local approach to the non-abelian Lubin-Tate theory in the special case of depth 0 or level $\mathfrak{p}$. We may relate his success to the above **Proposition 5.2.** Let

$$X \cong \mathcal{M}_1 \cong \mathrm{Spec}(R_{h\,1}),$$

which will be a scheme over $(\mathcal{O}_{\breve{k}}, \bar{\kappa}, \breve{k})$. Here $\kappa = \mathbb{F}_q$. He shows that its special fiber $X_s = X \times_{\mathrm{Spec}(\mathcal{O}_{\breve{k}})} \mathrm{Spec}(\bar{\kappa})$ contains a smooth affine variety over $\bar{\kappa}$ which is isomorphic to DL_h. DL_h is the Deligne-Lustzig variety for $GL_h(\mathbb{F}_q)$ associated to a non split torus T with $T(\mathbb{F}_q) \cong \mathbb{F}_{q^h}{}^*$. Here $(\mathbb{F}_q)^h \cong \mathbb{F}_{q^h}$. We will identify T with C, so we may consider the X as such a X/C. Therefore, there exists an open subgroup I_1 of the inertia group I_k such that, for all $\sigma \in I_1$ and all $i \in \mathbb{Z}$, $(\sigma-1)^{i+1} = 0$ on $H^i(X_{\bar{\eta}}, \bar{\mathbb{Q}}_\ell)$.

It turns out that

$$H^i(X_{\bar{\eta}}, \bar{\mathbb{Q}}_\ell) \cong H_c{}^i(\mathrm{DL}_h, \bar{\mathbb{Q}}_\ell) \qquad -(5.1).$$

Fix a character $\chi : \mathbb{F}_{q^h}{}^* \longrightarrow \mathbb{C}^*$ of $\mathbb{F}_{q^h}{}^*$ and suppose that χ is in general position. Here, when $\chi, \chi^q, \cdots, \chi^{q^{h-1}}$ are distinct then χ is called "in general position". If there exists $i \geq 1$ and $\chi' = \chi^{q^i}$ then denote $\chi \sim \chi'$. Put

$$\begin{aligned} \mathrm{DL}(\chi) &= \mathrm{Hom}_{\mathbb{F}_{q^h}{}^*}(\chi, H_c{}^{h-1}(\mathrm{DL}_h, \bar{\mathbb{Q}}_\ell)) \\ &\cong (\chi^* \otimes H_c{}^{h-1}(\mathrm{DL}_h, \bar{\mathbb{Q}}_\ell))^{\mathbb{F}_{q^h}{}^*}. \end{aligned}$$

$\mathrm{DL}(\chi)$ gives a representation:

$$(\chi^*(h) \otimes \pi(g) \otimes \eta(h))^{\mathbb{F}_{q^h}{}^*} = \pi(g); \quad h \in \mathbb{F}_{q^h}{}^*, g \in GL_h(\mathbb{F}_q).$$

So, $\mathrm{DL}(\chi)$ is a representation of $GL_h(\mathbb{F}_q)$. It turns out that $\mathrm{DL}(\chi)$ is an irreducible cuspidal representation of $GL_h(\mathbb{F}_q)$ and any irreducible cuspidal representation of $GL_h(\mathbb{F}_q)$ is given by $\mathrm{DL}(\chi)$.

$$\begin{array}{ccc} \{\text{irreducible cuspidal represen-} & \overset{1:1}{\longleftrightarrow} & \{\text{characters } \chi \text{ of } \mathbb{F}_{q^h}{}^* \\ \text{tations } \mathrm{DL}(\chi) \text{ of } GL_h(\mathbb{F}_q)\}/_{\cong} & & \text{in general position}\}/_{\sim} \end{array} \quad -(5.2).$$

We consider χ as an inertia character of I_k from the canonical surjection

$$I_k \twoheadrightarrow I_{\mathrm{t}} \twoheadrightarrow \mathbb{F}_{q^h}{}^*.$$

Then

$$\{\mathrm{DL}(\chi)\}/_{\cong} \overset{1:1}{\longleftrightarrow} \{\text{generic inertia characters } \chi \text{ of } I_k\}/\sim \quad -(5.2)'$$

Here "generic" corresponds with "in general position". From (5.1), we can show the above correspondence in a purely local manner.

For the proof of the Langlands correspondence in a purely local manner, it is important to think of automorphic inductions. Let E/k be a finite extension of the degree n. Let $\theta : E^* \longrightarrow \mathbb{C}^*$ be a character of E^*. We see that

$$\mathrm{rec}_E\colon \mathrm{Irr}(GL_1(E)) \xrightarrow{\simeq} \mathcal{G}_{1,\ell}(E); \quad \theta \longmapsto \rho_\theta.$$

When the representation $\rho_\theta \in \mathcal{G}_{1,\ell}(E)$ is obtained then the n-dimensional representation $\mathrm{Ind}^{W_k}{}_{W_E}\, \rho_\theta$ of W_k is obtained from $\mathrm{Ind}^{W_k}{}_{W_E}\colon \mathcal{G}_{1,\ell}(E) \longrightarrow \mathcal{G}_{n,\ell}(k)\,(W_E \subset W_k)$. On the other hand, if the automorphic induction $\mathrm{AI}_{E/k}(\theta) \in \mathrm{Irr}(GL_n(k))$ is obtained then we define

$$\mathrm{rec}_k(\mathrm{AI}_{E/k}(\theta)) = \mathrm{Ind}^{W_k}{}_{W_E}\, \rho_\theta.$$

Consider the Langlands correspondence, then it is conjectured that we can obtain the automorphic induction $\mathrm{AI}_{E/k}(\theta)$ for any finite extension E/k. However, without the assumption of the Langlands correspondence, we can concretely construct automorphic inductions for special cases, e.g., E being a cyclic extension of k. The following is an instance. Let E/k be an unramified extension of the degree n with uniformizer w. The E/k will be a cyclic extension of k. The residue field of E is $\mathcal{O}_E/\mathfrak{P} \cong \mathbb{F}_{q^n}$ ($\mathfrak{P} = (w)$). A regular tamely ramified character $\bar{\chi} : E^* \longrightarrow \mathbb{C}^*$, which satisfies $\mathrm{Stab}_{\mathrm{Gal}(E/k)}(\bar{\chi}) = \{1\}$ and $\bar{\chi}|_{1+w\mathcal{O}_E} = \{1\}$ is given. Here $\bar{\chi}|_{\mathcal{O}_E{}^*}$ is via a character $\chi\colon \mathbb{F}_{q^n}{}^* \longrightarrow \mathbb{C}^*$ of $\mathbb{F}_{q^n}{}^*$. The representation $\rho_{\bar{\chi}} \in \mathcal{G}_{1,\ell}(E)$ is obtained, and the n-dimensional representation $\mathrm{Ind}^{W_k}{}_{W_E}\, \rho_{\bar{\chi}}$ of W_k is obtained from $\mathrm{Ind}^{W_k}{}_{W_E}\colon \mathcal{G}_{1,\ell}(E) \longrightarrow \mathcal{G}_{n,\ell}(k)$ $(W_E \subset W_k)$. The supercuspidal representation $\pi(\bar{\chi})$ of $GL_n(k)$ is obtained as an induction of the representation $\rho(\bar{\chi})$ of $H = E^* \cdot GL_n(\mathcal{O}_k)$ given through the lift of $\mathrm{DL}(\chi)$ to the representation of $GL_n(\mathcal{O}_k)$. Define

$$\mathrm{rec}_k(\pi(\bar{\chi})) = \mathrm{Ind}^{W_k}{}_{W_E}\, \rho_{\bar{\chi}}.$$

From the canonical surjection $I_k \twoheadrightarrow \mathbb{F}_{q^h}{}^*$, we shall consider χ as the generic tame inertia character. By extending χ from I_k to W_E, a 1-dimensional representation of W_E is obtained. We can say that it is $\rho_{\bar{\chi}}$. Use $\mathrm{Ind}^{W_k}{}_{W_E}$ and consider $(5.2)'$, the n-dimensional representation $\mathrm{Ind}^{W_k}{}_{W_E}\,\rho_{\bar{\chi}}$ of W_k is obtained and it corresponds to $\mathrm{DL}(\chi)$. We have

$$\begin{array}{ccc} \{\text{irreducible cuspidal repressiontations of } GL_h(\mathbb{F}_q)\} & \longrightarrow & \mathcal{G}_{n,\ell}(k) \\ \Cup & & \Cup \\ \mathrm{DL}(\chi) & \longmapsto & \mathrm{Ind}^{W_k}{}_{W_E}\,\rho_{\bar{\chi}} \end{array} .$$

We may say that this correspondence is obtained in a local manner. We will also see that it is compatible with the above automorphic inductions. So we can show

$$\begin{array}{ccc} \mathrm{Irr}^{\mathrm{sc}}(GLn(k)) & \longrightarrow & \mathcal{G}_{n,\ell}(k) \\ \Cup & & \Cup \\ \pi(\bar{\chi}) & \longmapsto & \mathrm{Ind}^{W_k}{}_{W_E}\,\rho_{\bar{\chi}} \end{array}$$

in a purely local manner.

If we take the **Proposition 5.1** into account then what we have to do will be to prove the Jacquet-Langlands correspondence in a purely local manner. M. Strauch shows it.

Recall $\mathcal{M}_n \cong \mathrm{Spf}(R_{h\,n})$ where $R_{h\,n}$ is an $\mathcal{O}_{\breve{k}}$-algebra. The generic fiber of $\mathcal{M}_n$, $(\mathcal{M}_n)_\eta = \mathcal{M}_n \times_{\mathrm{Spec}(\mathcal{O}_{\breve{k}})} \mathrm{Spec}(\breve{k})$, is a formal scheme over $\mathrm{Spec}(\breve{k})$. Denote the blow-up of $\mathcal{M}_n$ by $(\mathcal{M}_n)'$. We have a morphism p: $Z = (\mathcal{M}_n)' \longrightarrow \mathcal{M}_n$ over $\mathcal{O}_{\breve{k}}$ and we denote the inverse image $x \in \mathcal{M}_n$ by $Y = p^{-1}(x)$. Let i_Z, j_Z be the inclusion $Y \xrightarrow{i_Z} (\mathcal{M}_n)'$ and $(\mathcal{M}_n)_\eta \xrightarrow{j_Z} (\mathcal{M}_n)'$. We obtain the following diagram:

$$\begin{array}{ccccc} Y & \xrightarrow{i_Z} & (\mathcal{M}_n)' & \xleftarrow{j_Z} & (\mathcal{M}_n)_\eta \\ \downarrow & & \downarrow p & & \| \\ x & \longrightarrow & \mathcal{M}_n & \xleftarrow{j} & (\mathcal{M}_n)_\eta \end{array} .$$

By the base change under $\mathrm{Spec}(\mathcal{O}_{\bar{k}^\wedge}) \longrightarrow \mathrm{Spec}(\mathcal{O}_{\breve{k}})$, we have

$$\begin{array}{ccccc} Y & \xrightarrow{\bar{i}_Z} & (\bar{\mathcal{M}}n)' & \xleftarrow{\bar{j}_Z} & (\mathcal{M}_n)_{\bar{\eta}} \\ \downarrow & & \downarrow \bar{p} & & \| \\ x & \longrightarrow & \bar{\mathcal{M}}n & \xleftarrow{\bar{j}} & (\mathcal{M}_n)_{\bar{\eta}} \end{array} .$$

Here $\bar{\mathcal{M}}n = \mathcal{M}_n \times_{\mathrm{Spec}(\mathcal{O}_{\breve{k}})} \mathrm{Spec}(\mathcal{O}_{\bar{k}^\wedge})$, $(\bar{\mathcal{M}}n)' = (\mathcal{M}_n)' \times_{\mathrm{Spec}(\mathcal{O}_{\breve{k}})} \mathrm{Spec}(\mathcal{O}_{\bar{k}^\wedge})$ and $(\mathcal{M}_n)_{\bar{\eta}} = \mathcal{M}_n \times_{\mathrm{Spec}(\mathcal{O}_{\breve{k}})} \mathrm{Spec}(\bar{k}^\wedge)$. $Y = \bar{p}^{-1}(x)$ will be a subscheme of the special fiber $(\mathcal{M}_n)'_s =$

$(\mathcal{M}_n)' \times_{\mathrm{Spec}(\mathcal{O}_{\breve{k}})} \mathrm{Spec}(\bar{\kappa})$ of $(\mathcal{M}_n)'$. If we denote by $R\psi\Lambda$ the nearby cycle sheaves, we may say that

$$H^i((\mathcal{M}_n)_{\bar{\eta}}, \Lambda) \cong H^i(Y, R\psi\Lambda).$$

We will give brief references concerning the construction of M_n in the sense of R. Huber. He associates to a locally noetherian formal scheme $\mathcal{X}$ an adic space $t(\mathcal{X})$;

$$\begin{aligned} t(\mathcal{X}) &= \mathrm{Spa}(R, R) \\ &= \{|\cdot| \mid |\cdot| \text{ is a continuous valuation of } R \text{ with } |a| \leq 1 \text{ for every } a \in R\} \end{aligned}$$

where $\mathcal{X} = \mathrm{Spf}(R)$ is affine. Here

$$sp_{\mathcal{X}}(|\cdot|) = \{a \in R \mid |a| < 1\}$$

is an open prime ideal of R, so a point in $\mathcal{X} = \mathrm{Spf}(R)$. We can obtain the map

$$sp_{\mathcal{X}} : t(\mathcal{X}) \longrightarrow \mathcal{X}; \ |\cdot| \longmapsto sp_{\mathcal{X}}(|\cdot|).$$

In our case $\mathcal{M}_n \cong \mathrm{Spf}(R_{hn})$ where R_{hn} is an $\mathcal{O}_{\breve{k}}$-algebra. Let

$$\bar{M}_n = t(\mathcal{M}_n)_a = \mathrm{Spa}(R_{hn}, R_{hn})_a.$$

- A point x of an adic space X is called analytic if there exists an open neighborhood U of x in X such that $\mathcal{O}_x(U)$ has a topologically nilpotent unit. Put $X_a = \{x \in X \mid x$ is analytic$\}$.

The generic fiber $(\mathcal{M}_n)_\eta$ is a formal scheme over $\mathrm{spec}(\breve{k})$, so we may say that $(\mathcal{M}_n)_\eta \cong \mathrm{Spf}(R'_{hn})$ where R'_{hn} is an $\breve{k}$-algebra. Denote the associated adic space to $(\mathcal{M}_n)_\eta$, which we can identify with the rigid generic fiber of $\mathcal{M}_n$, by M_n. Since an adic space is consist with the base change, we can obtain such an adic space.

Now $M_n \otimes_{\breve{k}} \bar{k}^\wedge$ will be the adic space associated to $(\mathcal{M}_n)_{\bar{\eta}}$. Thus we can say that

$$H^i(M_n \otimes_{\breve{k}} \bar{k}^\wedge, \Lambda) \cong H^i(Y, R\psi\Lambda).$$

Put $\gamma = (g, b^{-1}) \in GL_h(k) \times J^*$, both regular elliptic elements. It acts on $M_n \otimes_{\breve{k}} \bar{k}^\wedge$ and induces a morphism

$$\gamma\colon M_n \otimes_{\breve{k}} \bar{k}^\wedge \longrightarrow M_n \otimes_{\breve{k}} \bar{k}^\wedge.$$

We shall compute the trace of γ, $\mathrm{tr}((g, b^{-1}) | H^i(M_n \otimes_{\breve{k}} \bar{k}^\wedge, \Lambda))$. It turns out that

$$\mathrm{tr}((g, b^{-1}) | H^i(M_n \otimes_{\breve{k}} \bar{k}^\wedge, \Lambda)) = \mathrm{tr}((g, b^{-1}) | H^i(Y, R\psi\Lambda)).$$

We will compute $\mathrm{tr}((g, b^{-1}) | H^i(Y, R\psi\Lambda))$. Consider $sp_{(\mathcal{M}_n)'}\colon t((\mathcal{M}_n)') \longrightarrow (\mathcal{M}_n)'$. We can say that $\bar{M}_n = \varprojlim t((\mathcal{M}_n)')$, thus it deduces

$$sp: \bar{M}_n \longrightarrow (\mathcal{M}_n)'.$$

We have a morphism $Y \xrightarrow{i_Z} (\mathcal{M}_n)'$. Let $\partial M_n = \bar{M}_n - M_n$. From $\partial M_n \subseteq \bar{M}_n$, $sp(\partial M_n) \subseteq (\mathcal{M}_n)'$. We define

$$\partial Y = i_Z^{-1}(sp(\partial M_n)).$$

Put $Y = \partial Y \cup (Y - \partial Y)$. Here $Y - \partial Y$ is open in $(\mathcal{M}_n)'_s$. We can use Lefschetz-Verdier trace formula:

$$\sum_i (-1)^i \operatorname{tr}((g, b^{-1}) | H^i(Y, R\psi\Lambda)) = \#\mathrm{Fix}\gamma + \text{the remainder}.$$

If γ has no fixed points on ∂Y then the remainder $= 0$. The fact that γ has no fixed points on ∂M_n shows that γ has no fixed points on ∂Y.

Theorem 5.1. Let $g \in GL_h(k)$, $b \in J^*$ be both regular elliptic elements.

$$\operatorname{tr}((g, b^{-1}) | H_c^*(M_n)) = \sum_i (-1)^i \operatorname{tr}((g, b^{-1}) | H_c^i(M_n \otimes_{\bar{k}} \bar{k}^\wedge, \mathbb{Q}_\ell))$$

is equal to the number of fixed points of (g, b^{-1}) on $M_n \otimes_{\bar{k}} \bar{k}^\wedge$, which is finite.

Let π be an irreducible supercuspidal representation of $G = GL_h(k)$. The character of π, which is denoted by χ_π, is a locally constant function on the set of elliptic regular elements in G. Put the representation $\vartheta = \mathrm{JL}(\pi)$ of J^* that corresponds to π via the Jacquet-Langlands correspondence. Let $g \in GL_h(k)$ and $b \in J^*$ be the regular elliptic elements with the same characteristic polynomial. Then the following character relation

$$\chi_\vartheta(b) = (-1)^{h-1} \cdot \chi_\pi(g)$$

holds. We will show the character relation in a purely local manner.

Now, $\mathrm{Hom}_G(H^i{}_{\mathrm{LT}}, \pi)$ gives a representation:

$$(\pi^*(g) \otimes \vartheta^*(b) \otimes \rho^*(\sigma) \otimes \pi(g))^G = \vartheta^*(b) \otimes \rho^*(\sigma);\ g \in GL_h(k), b \in J^*, \sigma \in W_k.$$

We may say that $\mathrm{Hom}_G(H^i{}_{\mathrm{LT}}, \pi)$ is a finite-dimensional smooth representation $\vartheta^*(b)$ of J^*. We can say that $\vartheta^*(b^{-1}) = \vartheta(b)$. We will consider

$$\mathrm{Hom}_G(H^*{}_{\mathrm{LT}}, \pi) = \sum_i (-1)^i \mathrm{Hom}_G(H^i{}_{\mathrm{LT}}, \pi).$$

We will compute $\operatorname{tr}(\mathrm{Hom}_G(H^*{}_{\mathrm{LT}}, \pi))$. Unless $i = h-1$, no supercuspidal representation of G appears in $H^i{}_{\mathrm{LT}}$ as a sub-quotient. Thus we may say that $\operatorname{tr}(\mathrm{Hom}_G(H^*{}_{\mathrm{LT}},$

$\pi)) = \mathrm{tr}((-1)^{h-1}\mathrm{Hom}_G(H^{h-1}{}_{\mathrm{LT}}, \pi))$. Let f be a compact supported function on J^* and let g_b be the regular elliptic element of G whose characteristic polynomial is same as that of b. Now suppose that $\mathrm{supp}(f)$ is contained in the set of regular elliptic elements of J^*. We can use the above **Theorem 5.1** and we can show that

$$\mathrm{tr}(f \mid \mathrm{Hom}_G(H^*{}_{\mathrm{LT}}, \pi)) = h \cdot \int_{J^*} \chi_\pi(g_b) f(b)\, db\,.$$

We replace f by a sequence of compactly supported functions on J^* whose support converges to $\{b\}$ and whose integral is 1, for example a sequence of

$$\delta_b(t) = \begin{cases} \infty \cdots t = b \\ 0 \cdots t \neq b \end{cases},$$

then we can say that

$$\mathrm{tr}(b \mid \mathrm{Hom}_G(H^*{}_{\mathrm{LT}}, \pi)) = h \cdot \chi_\pi(g_b).$$

On the other hand, if we consider that $b = (b^{-1})^{-1}$, we may say that

$$\mathrm{tr}(b \mid \mathrm{Hom}_G(H^*{}_{\mathrm{LT}}, \pi)) = \mathrm{tr}(b \mid (-1)^{h-1}\mathrm{Hom}_G(H^{h-1}{}_{\mathrm{LT}}, \pi)) = (-1)^{h-1}\chi_{\mathrm{Hom}_G(H^{h-1}{}_{\mathrm{LT}},\, \pi)^\vee}(b).$$

It turns out that

$$\chi_{\mathrm{Hom}_G(H^{h-1}{}_{\mathrm{LT}},\, \pi)^\vee}(b) = (-1)^{h-1}\, h \cdot \chi_\pi(g_b).$$

and put $\chi_{\mathrm{Hom}_G(H^{h-1}{}_{\mathrm{LT}},\, \pi)^\vee}(b) = h \cdot \chi_\vartheta(b)$. We can obtain our desired character relation.

Let L be a global field. Here,

- $\mathcal{A}(n, L)$ is the class of automorphic representations of $GLn(\mathbb{A}_L)$,
- $\mathcal{G}(n, L)$ is the set of equivalence classes of n-dimensional semi-simple ℓ-adic representations of $\mathrm{Gal}(\overline{L}/L)$.

The global Langlands program says that the following correspondence:

$$\begin{array}{ccc} \mathcal{A}(n, L) & \xleftrightarrow{1\text{–}1} & \mathcal{G}(n, L) \\ \Cup & & \Cup \\ \Pi & \longleftrightarrow & R_\Pi \end{array}$$

exists. If the global field L is totally real or a CM-field $L_0(\sqrt{r})$ for some totally real number field L_0 and some totally negative $r \in L_0$ then the above correspondence exists. Fix an automorphic representation $\Pi = \otimes_v \Pi_v$ of $GLn(\mathbb{A}_L)$. We have seen that the ℓ-adic representation $\mathrm{rec}_{n,L}(\Pi)$ of $\mathrm{Gal}(\overline{L}/L)$, when it exists, should have the property that

$$\mathrm{rec}_{n,L}(\Pi)\,|_{W_{L_v}} = \mathrm{rec}_{n,L_v}(\Pi_v)$$

for almost all v such that $\Pi_v \in \mathcal{A}^{unr}(n, L_v)$. Here, $\mathcal{A}^{unr}(n, L_v)$ is the class of unramified representations of $GLn(L_v)$. Denote the unramified subset of $\mathcal{G}(n, L_v)$ by $\mathcal{G}^{unr}(n, L_v)$. We can define a bijection

$$\mathrm{rec}_{n,L_v}\colon \mathcal{A}^{unr}(n, L_v) \xrightarrow{\simeq} \mathcal{G}^{unr}(n, L_v).$$

It enables us to state that

$$\mathrm{rec}_{n,L_v}\colon \mathcal{A}(n, L_v) \xrightarrow{\simeq} \mathcal{G}(n, L_v).$$

A local field k is the completion of a number field $K/\mathbb{Q}$, i.e., there exists a place v and $k = K_v$. We will think of $K = \mathbb{Q}(\theta)$. Let $\theta = a + b\sqrt{-1}$ $(a, b \in \mathbb{R})$. Denote its complex conjugate by $\overline{\theta} = a - b\sqrt{-1}$. Let $r = (\theta - \overline{\theta})^2 \leq 0$. Then $L_0 = \mathbb{Q}(\theta + \overline{\theta}, r)$ is totally real. So $L_0(\sqrt{r})$ is totally real or a CM-field. Since $(\theta + \overline{\theta}) + (\theta - \overline{\theta}) = 2\theta$, $\theta \in L_0(\sqrt{r})$. It turns out that $\mathbb{Q}(\theta) \subseteq L_0(\sqrt{r})$. We can say that there exists a totally real or a CM-field L and there is a place w of L such that $v \,|\, w$.

We shall ignore some subtle problems in the following discussion. Now, we shall consider that L_w is an extension of k. Namely, there exists a totally real or a CM-field L such that L_w/k for an arbitrary given local field k. Let $d = [L_w : k]$. It holds that

$$L_w = L_w \otimes_k k.$$

We have

$$\mathrm{Res}^k{}_{L_w} \colon \mathcal{G}(n,\,k) \longrightarrow \mathcal{G}(n,\,L_w); \quad \rho_k \longmapsto \rho_{L_w}$$

since $W_{L_w} \subset W_k$. On the other hand, we have

$$\mathrm{Res}_{L_w/k} \colon \mathcal{G}(n,\,L_w) \longrightarrow \mathcal{G}(n,\,L_w \otimes_k k); \;\; \rho_{L_w} \longmapsto (\mathrm{Ind}^{W_k}{}_{W_E}\, \rho_{L_w})^d.$$

We can say that $\mathrm{Res}_{L_w/k}$ is a bijection. Let $\mathbb{L}_n = \mathrm{Res}_{L_w/k}\, GLn$. The automorphic representation Π_w of $GLn(L_w)$ is identified with the automorphic representation Π of $\mathbb{L}_n(L_w)$. Thus, let $\Pi = (\pi)^d$ for $\pi \in \mathcal{A}(n,\,k)$ then $(\pi)^d$ is identified with Π_w. Since it holds that $\mathcal{A}(n,\,L_w) \overset{1:1}{\longleftrightarrow} \mathcal{G}(n,\,L_w)$,

$$(\pi)^d \;\overset{1:1}{\longleftrightarrow}\; \rho_{L_w} \;\overset{1:1}{\longleftrightarrow}\; (\mathrm{Ind}^{W_k}{}_{W_E}\, \rho_{L_w})^d.$$

So, we see that

$$\mathcal{A}(n,\,k) \ni \pi \;\overset{1:1}{\longleftrightarrow}\; \mathrm{Ind}^{W_k}{}_{W_E}\, \rho_{L_w} \in \mathcal{G}(n,\,k).$$

It enables us to state that

$$\mathcal{A}(n,\,k) \ni \pi \;\overset{1:1}{\longleftrightarrow}\; \rho_k \in \mathcal{G}(n,\,k).$$

Put $\mathrm{rec}_{n,K} = \prod_v \mathrm{rec}_{n,K_v}$. It realizes the global Langlands correspondence. We have seen that it is obtained via a totally real or a CM-field L. Thus, it must become our problem to show the global Langlands correspondence independently of the field L. We have seen that the local Langlands correspondence is shown in a purely local manner. This means that the global Langlands correspondence is obtained independently of the field L.

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