Electric Field and Divergence Theorem

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The divergence theorem states that the surface integral of the flux is equal to the volume integral of the divergence of the flux. This is not true if there is singularity in the volume integral. One example is the electric field flux described by Coulomb's law. The divergence of the electric flux is zero or undefined. Another example is the gravitational force. Consequently, Gauss's flux theorem is not applicable to the divergence of the electric field.

I. INTRODUCTION

In 1828, George Green privately published a number of integral theorems relating volume integrals to surface integrals[1]. Michael V. Ostrogradskii also presented a paper containing divergence theorem to St. Petersburg Academy in 1828 and published the paper in 1831[2].

The divergence theorem is important particularly in electrostatics and fluid dynamics. The theorem states that the surface integral of a flux vector is equivalent to the volume integral of the divergence of this flux vector.

However, the volume integral can not be evaluated if there is any singularity in the volume. The volume integral may be a random value for certain flux such as the electric field flux described by Coulomb's law[3] while the surface integral is a fixed value.

II. PROOF

A. Divergence Theorem

The divergence theorem states that

$$\iint \vec{E} \bullet \, d\vec{S} = \iiint \vec{\nabla} \bullet \vec{E} \, dV \tag{1}$$

The theorem assumes there is no singularity in both integrals. $\vec{E}$ should be differentiable over the surface. $\vec{\nabla} \bullet \vec{E}$ should be differentiable over the volume.

For example, if

$$\vec{\nabla} \bullet \vec{E} = \frac{1}{r} \tag{2}$$

diverges at $r = 0$, the divergence theorem is not applicable to $\vec{E}$.

Furthermore, the divergence is zero or undefined for any flux that follows the inverse square law. For example, gravitational force and electric force.

B. Coulomb's Law

in 1785, the French physicist Charles-Augustin de Coulomb published his first three reports of electricity and magnetism where he stated his law. According to his law, the electric force from a point charge of Q to a point charge of q is represented by

$$\vec{F} = k\frac{Qq}{r^3}\vec{r} \tag{3}$$

with $\vec{r}$ as the displacement vector from charge Q to charge q.

The electric field from the point charge of Q toward point charge of q is defined as

$$\vec{E} = \frac{\vec{F}}{q} = k\frac{Q}{r^3}\vec{r} \tag{4}$$

$$E = k\frac{Q}{r^2} \tag{5}$$

The divergence of $\vec{E}$ in spherical coordinate is

$$\vec{\nabla} \bullet \vec{E} = \vec{\nabla} \bullet \frac{kQ}{r^3}\vec{r} \tag{6}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{kQ}{r^2}) \tag{7}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(kQ) \tag{8}$$

$$= 0 \tag{9}$$

The divergence of the electric field is zero except at $\vec{r} = 0$. The divergence can be any value if $\vec{r} = 0$.

From equations (6,9), the volume integral of the divergence of the electric field is undefined.

$$\iiint \vec{\nabla} \bullet \vec{E} \, dV = undefined \tag{10}$$

By definition, the electric field is in the same direction of the electric force. Coulomb's law requires two charges for the existence of the electric force. Without point charge of q, there is neither force nor field from point charge of Q.

The volume integral of the divergence of the electric field is undefined.

C. Charge Distribution

In electrostatics, the electric charge is distributed over a finite volume. Let $q(\vec{s})$ be the charge density at $\vec{s}$. From equation (4), the electric field can be represented as

$$\vec{E}(\vec{r}) = k \iiint \frac{q(\vec{s})}{(|\,\vec{r}-\vec{s}\,|)^3}(\vec{r}-\vec{s})\,dV_s \tag{11}$$

Evaluate divergence in Cartesian coordinate,

$$\vec{\nabla} \bullet \frac{\vec{r}-\vec{s}}{(|\,\vec{r}-\vec{s}\,|)^3} \tag{12}$$

$$= (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \bullet \frac{(x-s_x, y-s_y, z-s_z)}{(|\,(x-s_x, y-s_y, z-s_z)\,|)^3} \tag{13}$$

$$= 0 \tag{14}$$

At $\vec{r}=0$, the evaluation produces arbitrary value.

From equations (11,12,14), the divergence of $\vec{E}(\vec{r})$ is

$$\vec{\nabla} \bullet \vec{E}(\vec{r}) \tag{15}$$

$$= k \iiint q(\vec{s}) \vec{\nabla} \bullet \frac{\vec{r}-\vec{s}}{(|\,\vec{r}-\vec{s}\,|)^3}\,dV_s \tag{16}$$

The divergence of the electric field is either zero or undefined for any charge distribution.

D. Capacitor

Let the charge be distributed uniformly over a sphere. A capacitor can be formed with two concentric charged spheres. The electric force exists between the spheres if the spheres carry charges of opposite sign. The electric field is zero outside the region between the spheres.

Apply the divergence theorem to this capacitor by choosing a sphere of radius R enclosing the inner charged sphere but not the outer charged sphere. Due to spherical symmetry, the electric field is a function of the radial distance from the center of both charged spheres.

$$\vec{E}(\vec{r}) = E(r)\frac{\vec{r}}{r} \tag{17}$$

The surface integral is a fixed value.

$$\iint \vec{E} \bullet\, d\vec{S} = \iint E\,dS = E(R)4\pi R^2 \tag{18}$$

$$\neq 0 \tag{19}$$

From equation (16), the divergence of the electric field for any distribution of charge is zero or a random number. The volume integral over this sphere is a random number.

$$\iiint \vec{\nabla} \bullet \vec{E}\,dV = undefined \tag{20}$$

From equations (19,20),

$$\iint \vec{E} \bullet\, d\vec{S} \neq \iiint \vec{\nabla} \bullet \vec{E}\,dV \tag{21}$$

The surface integral is not equal to the volume integral because the volume integral is a random number.

The divergence theorem is not applicable to a concentric capacitor.

III. CONCLUSION

The divergence theorem describes differentiable flux. The theorem fails if the divergence of the flux becomes singular in the volume integral.

The theorem is not applicable to the electric field flux described by Coulomb's law because the divergence of the electric field is zero or undefined for any charge distribution.

Gauss's flux theorem[4] is based on the divergence theorem. Consequently, Gauss's law[4] in differential form is invalid for electric field.

[1] Charles H.Stolze: "A history of the divergence theorem". https://www.sciencedirect.com/science/article/pii/0315086078902124

[2] Victor J. Katz (May1979) "The history of Stokes' theorem," Archived April 2, 2015, at the Wayback Machine Mathematics Magazine, 52(3): 146156; for Ostragradsky's proof of the divergence theorem, see pages 147148. https://web.archive.org/web/20150402154904/http://www-personal.umich.edu/ madeland/math255/files/Stokes-Katz.pdf

[3] Coulomb (1785a) "Premier mmoire sur llectricit et le magntisme, " Histoire de lAcadmie Royale des Sciences, pp. 569577 https://books.google.com/books?id=by5EAAAAcAAJ&

pg=PA569#v=onepage&q&f=false
[4] Duhem, Pierre. Leons sur l'lectricit et le magntisme (in French). vol. 1, ch. 4, p. 2223. shows that Lagrange has priority over Gauss. Others after Gauss discovered "Gauss' Law", too. https://archive.org/stream/ leonssurllec01duheuoft#page/22/mode/2up
[5] Eric Su: List of Publications, http://vixra.org/author/eric_su