PHYSICAL PRINCIPLES IN REVEALING THE WORKING MECHANISMS OF BRAIN

PART ONE

by

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Abstract. This work is addressed to a wide range of scientists who approach the research of the human brain from different points of view. Our main point is that no matter of the angle of approach in the brain research, a scientist has to be aware of the physical possibility of working of the brain. We describe this possibility by modeling the brain as light. The essential physical property of this model is fractality. It will be physically explained for the light itself and then applied as such to the brain. The main brain functions: the memory, acquiring information, and handling this information are then explained as scale transient fractal phenomena. This physical model is thereby useful in guiding any research on brain, no matter of its nature.

Keywords: neuron, light ray, universe, luxon, Madelung fluid, theory of interpretation, theory of memory, holography, hologram

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1. Introduction

Our task with the present work is twofold: first, to consider critically what we can carry, from the fundamental physics at large, over to the specific physics of brain; secondly, with a proper selection thus done, to explain the basic functions of the brain, *i.e.* acquiring and handling the information, and, more importantly, the memory. Useless to say, these two tasks are in fact heavily entangled with each other, so that our presentation follows them only as guidance: at any point of discourse we have in mind both the physics and the world of brain on equal footing.

Up to this point in our life, the physics can be categorized as the science describing what we can perceive outside ourselves, *i.e*. the material universe. Transposed over to the brain, this science has created that part anatomy of brain, which defines it as structured matter: gray matter, white matter, neurons, axons, dendrites and such. But this means dead matter, for here we have to do with the matter accessible to our senses, either in a mediated way, or directly. Once we need to describe the functioning brain, we have to define it as a universe, and from this moment on the trouble starts brewing: *this is not a physical universe as we know it*. For, in the case of a physical universe, the task of describing it is a lot easier: by our senses, we have pretty much at our disposal the structural formations of matter representing the constitutive parts of the universe – atoms, molecules, planetary system, galaxy, metagalaxy, *etc*. – together with some of their possible connections, and their space and time scales' dependence which explain physically the constitution of the universe as a whole. This is not the case for the living brain: it is a universe that cannot be studied but only from outside on both the accounts of *functionality* and *connectivity*. So one of the fundamental problems of physics becomes critical here: *how de we define a universe in general*? We have an answer to this question, and this answer provides a criterion of definition of brain as a universe. This definition, in turn, provides criteria of choice of the physical principles involved in the physics of brain, and their adaptations to this physics.

As a living universe, the brain produces and processes what we know physically – and handle ourselves routinely from the medical point of view – as *electric* and *magnetic fields*. Indeed, from a noninvasive point of view – the only way to preserve life during research – we have at our disposal for the study of brain two main methods, generated by the nature of the brain activity. The first – the electro-encephalography (EEG) – exploits the electrical activity of the brain, while the second – the magneto-encephalography (MEG) – exploits the magnetic activity of the brain. If we do not consider the magnetic field as 'invasive', then the magnetic resonance imaging (MRI) and functional magnetic resonance imaging (fMRI) can also count as valid methods of study under 'noninvasive' description. The experience generated by these methods has long ago produced pertinent general conclusions about the universe we call brain.

We can only infer that these fields are involved in both the connectivity and the physical structure of the local neural populations, synchronized as it were, in order to form structural constitutive units in the human brain. Concepts like 'amount of synchrony', 'coherence', 'time series', have a precise physical meaning. There seems to be no doubt that the physics is involved here through electrodynamics. But, according to the orthodox view on this part of physics, such an application would mean that the fields are always considered as a consequence of some motion, and the induction phenomenon is orderly in the brain. We challenge this idea in a specific way, with the hope that there is something positive in 'statics' rather than only in 'kinematics', or even 'dynamics': electro-statics and, above all, magneto-statics, properly handled are liable to show a way to solution of functionality of the brain. We relate this statics to the *concept of phase*, and then attach to it the concepts listed above and then some others to be presented in due time.

1.1 The Strategy of Approach

The main point to be observed from the physics we have at our disposal thus far, is then its way to construct the image of the universe around us. The light, in any and all of its instances, is the essential structure of this universe, allowing the explanation of its very structure. The light gives the connections in the universe, the light stores the information transmitted in the universe in the form of a memory, the light is the epitome of interactions between the fundamental structures of the universe. Most of these properties have been revealed based on the idea of confinement of light, experimentally realized in the form of a Wien-Lummer enclosure, which made light into a thermodynamical system (Wien & Lummer, 1895). The research of this system was able to produce the quantum physics with all its consequences. Somewhere along this path, a fundamental property of light has been discovered: *the invariance with respect to dimensions of the Wien-Lummer enclosure*. This means that the light as a thermodynamical system has some special properties which are the same in microcosmos, at the scale of the laboratory, and in the universe at large, which thus becomes simply a Wien-Lummer box of a particular dimension.

The world of brain is naturally confined in such an enclosure: *the skull*. This is our starting point with the present research, but there is an essential difference with respect to the physical case of thermodynamic light that inspired us: *there is no scale transition here*. All skulls of human population have just about the same dimensions, so that the variation in dimensions can be only of a statistical nature. The problem then becomes apparently simpler: *describe those processes that are only statistically invariant*, in order to present the connectivity of the brain from a physical point of view. Then we find that the essential statistically invariant description of light, which can be safely translated into brain is the concept of *light ray*. The theory of light started with this concept in the first place, but we shall show here that the physics of brain adds more to it, along the idea of fundamental physical structure.

1.2 The Fundamental Constitutive Unit of Brain

The description of living brain should start from the functionality of its fundamental anatomical unit: *the neuron*. This is to be taken as one or more tubes attached to a soma, and this is the concept that epitomizes in the case of brain world what we know as a light ray. The problem comes down to the statistical description of the light ray, and we shall concentrate on it, in order the write down the required mathematics. The concepts of memory, connection between neurons, and the structure of the electro-magnetic fields involved in the brain physics should then come out only naturally along this line, and we show here that they are coming indeed.

2. The Physical Description of a Universe

Rarely, if ever, is theoretical physics concerned with the definition of coordinates: the usual consent is that the coordinates exist, they do have the meaning we happen to assign to them, and the theoretical results can be expressed in such a way that, when it comes to verification, there is always a correspondence with the reality of things described by those coordinates. It is on this state of the case that the modern global positioning on Earth came to remind us that the special relativity, with all its specific requirements, actually represents a modality of defining the coordinates physically, by a condition of equivalence between them. This condition is always tacitly assumed in the classical cases, but never explicitly stated: the space coordinates are defined, first and foremost, by coordinate lines, along which we need to pinpoint some values uniquely corresponding to a position in space. However, for a holistic theory of the universe – of the kind we need to consider in order to describe the living brain – such a definition of the coordinates is not enough.

First of all, the brain is confined to skull, and if it is that some analogy with the radiation defining the relativistic coordinate systems to be involved here, one needs a description of radiation from thermodynamical point of view, which involves an enclosure containing it. Then we need to choose that description of the thermal radiation which proved to be invariant to the changing in dimensions of the enclosure, and imposed the modern cosmology. Historically, such an addition to the reference frame was realized in the form of Wien-Lummer enclosure for radiation studies (Wien & Lummer, 1895). This device has had a crucial role in establishing the radiation laws, and their concordance with the observed properties of the radiation at large. That concordance, in the form of Wien's displacement law, contributed in establishing the quantum theory as a natural theory of light as we know it today.

It is along this line of historical development of physics, that the theories of the light ray have been improved, until they reached the discovery of a new phenomenon, namely the *holography*, to be added to classical ones: reflection, refraction and diffraction. The holography allows then a natural description of the concept of memory, a description whose main promoter was the renowned psychiatrist and neurosurgeon Karl Pribram. And the holographic theory of the brain is entirely based on the quantum theory, once it is based on the idea of a hologram (Pribram, 2007). Therefore, if we need to involve the physics in the study of brain, we need first and foremost a coordinate system adapted to the skull, as representing the enclosure of the universe called 'brain'. This enclosure is our reference frame.

2.1 The Idea of Coordinate System in a Universe

Let us start with the idea of coordinate system. In order to describe the spatial position in a certain reference frame, one always needs such a concept, usually connected to a particular geometry that offers the meaning of coordinates. This is the typical case in physics, and it even became cursory, to the point where sometimes the coordinates are only mentioned, with no precaution of defining them in a way or another. The modern idea of global positioning came to impose a closer consideration of the concept of *space position* itself, in the definition of which the idea of light ray needs to be taken as a fundamental concept. As it happens, the skull correlations revealed in EEG and MEG (Pribram, 1998), seem to uncover the fact that such a concept of ray, is to be somehow connected to the neuron. We take this idea for granted based on the consideration that follow.

The best idea of definition of a system of coordinates, in our opinion, is that of Bartolomé Coll who aims at defining physically the coordinate lines, and actually builds a general natural philosophy, as it were, to be followed in such a construction in any universe (Coll, 1985). We shall apply Coll's philosophy, but only for the three-dimensional case. It is in order to make this philosophy amenable for the physics of brain, that we reproduce here three essential endnotes from the Coll's work just cited. First, comes the idea of *lines of coordinates*:

> Typically, the definition domains of such systems correspond to *world tubes* obtained by evolution of *space-like tetrahedral figures*, over whose four faces, at every instant, *light beams fall on*. (Coll, 1985, Endnote [6], our emphasis)

In the present work we use 'tetrahedral figures' in Euclidean reference frames where only one of the faces of tetrahedron is to be taken into consideration. This way eight tubes will be constructed for each and every reference frame. The structure of these light beam should be, and it is indeed, as 'physical' as possible. For us it is the classical light ray, to be described in detail in due time, and completed with a physical interpretation due to Louis de Broglie (de Broglie, 1927, b, c). The word 'interpretation' is taken here in the precise meaning necessary for the construction of wave mechanics (Darwin, 1927). Only, de Broglie's 'wave phenomenon called material point' gets here baptized by Bartolomé Coll as 'luxon', a name that we adopt without reserve. Quoting:

> Here we consider light in the *geometric optics* approximation, that is to say, *as a fluid* of point like "luxons" (Coll, 1985, Endnote [7], our emphasis).

Therefore, along the world tube representing a light beam, particularly a light ray, the luxons are traveling, and the light wave is laterally limited by the tube. Louis de Broglie then defines the world tube as a *capillary tube*, with the simultaneous luxons defining a surface evolving along the tube, just as Newton did for his definition of the light ray (Newton 1952, p. 1), thereby generating the whole physics of light. Among other things it becomes possible to define a physical coordinate line with respect to this capillary tube: it is the cordinate normal to the surface determined by simultaneously travelling luxons. Quoting, again:

> … the theoretical or experimental conclusion that a specific physical field, under particular conditions, depends only on the variable *r* is void or, at least, confuse, if the nature of the coordinate line "*r* variable" is not precised (radial, cylindrical, angular or other). And, from the experimental point of view, we need to describe *the physical procedure for its construction*. It is this sense that has, for us here, the word *operational* (Coll, 1985, Endnote [4], our emphasis).

This is the meaning of the word 'operational' for us too, all along the present work. The neuron will be a de Broglie capillary tube along whose dendrites and axons, some luxons are circulating. Only, for the neuron, the luxons must be particularly defined, and this is the point where the cosmology enters the stage. For, we shall carry out this definition in relation with the physical structure of the universe represented by the living brain, after the physical example of the universe at large. But, let us first see what the de Broglie light ray involves from a physical point of view.

2.2 The Louis de Broglie's Light Ray

Our first task here is, therefore, to describe a light ray – the neuron – which can be imagined as a tube of trajectories defined by the motion of some material points – which, following Bartolomé Coll, we label generically as *luxons* – along it. As we said, this task has already been accomplished in physics by Louis de Broglie who, aiming at proving a certain noncontradiction between the concepts of wave and particle, has actually brought a much greater service to knowledge in general (de Broglie, 1927). Let us first see the facts.

At the time when he produced the works just cited, Louis de Broglie was engaged in proving explicitly that there is no gap between geometrical optics and quantum theory. The specific problem at that time was, in de Broglie's idea, to prove that the light can be seen as a flux of photons, and he intended to show that this image contradicts neither the optical nor the mechanical rules of thinking alike. The optical rules were considered all concentrated in the representation of propagation of light, as described, for the case of vacuum, by the D'Alembert equation:

$$\Delta u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{2.1}$$

In this context, Louis de Broglie took notice of the fact that an *optical solution* of the equation (2.1) should be written in the form:

$$u(\boldsymbol{x},t) = a(\boldsymbol{x})e^{i\omega[t-\phi(\boldsymbol{x})]} \tag{2.2}$$

which must then be submitted to some space constraints, evidently mandatory in optics by the presence of screens, diopters, or some other obstacles met by light in space. Then, de Broglie was forced to consider the light as a fluid of particles, for the incarnation of which the best candidate seemed at the moment the idea of photon, 'floating in the air' so to speak, for at that moment of time the photon was just getting baptized (Lewis, 1926). The light ray should therefore be taken as describing a fluid of particles.

Now, by taking the light quanta as those material particles able to explain, from a classical point of view, the particulate structure of light, de Broglie noticed that one needs to assume a solution of equation (2.1), having nonetheless not only the phase, but also the amplitude time dependent:

$$u(\boldsymbol{x},t) = f(\boldsymbol{x},t)e^{i\omega[t-\phi(x)]} \tag{2.3}$$

Here ϕ is the same function as in (2.2), embodying the earlier idea of Louis de Broglie himself, that the corpuscles and their representative waves have *the same phase* (de Broglie, 1923). Why should now the amplitude be variable with time?

De Broglie gives an explanation in the English version of the work cited (de Broglie, 1926c), and this can be summarized as follows: such an elementary particle must be described by a field satisfying the Klein-Gordon equation, not the D'Alembert's. This defines, choosing words of de Broglie himself, «the wave phenomenon called 'material point'», and can be written as:

$$\Delta u(\boldsymbol{x},t) - \frac{1}{c^2}\frac{\partial^2 u(\boldsymbol{x},t)}{\partial t^2} = \frac{\omega^2}{c^2}u(\boldsymbol{x},t); \quad \hbar\omega \equiv m_0 c^2 \tag{2.4}$$

The last identity in this equation represents de Broglie's initial idea from 1923, *apparently* prompted by the relativistic mechanics, according to which one can associate, via energy, a frequency to a classical material point: the *de Broglie's frequency*. Now, the fundamental solution of equation (2.4), based on which one can build the general solution as a linear combination according to mathematical rules, is taken by de Broglie in the general form:

$$u_v(\boldsymbol{x},t) = \frac{u_v}{\sqrt{x^2+y^2+\gamma^2(z-vt)^2}} \cdot \sin\left[\omega\left(t - \frac{\beta}{c}z\right)\right] \tag{2.5}$$

with a_v a constant, $\beta \equiv v/c$ and $\gamma^2(1-\beta^2) \equiv 1$, and the direction of motion chosen as the axis z of the reference frame. No doubt, the general solution of (2.4) can be taken as being a linear combination of waves of the form (2.5), having different velocities. However, any of these should have a spacetime singularity: at the event that locates the 'material point' in motion with respect to a origin of space coordinates and time, its amplitude becomes infinite. Thus, when considering the classical material point a 'wave phenomenon', if this wave phenomenon is classically located as an event, *i.e. interpreted* as a particle, the representative wave of this particle has a specific singularity at its location: its amplitude becomes infinite. In other words, by *interpretation*, the very concept of wave acquires here a differentia, for the particle itself gets new properties above and beyond its usual classical depiction as a position endowed with mass or charge: it is a *singularity of the wave amplitude, whereby this one becomes infinite*.

However, from a 'phenomenological' point of view, we might say, the things are to be presented in quite a different manner. The wave is here a light wave, and it should be the locus, in a proper geometrical sense, of an ensemble of events representing the «wave phenomena called 'material points'». The linearity of the Klein-Gordon equation allows a superposition of wave phenomena represented by (2.5) *with different velocities*, but there is a problem: as all of the material points move with the speed of light, one has $\beta \to 1$, and thus $\gamma \to \infty$, for all the waves of this type. So the resultant wave, if represented by a linear combination of such 'wave phenomena', must have a rather *vanishing amplitude* no matter where the material points representing the light are located in space and time. Thus, while the classical the trajectory of the material point is the locus of successive positions of a material point in motion, in a wave representation of 'the phenomenon called material point' it is simply the locus of the events where the *amplitude of the wave vanishes*, *no matter of the time sequence and space locations of these events*. Therefore, along the space line (continuously or not) considered on behalf of a trajectory of those 'wave phenomena representing a classical material point', as Louis de Broglie has characterized this concept, not only the amplitude happens to have discriminated values, but the phase should also be arbitrary according to this optical representation, as the amplitude vanishes.

A little digression may be in order here: the phase and the amplitude of the optical elongation (2.5), which allowed the preceding speculations, are quite particular. However, only as such particulars allowed they the very construction of the special relativity, based on D'Alembert equation (Lorentz, 1904, 1916). Thus, against these speculations one might raise the objection that the representation (2.5) is just as… special as the relativity is, and a general definition of the optical signal in the form $A(\boldsymbol{x},t)\cdot e^{i\phi(\boldsymbol{x},t)}$, may render them obsolete. Two things have to be considered, however, when engaging along this line of thought. First, for a general signal like the above one, satisfying any desires of generality for both amplitude and phase, we do not have an equation of motion: in wave-mechanical terms, such a signal *is not interpretable yet* (Darwin, 1927). It is only when such a function becomes interpretable by ensembles of free particles that it becomes useful to physics. Uncovering a general equation to be satisfied by such a general function is vital for the natural philosophy in general, for then, based on it, one can argue just 'how special' is the special relativity, *viz*. can give some reasons for a general… special relativity, so to speak, ideally even to find its general formulation. As it turns out, such a function satisfies the free particle Schrödinger equation (Schleich, Greenberger, Kobe, & Scully, 2013), given some natural conditions that define what came to be known in physics as a *Madelung fluid* (Madelung, 1927). Secondly, as Dirac once has noticed in his works which inspired a certain wave-mechanical approach to the idea of magnetic charge (Dirac, 1931, 1948), a general spatial geometric locus of zero amplitude of the signal might be instrumental for the condition of quantization based on the concept of wave function satisfying a Schrödinger equation for the free particle. Therefore, we should not avoid such a line of thought, by any means.

However, Louis de Broglie has elaborated on another observation, in concordance with his own idea of *phase waves*. Notice the fact that if the amplitude function f is to have any mobile singularities whatsoever, we need to decide the nature of them: are they spacetime singularities or 'phenomenological' singularities. In order to decide their nature, notice that they have to move *across the surface of constant phase*, particularly normally to this surface. In this case the nature of singularity can be directly settled by the amplitude, for the speed of a material point in a position M at the time t is, according to the mathematical laws, necessarily of the form:

$$U(\boldsymbol{x},t)=\left.\frac{\partial_t f(\boldsymbol{x},t)}{\partial_n f(\boldsymbol{x},t)}\right|_{M,t} \tag{2.6}$$

Here the variable n is taken *along the very trajectory of the material point* – the symbol n is here intended to suggest the idea of 'normal' to the wave surface – and the partial derivative upon time ($\partial_t f$) as well as that along the normal direction ($\partial_n f$), are taken in the position M at the moment t.

In order to use this definition, we need a few partial results of our optical and wave-mechanical representations. Thus, substituting (2.2) and (2.3) in equation (2.1), and making the imaginary parts of the relations thus obtained vanish (the physical results have to be real at any rate!), one can find the following equations connecting the *optical amplitude A* and *particle amplitude f* to *phase ϕ*:

$$\frac{2}{A}\frac{\partial A}{\partial n}\equiv\frac{1}{A^2}\frac{\partial(A^2)}{\partial n}=-\frac{\Delta\phi}{\partial_n\phi} \tag{2.7}$$

and

$$c^2\left(2\frac{\partial\phi}{\partial n}\frac{\partial f}{\partial n}+f\Delta\phi\right)=-2\frac{\partial f}{\partial t} \tag{2.8}$$

Then we simply have, as de Broglie noticed, that the equation (2.7) will describe the diffraction phenomena *according to physical optics*, while the equation (2.8) will describe the diffraction phenomena *according to* *quantum theory*, *i.e. by an ensemble of particles*, even though with this last concept taken as a classical material point. It should be indeed all about diffraction, forasmuch as we have to deal here with a space locus of events distributed in space, and not with a classical trajectory *per se*. Therefore, this is indeed an *interpretation of the wave* in the acceptance of the definition given by Charles Galton Darwin.

However, in the French version of his work (de Broglie, 1927, c), Louis de Broglie assumes that if, as one *approaches at constant time* a light particle following its trajectory, the function f varies as the reciprocal distance to that particle, then in the position M of the particle the *ratio between* f *and* $(\partial_n f)$ *vanishes*. This fact obviously generalizes the one represented by the equation (2.5), so that it can be taken as typical for the wave mechanics. Under this condition of space behavior of the amplitude, the equation (2.8) gives a special expression for the light particle velocity in a certain position, and this expression befits the classical character of phase. Indeed, using the equation (2.6) and the Louis de Broglie's condition of 'approaching the point at constant time' in the form: $f/(\partial_n f) \to 0$, the formula for this velocity reveals the important fact that *the phase should be a potential of velocities*, *i.e.* it should assume the very *classical role of the variable of action*:

$$U(\boldsymbol{x},t) = c^2 \left.\frac{\partial\phi}{\partial n}\right|_{M,t} \tag{2.9}$$

Thus, the only thing left for explanation in this case, would be the construction of *a physical light ray*, and this can be classically understood as a thin pencil of trajectories of classical material points. So, de Broglie came to the idea that an *infinitely thin tube confining an ensemble of trajectories of light particles* would be able to do the job. Thus, the classical Newtonian image – or to be more precise, the Hookean image – of the physical light ray takes, within de Broglie's description, a geometrically precise modern shape: *a generalized cylinder, whose area of any transversal section is variable with the position along the ray*.

And so it comes that de Broglie *assimilates a physical ray with a capillary tube of variable cross-section* σ, and he describes this tube by the known physical principles of the *theory of capillarity*. Assuming, for instance, that the flux of light particles is conserved along the ray – an assumption that can, in general, be taken as the fundamental attribute of the concept of ray within the theory of fluids – the equation representing this situation:

$$\rho U\sigma = const \tag{2.10}$$

should be satisfied, where ρ means the Newtonian volume density of the particles of light. Taking the logarithmic derivative in the direction of the ray, one can find

$$\frac{1}{\rho}\frac{\partial\rho}{\partial n} + \frac{1}{U}\frac{\partial U}{\partial n} + \frac{1}{\sigma}\frac{\partial\sigma}{\partial n} = 0 \tag{2.11}$$

By a "known theorem of geometry", as de Broglie declares, one can calculate the last term here. Now, because the physical ray is a space construction, it would be hard to decide *the meaning of* $\partial/\partial n$ – is it effectively variation along the ray itself, or along the normal to the wave surface as de Broglie assumes!? – but to a good approximation we can take that it means variation along the normal, to start with. It is, indeed, *only in this case that we can take advantage of that 'known theorem'* to which de Broglie alluded, and according to which the last term in (2.11) is the double of the mean curvature of the surface $\phi = const$ in a given position (Mazilu, Agop, & Mercheș, 2019, Chapter 3). That quantity has as expression the sum of the principal curvatures of the surface:

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{\Delta\phi - \partial_n^2\phi}{\partial_n\phi} \tag{2.12}$$

Here $R_{1,2}$ are the radii of curvature of the principal sections of the surface. With (2.9) and (2.12), the equation (2.11) now takes the form

$$\frac{1}{\rho}\frac{\partial \rho}{\partial n} = -\frac{\Delta \phi}{\partial_n \phi} \tag{2.13}$$

and comparing this with (2.7), one finds

$$\rho = const \cdot A^2 \tag{2.14}$$

Thus, Louis de Broglie has the essential result of interpreting the physical optics based solely on diffraction phenomena *without making any reference to the idea of harmonic oscillator* and its classical dynamics in order to calculate the intensity of light. Indeed, the equation (2.14) shows that the density of the light particles conceived as classical material points, or localized quanta as it were, is proportional with the intensity of the classical theory of light. The interference phenomena are, therefore, explained by the corpuscular theory, just as well as the difraction phenomena, provided we add to the wave mathematical image a necessary property deriving from the wave representation of classical material point: the ratio between the amplitude of the wave and the normal derivative of its wave surface, taken at constant time, vanishes in the position of the 'wave phenomenon called material point'.

An issue is still lurking in the background though, even a fundamental gnoseological issue for that matter: was this effort of mathematics and stretch of imagination necessary at all for our knowledge? From the point of view of the continuity of the knowledge, the answer is definitely affirmative. Indeed, making reference to the harmonic oscillator in the case of light – in order to interpret the intensity of light, for instance, to say nothing of some other physically fundamental necessities – is, by stretching a little the meaning of word, 'illegal'. For, as a purely dynamical system, the harmonic oscillator is a dynamical system described by forces proportional with displacements (Hooke-type elastic forces), and in the case of physical optics the second principle of dynamics is only incidental, being introduced only by a property of transcendence of the second order ordinary differential equation: it describes any type of periodic processes. And the fact is, that in the foundations of modern physical optics, the periodic processes of diffraction have more to do with the theory of statistics than with the dynamics (Fresnel, 1827).

This is, however, not to say that the harmonic oscillator is to be abandoned altogether, as a model, because it is not the case, either from experimental point of view, or theoretically. All we want to say is that we need to find its right place and form of expression in the theory, and this is indicated again through the order imposed by the measure of things, this time as their mass. Indeed, dynamically, the second order differential equation involves a finite mass. On the other hand, for the light, the mass – in any of its two capacities, gravitational or inertial – is conspicuously absent, and if the second order differential equation is imposed by adding the diffraction to the phenomenology of light, this means that it describes actually *a transcendence between finite and infinitesimal scales of mass*. As it turns out though, a universe with constitutive rest particles having negligible mass, is just as legal as a universe having negligible charge, like the universe we live in. The brain is just such a universe!

2.3 The Madelung Fluid of Luxons

There is not too much to say over what was just said in the previous §2.2, in order to catch the general idea that the physical theory of a classical light ray, as completed by Louis de Broglie, can be taken as the theory of a physical ray in general. One just needs a few further additions in order to bring it in the position of helping in operationally defining a coordinate system, according to the ideas of Bartolomé Coll and his collaborators (Coll, 1985, 2001; Coll & Morales, 1991). Of these additions we can recognize a few right away.

First of all comes the interpretation: Louis de Broglie just showed that adding particles to the optical idea of waves, produces the optical formula according to which the density should be proportional to the square of the amplitude of signal representing a wave. Then everything comes down to the *interpretation of the wave function*, which should be part and parcel of a *general interpretation process*, and this is the moment where the idea of ensemble makes its proper entrance in the argument. Like in all of the classical cases, the ensemble enters first by its historical element – the classical material point – just as in the quintessential physical case of classical ideal gas. It is time now that we turn to Charles Galton Darwin for a brilliant choice of the proper words characterizing the physical situation:

> It is almost impossible *to describe the result of any experiment except in terms of particles* – a scintillation, a deposit on a plate, etc. – and this language is quite *incompatible with the language of waves*, which is used in the solution. A necessary part of the discussion of any problem is therefore *the translation of the formal mathematical solution, which is in wave form, into terms of particles*. We *shall call this process the interpretation*, and only use the word in this technical sense [(Darwin, 1927); *our Italics*]

Notice, in this context, the necessity of the presence of a surface in order to support, as it were, the records of a certain experiment. This calls for a second addition we have in mind in order to properly complete the theory of physical rays according to de Broglie's philosophy, namely the concept of *physical surface*. To Louis de Broglie this was a vave surface limited by a capillary tube, and thus having a limited surface area.

The definition of a physical surface does not involve just its geometrical properties, but should also include some physical ones. For instance the surface in a fluid has superficial tension, and this varies with the electric and magnetic state of the fluid. Better yet, in order to describe a holographic universe like the brain, one needs to insure physical properties involving the idea of memory, and this needs a special description of the physics of a surface. We shall turn to these issues in due time.

Meanwhile we just need to find what is happening if in the de Broglie's theory we do not use the idea of frequency. Fact is that any signal represented in the complex form

$$\psi(\boldsymbol{x},t) = A(\boldsymbol{x},t)\cdot e^{i\phi(x,t)} \tag{2.15}$$

is the solution of an differential equation resembling the free particle Schrödinger equation, provided some specific conditions are satisfied (Schleich, Greenberger, Kobe & Scully, 2013). Indeed, the complex form we are talking about here assumes an *implicit time and space dependence* for the phase, as well as for the amplitude of the signal, resulting in *the identity*:

$$\begin{aligned}\frac{i\dfrac{\partial\psi(\boldsymbol{x},t)}{\partial t}+\beta\cdot\nabla^2\psi(\boldsymbol{x},t)}{\psi(\boldsymbol{x},t)} &= -\left(\frac{\partial\phi}{\partial t}+\beta\cdot(\nabla\phi)^2-\beta\cdot\frac{\nabla^2 A}{A}\right)\\ &\quad+\frac{i}{2A^2}\cdot\left(\frac{\partial A^2}{\partial t}+2\beta\nabla\cdot(A^2\nabla\phi)\right)\end{aligned} \tag{2.16}$$

where β is a constant having the physical dimensions of a rate of area (m^2/s). The 'specific conditions' necessary in order that $\psi(\boldsymbol{x},t)$ be a solution of the time dependent Schrödinger-type equation for the free particle, to wit:

$$i\frac{\partial\psi(\boldsymbol{x},t)}{\partial t}+\beta\cdot\nabla^2\psi(\boldsymbol{x},t)=0 \tag{2.17}$$

are in fact the two known equations that guarantee the vanishing of the right hand side of equation (2.16), so that (2.17) can take place:

$$\begin{aligned}\frac{\partial\phi(\boldsymbol{x},t)}{\partial t}+\beta\cdot[\nabla\phi(\boldsymbol{x},t)]^2-\beta\cdot\frac{\nabla^2 A(\boldsymbol{x},t)}{A(\boldsymbol{x},t)}&=0\\ \frac{\partial A^2(\boldsymbol{x},t)}{\partial t}+2\beta\nabla\cdot[A^2\nabla\phi(\boldsymbol{x},t)]&=0\end{aligned} \tag{2.18}$$

The first of these equations is the equivalent of classical Hamilton-Jacobi equation. The second one, is a continuity equation for the cases where the square of the amplitude of the function $\psi(\boldsymbol{x},t)$ can be taken as a density. Obviously, according to Louis de Broglie's theory just presented above, the classical light ray is one of such cases, if it is conceived as a flux of luxons – to use our agreed terminology – in a thin capillary tube, and if the *moving phase surface* in this capillary tube is the *locus of vanishing of the ratio between the amplitude of the wave and its normal derivative*.

Now, an interesting turn of the tide in physics comes with these conclusions. Usually, de Broglie's doctrine is connected with the objective correlation between particle and wave. On the other hand, the Schrödinger doctrine is usually connected with the subjective correlation, according to which the wave describes the probability of presence of a particle in a place from the universe. And here we are now, with the conclusion of a necessary logical completion of the definition of classical light ray, enforced, as it were, by the phenomenology. This asks for a precise connection between the amplitude of the objective signal representing the light, and the density of the fluid of classical particles helping in the physical interpretation of light. To wit, according to Louis de Broglie, the density must be proportional with the square of the amplitude. Then notice that the de Broglie's conclusion is pending on the admission of the fact that *the phase of signal is linear in time*, as a consequence of adding the diffraction phenomenon to the classical Newtonian phenomenology of light, based on reflection and refraction phenomena. Now, if we give up this condition, which comes down to *assuming free particles for interpretation* and, consequently a functionally arbitrary phase as in equation (2.15), the signal representing the light ray *must satisfy the Schrödinger equation* (2.17).

Here, however, *the classical potential* suggested by the the first equation from (2.18) taken as a Hamilton-Jacobi equation for the phase of function $\psi(\boldsymbol{x},t)$, is only defined by the amplitude of this function, through equation

$$V(\boldsymbol{x},t) \equiv -\beta \cdot \frac{\nabla^2 A(\boldsymbol{x},t)}{A(\boldsymbol{x},t)} \tag{2.19}$$

This can be taken as a *stationary* Schrödinger equation, but it is not a consequence of the *nonstationary* Schrödinger equation, as in the classical case. In other words, the nonstationary equation (2.17) is a mathematical tool of unquestionable existence and necessity – just like any other mathematical concept used by physics – as long as the wave function is complex. The problems arise with the equation (2.19), and they concern only the mathematical structure of the functions representing the amplitude and the phase of the wave function. These should tell us what is a free material particle from the point of view of wave mechanics. When it comes to considering the potential, the phase of wave function cannot be identified with the classical action quite unconditionally: this last one must satisfy the above constraints. Only in this instance can one have the freedom of using the potential in a two-way reasoning: either as a given function and then helping to find the amplitude – as it is currently used, almost exclusively, in physics – or, once the amplitude known, it helps to find the potential. Notice, however, that the potential can be calculated through equation (2.19) in any location where the wave reaches. For all intents and purposes, this is a property of holography. Rephrasing it, the theory of Louis de Broglie adds the *phenomenon of holography* to the phenomenology of light, just as, historically speaking, the theory of Augustin Fresnel added the phenomenon of difraction of light to the classical phenomenology, based only on reflection and refraction phenomena. In other words, *the hologram is part and parcel of the light phenomenon*, just like reflection, refraction and diffraction: we are not to forget this when describing the light; but mostly we are not to forget it when we use the light as a model, especially as *a model of brain*.

Thus, we can say that the equation (2.17) is indeed a universal instrument of our knowledge, once it ensues mathematically from a necessary complex form of the wave function. Therefore, the Schrödinger nonstationary equation for the free particles should be considered as essential, once it involves no other assumptions than the complex form of the wave function. However, classically, these particles may not be free, as the first of the equations (2.18) shows. In the very process of interpretation we need to provide the means of assembling them into physical structures. These are then described by the equation (2.19), giving the potential in terms of the amplitude of wave. Inasmuch as the potential shows classically where the forces responsible for these structures are to be found, the universe thus described is holographic. Consequently, if the function $\psi(\boldsymbol{x},t)$ itself represents an *ensemble of free particles* as required for a proper physical interpretation, these are free particles *not from classical point of view, but from the Schrödinger equation point of view*: classically they can be anything along the line of physical freedom. Like the luxons of Bartolomé Coll.

The main point of this image of interpretation is the shape of the tube representing the light ray: it needs not be a straight cylinder, as de Broglie himself thought of it, but a canal surface in general, of variable cross section should be the best image. A classical example may come in handy here, for building a conception: *the classical hydrogen atom*. Assuming space extension of the electron, this one describes a virtual de Broglie ray around the nucleus: *a toroidal ray* as it were. The electron can then be interpreted as an ensemble of luxons held together by 'forces' of the de Broglie type, describing the behavior of the classical amplitude. It is these forces, then, that are responsible for the cohesion of the electron. Some similar forces should then be responsible for the cohesion of the proton in the nucleus of the model.

The classical argument of interpretation of the wave function brought out by Erwin Madelung, assumes the classical objective existence of a potential (Madelung, 1927). As a result the potential appears as 'updated' by interpretation, with a term like (2.19), representing what Madelung calls the 'quantum contribution'. In this spirit, the theory here begs for a kind of generalization of the modern principle of asymptotic freedom: *in a region of pure quantum forces the wave function describes an ensemble of free particles*. For, indeed, the stationary Schrödinger equation (2.17) shows that the continuum described by the function (2.15), as interpreted by a swarm of free particles in the manner proposed by Madelung, appears as an ensemble of particles evolving under a purely quantal potential: *the whole potential*, not just part of it, is in fact a 'quantum contribution' as Madelung defines it. By its amplitude, the wave carries a memory that can be reproduced as a physical structure in any point of space. Thus, the description of the continuum by the function (2.15) is a purely undulatory description of a region of spacetime. Its interior however, is a swarm of free classical particles described by an equation of continuity for a density proportional to the square of the modulus of function (2.15). Each one of these classical particles has a momentum given by the gradient of phase of (2.15). However, a question still stands: *what this very region represents*? The answer to this question cannot be given classically, for the classical argument has already been exhausted. But Louis de Broglie taught us that it involves *the frequency* and *the surface concepts*, on which we have to further elaborate in due time.

2.4 First Characterization of Direction Along the Ray

We are now in position to state that this interpretation is related to the SL(2,R)-type algebraical structures, which will be revealed in its details further in the present work. Both the classical mechanics and the general relativity contain a clear possibility of such an interpretation, for the case of the so called free fall in a gravitational field. It thus becomes obvious that we need to put this interpretation under the concepts related to the nonstationary Schrödinger equation for the free particle, insofar as this equation is a fundamental mathematical instrument.

Fact is that the nonstationary Schrödinger equation for the free particle admits, besides the clasical Galilei group proper, an extra set of symmetries (Niederer, 1972) that, in general conditions, can be taken in a form involving *just one space dimension and time*, described as a SL(2,R) type group action in two variables with three parameters (de Alfaro, Fubini, & Furlan, 1976). Limiting the general conditions, the space variable can be chosen as *the radial coordinate in a free fall*, as in the case of Galilei kinematics, which can also be extended *as such* in general relativity, for instance in the case of free fall in a Schwarzschild field (Herrero & Morales, 1999, 2010). The essentials of the argument of Alicia Herrero's and Juan Antonio Morales' work just cited, are delineated based on the fact that the radial motion in a Minkowski spacetime should be a conformal Killing field, which is a three-parameter realization of the $\mathfrak{sl}(2,\mathrm{R})$ algebra in time and the radial coordinate. This is a Riemannian manifold of the Bianchi type VIII (or even type IX, forcing the concepts a little) when taking the stand of one of the epoch-making, and widespread, nomenclatures of the theory of general relativity (Bianchi, 2001). The bottom line here is that, as long as the general relativity is involved, the nonstationary Schrödinger equation must be taken to *describe the cosmological continuity of matter*. And since, as a universal instrument of knowledge, this Schrödinger equation is referring to free particles, we need to show what kind of freedom is this in classical terms. These classical terms are regulated by a Riemannian $\mathfrak{sl}(2,\mathrm{R})$ type structure.

In order to show this, it is best to start with the finite equations of the specific action of SL(2,R) group, and build gradually upon these (Mazilu & Porumbreanu, 2018), in order to discover the connotations we are seeking for. Working in the variables (t, x) representing the time and the space variable respectively, the finite equations of this group are given by the transformations:

$$t \to \frac{\alpha t+\beta}{\gamma t+\delta}, \quad x \to \frac{x}{\gamma t+\delta} \tag{2.20}$$

This transformation is, indeed, a realization of the SL(2,R) action in two variables (t, x), with three essential parameters (one of the four constants α, β, γ and δ is superfluous here). Every vector in the tangent space $\mathfrak{sl}(2,\mathrm{R})$ is a linear combination of the three fundamental vectors, the infinitesimal action generators:

$$\boldsymbol{X}_1 = \frac{\partial}{\partial t}, \quad \boldsymbol{X}_2 = t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x}, \quad \boldsymbol{X}_3 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} \tag{2.21}$$

These satisfy the *basic structure equations*:

$$[\boldsymbol{X}_1, \boldsymbol{X}_2] = \boldsymbol{X}_1, \quad [\boldsymbol{X}_2, \boldsymbol{X}_3] = \boldsymbol{X}_3, \quad [\boldsymbol{X}_3, \boldsymbol{X}_1] = -2\boldsymbol{X}_2 \tag{2.22}$$

which we take as standard commutation relations for this type of algebraic structure, all along the present work. The exponential group has an invariant function, which can be obtained as the solution of a partial differential equation:

$$(c\boldsymbol{X}_1 + 2b\boldsymbol{X}_2 + a\boldsymbol{X}_3) f(t,x) = 0$$

which, in view of (2.21), means

$$(at^2+2bt+c)\frac{\partial f(t,x)}{\partial t} + (at+b)x\frac{\partial f(t,x)}{\partial x} = 0 \tag{2.23}$$

The general solution of this equation is a function of the arbitrary values of the ratio:

$$\frac{x^2}{at^2+2bt+c} \tag{2.24}$$

which represents the different paths of transitivity of the action described by operators from equation (2.21).

In order to draw some proper conclusions from these mathematical facts, let us go back to the transformation (2.20) and consider it from the point of view of classical physics. The first principle of dynamics offers a special content to the classical time in its capacity of sequence: it is represented by the uniform motion of a free classical material particle. Such a particle is free as long as no forces act upon it. The equation (2.24) faithfully records this idea in an obvious form: the paths of transitivity of the group (2.20) are given by the 'radial motion' of a free classical material point, no question about that, for the quadratic polynomial in time represents a uniform motion in an arbitrary direction in space. Questions rise, however, and on multiple levels at that, when noticing that the general solution of equation (2.23) is an *arbitrary function* of the ratio (2.24). For once, we are compelled to notice that the *content of time* in (2.20) is not classical anymore, at least not in general, being a ratio of coordinates representing *two uniform motions*. Likewise, the second equation (2.20) can be taken as representing the *content of spatial coordinate* of the motion in terms of the classical coordinate of a uniform motion: as long as the some forces act, the space coordinate along the direction of action is not linear anymore, but its ratio to a linear motion can be taken as a coordinate. This much, at least, can be put in the common charge of the *wave mechanics* and *general relativity*, regarding an 'updating' of the idea of time and space contents for the necssities of constructing a proper light ray. But there is more to it, regarding the concept of freedom, because at this point we start to notice some apparently unrelated facts from the past, which seem to pick up concrete shapes, all converging to the ratio from equation (2.24).

First along this line, comes the second of Kepler laws, *viz*. that law serving to Newton as a means to introduce the idea of a center of force: if, with respect to such a material point, a motion proceeds according to the second Kepler law, then the field of force should be Newtonian. The wave mechanics shows that this law means more than it was intended for initially, namely that it should have a statistical meaning, according to the idea of Planck's quantization (Mazilu & Porumbreanu, 2018). Indeed, if x denotes the distance of the moving material point from the center of force, we have

$$x^2 d\theta = \dot{a}\cdot dt \quad \therefore \quad x^2 = \dot{a}\cdot\frac{dt}{d\theta} \tag{2.25}$$

where θ is the central angle of the position vector of the moving material point with respect to the center of force. In this form the law usually serves as a transformation in the mathematical treatment the central motion, defining a new time that can be identified physically as the eccentric anomaly θ. However, from the point of view of the physical content of time, the second equality in equation (2.25) tells us much more if we take the argument out of the mathematical form of the classical Kepler problem.

To wit, consider again the classical hydrogen atom: an *extended body* revolving in a central field of Newtonian forces. It can be imagined as a swarm of classical material points, and such a swarm illustrates the classical laws, provided it is considered as a swarm of free material points in the classical sense of the word (Larmor, 1900). In the first of equations (2.20) this requirement would mean that the material points are considered *simultaneously*. Then each material point can be located in the swarm by four homogeneous coordinates ($\alpha,\beta,\gamma,\delta$), or three nonhomogeneous coordinates, if the equations (2.20) represent the content of time and radial coordinate for the space region covered by this body. The simultaneity in the motion of the swarm of material points can be differentially characterized, giving a Riccati equation in pure differentials:

$$d\frac{\alpha t+\beta}{\gamma t+\delta}=0 \quad \therefore \quad dt=\omega^1 t^2+\omega^2 t+\omega^3 \tag{2.26}$$

Thus, for the description of the *extended body in motion as a succession of states of the ensemble of simultaneous material points*, it suffices to have three differential forms, representing a coframe of the $\mathfrak{sl}$(2,R) algebra:

$$\omega^1=\frac{\alpha d\gamma-\gamma d\alpha}{\alpha\delta-\beta\gamma},\quad \omega^2=\frac{\alpha d\delta-\delta d\alpha+\beta d\gamma-\gamma d\beta}{\alpha\delta-\beta\gamma},\quad \omega^3=\frac{\beta d\delta-\delta d\beta}{\alpha\delta-\beta\gamma} \tag{2.27}$$

That this coframe refers to such an algebra, can be checked by direct calculation of the Maurer-Cartan equations which are characteristic:

$$d\wedge\omega^1-\omega^1\wedge\omega^2=0,\quad d\wedge\omega^2+2(\omega^3\wedge\omega^1)=0,\quad d\wedge\omega^3-\omega^2\wedge\omega^3=0 \tag{2.28}$$

Élie Cartan has shown that under these conditions one can prove that the right hand side of equation (2.26) is an exact differential (Cartan, 1951), therefore it should always have an integral. The Cartan-Killing metric of this coframe is given by the quadratic form $(\omega^2/2)^2 - \omega^1\omega^3$, so that a state of an extended orbiting body in the Kepler motion, can be organized as a metric phase space, a Riemannian three-dimensional space at that. The geodesics of this Riemannian space, are given by some conservation laws of equations

$$\omega^1 = a^1 \cdot d\theta, \quad \omega^2 = 2a^2 \cdot d\theta, \quad \omega^3 = a^3 \cdot d\theta \tag{2.29}$$

where $a^{1,2,3}$ are constants and θ is the affine parameter of the geodesics, so that, along these geodesics the differential equation (2.26) is an ordinary differential equation of Riccati type:

$$\frac{dt}{d\theta} = a^1 t^2 + 2a^2 t + a^3 \tag{2.30}$$

This equation can be identified with (2.25), provided its right hand side is proportional to the square of a 'radial coordinate' of a free classical material point. Mathematically this requires an ensemble generated by a *harmonic mapping* between the positions in space and the material points, with the square of the radial coordinate x, measuring the variance which describes the distribution of material points in space.

2.5 Necessities of Improvement for Adaptation

In order to be able to adapt a physical theory into describing the living brain, a few necessities of general improvement are in order, on which we need to work specifically.

The basic idea is that we have to construct a *coordinate system* for the matter of brain, and this should be quite specific. However, it should be based on the general idea of light ray, as always was the case in physics. Only, the concept of light ray has to be improved itself, in order to account for the whole host of properties naturally invested in it, to which it answered only sporadically thus far. For, only having the whole set of properties at our disposal, shall we be able to chose those adequate in the description of the brain. As a matter of fact, this was the gist of history, and we just have to follow it faithfully: the properties of the light ray have been discovered sporadically along the history, for timely necessities of explanation. Rounding the concept with all its differentiae just gives us the chance to choose those appropriate from among them.

The pinnacle of the concept of light ray, seems to be the concept that we connected here with the name of Louis de Broglie. This concept of light ray harmoniously updates the classical concept due to Newton, up and above its classical update due to Fresnel. Happily enough, it turns out that the model responds to the classical concept of duality wave-particle, and also contains the Schrödinger theory under a remarkable condition: *the light ray thus described must possess the property of holography*. If it is to reveal here a historical continuity, one can say that while the classical Newtonian light ray has been defined based on just the classical phenomena of *reflection* and *refraction*, it needed to be completed, and Fresnel did it by introducing a new phenomenon: the *diffraction* of light. This, as it turns out, is not a systematic completion, inasmuch as the de Broglie's light ray reveals another phenomenon to be added to our experience with the light: the *holography*.

Now, in order to use this concept of light ray for the brain, in which case the skull is the only global reference frame that we can physically recognize immediately, we need to develop, first and foremost, the statistical side of the theory of light ray. And this we shall start to do right away. As it turns out, the whole theory originates naturally from the requirement of de Broglie that the amplitude of the wave *at a certain time* goes inversely with the distance to the wave surface along its normal.

3. Phase Related to Signal Recording in a Universe

The physical description of a living brain involves almost exclusively recorded signal at locations on the skull. Now, a recorded signal can always be thought of as representing a wave function, independently of the idea of interpretation, as it were, *i.e.* independently of the fact that it is a solution of Schrödinger equation or not. However, in order to have a meaning for this signal we still need to treat the recording as a complex function of the form $Ae^{i\phi}$. Then, a meaning could be extracted from the data, if this complex function is taken to represent harmonic oscillator in a given space point $\boldsymbol{x}$. In this case both the amplitude A and the phase ϕ are to be considered as functions of a *time sequence*. The experimental practice asks here for a certain analysis of signal in the time domain, allowing us to assign physical properties to the magnitudes extracted from the recording.

A classically mandatory parameter in this analysis is *the frequency*, which can be extracted in a variety of ways from the data. The most common kind of frequency to be extracted from a recorded continuous signal is the *instantaneous frequency* (Mandel, 1974). This kind of frequency can be calculated as the first time derivative of the phase of signal function: it coincides with the regular frequency only in the classical cases where the phase is linear in time, like that considered by Louis de Broglie in his construction of the light ray, as shown before. Now, denoting $q(t) \equiv A(t)e^{i\phi(t)}$ the signal function in this instance, *i.e.* as a local – at an arbitrary location in space – 'elongation' representing the recorded signal, we want to associate this signal with a mechanical oscillator in order to have an acceptable physical interpretation of the parameters extracted from the data, especially of the phase as a function of time. This association comes down to the following equivalences, representing connections between amplitude and phase as functions of time at a certain position:

$$\ddot{q}(t)+2\lambda\dot{q}(t)+\omega_0^2 q(t)=0 \quad \therefore \quad \begin{array}{c} \dfrac{\ddot{A}}{A}+2\lambda\dfrac{\dot{A}}{A}+\omega_0^2=\dot{\phi}^2 \\ \dfrac{\dot{A}}{A}+\dfrac{\ddot{\phi}}{\dot{\phi}}+\lambda=0 \end{array} \tag{3.1}$$

from which, denoting by {*,*} the *Schwarzian derivative* of the first symbol in curly brackets with respect to the second one (Needham, 2001), we have:

$$\omega_0^2-\lambda^2=\dot{\phi}^2+\frac{1}{2}\{\phi,t\} \tag{3.2}$$

According to the procedures of time-frequency analysis of the signals (Cohen, 1995), the Schwarzian derivative of the phase in the right hand side of this equation, is bound to represent some statistical properties of the frequency thus defined. Specifically, as we shall document later on here, we are naturally led to think of the *variance* of signal frequency thus defined. This means that, the signal having a well defined instantaneous frequency *as an exact mechanical frequency* of a damped harmonic oscillator, should have $\{\phi, t\} = 0$. This would mean

$$\phi(t)=\frac{\alpha t+\beta}{\gamma t+\delta} \quad \therefore \quad \dot{\phi}(t)=\frac{\alpha\delta-\beta\gamma}{(\gamma t+\delta)^2} \tag{3.3}$$

Therefore the most general signal having *mechanically well defined parameters* should be of the form

$$q(t)=(at+b)e^{-\lambda t}\exp\left(i\frac{\alpha t+\beta}{\gamma t+\delta}\right) \tag{3.4}$$

Now, for a proper choice of the arbitrary constants of integration a and b, the equation (3.4) is, in fact, a special connection between some group variables to be exhibited as we go on with this work. But let us recount the results here, in order to better realize what we have acquired thus far.

3.1 Statistics in Defining a Harmonic Oscillator

Notice an apparent 'contradiction' above: according to equation (3.2) the instantaneous frequency of a mechanically defined signal must be a constant in time, which is not the case with the phase and frequency from equation (3.3). In fact, the vanishing of the Schwarzian of phase means a mechanically defined instantaneous frequency, indeed, at any time along the signal. The equation (3.2) should be read as it was conceived to start with, and then there is not any real contradiction: it defines *the mechanical parameters* ω_0 and λ from *the recorded data*. Taken as such it only shows that it is impossible to solve this task in a one-to-one way. All we can do is to find the quadratic expression from the right hand side of the equation (3.2) as a function of time, using the instantaneous frequency defined in equation (3.3). The two mechanical parameters are thus defined up to an arbitrary hyperbolic rotation. However, we drew attention to this possible issue, inasmuch as the usual physical point of view is that *there is always a local harmonic oscillator* as part of the structure we are studying, whose parameters ω_0 and λ are known. This *a priori* philosophy is always a source of possible contradictions. It is our opinion that the data must be treated more realistically, and hence needs a necessary settling of the physical concepts themselves, before even starting to use a theory or another.

A first incentive along this path, is the fact that the phase itself is defined up to a homographic transformation, in view of the invariance of Schwarzian to this specific transformation (Needham, 2001). That is to say, the condition $\{\phi,t\}=0$ describes a class of functions ϕ, which can be obtained from one another through linear fractional transformations. The 'contradiction' above can then be clarified by the curvature properties of the Schwarzian [see (Flanders, 1970); see also (Duval & Ovsienko, 2000), for a modern treatment of the problem]. Maintaining the mathematical guise here, we can simply declare the obvious: the recorded signal is revealed with the aid of a local clock, which should be a periodic motion, or a known periodic process in general, and the definition (3.3) should be read in reverse actually. To wit, it provides the times recorded by the clock of known frequency, rather than the frequency when the time is known, according to equation:

$$\xi(t) \equiv \frac{\sqrt{\alpha\delta-\beta\gamma}}{\sqrt{\dot{\phi}(t)}} \tag{3.5}$$

and this gives a specific Schwarzian curvature in the form:

$$\frac{\ddot{\xi}(t)}{\xi(t)} = -\frac{1}{2}\sqrt{\alpha\delta-\beta\gamma}\,\{\phi,t\} \tag{3.6}$$

This vanishes only when the time is so chosen that the phase is homographic in time. However, the property represented by equation (3.6) is a general property independent of the functional form of the function $\phi(t)$ and its role in the physical theory. More to the point, given *any* such function, not just a homographic one, the function defined by $\xi(t) = C\cdot[(d/dt)\phi(t)]^{-1/2}$, satisfies the second order differential equation

$$\frac{\ddot{\xi}(t)}{\xi(t)} = -\frac{1}{2}C\{\phi,t\}$$

involving the Schwarzian derivative. In other words, for continuous well-behaving functions, this second order differential equation actually represents a connection between the two kinds of derivatives used in any calculational process in which they happen to be involved. If, then, the frequency of the local oscillator is known, the phase must depend on time in a specific way. We shall return to this issue later on.

Taken face value, however, the physical theory of harmonic oscillator reveals one of the most interesting properties of the energy, related to the conservation law characterizing this physical quantity, from which an observation noticed and exploited especially by Louis de Broglie ensues. Namely, the equation of motion of the damped harmonic oscillator, used in (3.1) for extracting the physical properties of the oscillator from our signal, is able to tell us exactly which one of the two parameters concerns a statistics related to the time sequence only. This characterization can be easily obtained for the harmonic oscillator, due to the fact that the two kinds of mechanical energy involved in the physical description of this simple system – kinetical and potential – are well defined. As strange as it might seem at a first sight, this is quite a rare situation in physics in general, and it can be positively used in the manner that follows.

The physics knows here about the well-established result that the equation of motion of the *undamped* harmonic oscillator is a direct consequence of the property of stationarity of the *time average of the difference between the kinetic and potential energies* – the Lagrangian – over the whole period of the motion. Indeed, the action between two moments of time, from which we extract the equation of motion of the oscillator, is the time integral of the Lagrangian between the two time moments. Assuming, then, that the *physical time is a uniformly distributed statistical variable*, the physical action can be, indeed, construed as a time average of the difference between kinetic and potential energies, as de Broglie once noticed (de Broglie, 1961, 1962). Therefore, one can say that the undamped harmonic oscillator *is a system which distributes the two kinds of interaction mechanical energies* – kinetic and potential – in such a way that the average of their difference over any time sequence included in the period of motion is stationary.

Before going any further, let us stop for a moment, in order to pinpoint an important idea already mentioned quite a few times in different junctures of our discussion in this work. Namely, the concept of time here deviates significantly from the regular time concept of classical dynamics, by assuming a differentia which brings it closer to the *time of special relativity*. Let us emphasize once again that, if we need to describe a general concept of time, then we have to assume that this concept must have *two* differentiae: that revealed first in the classical case, related to the property of continuity of motion, and the one associated mainly with the special relativity in describing an electrodynamical universe, whereby the time is a parameter of *global ordering of events*. It is in this last instance that the time is defined by the idea of sequence, which is a special case of an ensemble of time moments. As the Feynman's development of quantum electrodynamics shows, such an ensemble may not even be necessarily a causal sequence in the classical sense. All it needs is only to remain *deterministic* from the physical point of view (Feynman, 1949).

Now, continuing on with our discussion of the time statistics related to the Lagrangian of harmonic oscillator, not quite the same mathematical argument can be applied in obtaining the classical equation of motion of a *damped* harmonic oscillator [the left hand side of the equation (3.1), for instance]. This one does not involve in its physical structure *a direct, uncontrolled transition*, as it were, between the kinetic and potential energies, as reflected in the time average of the Lagrangian. However, the physics underlying the case can still be saved by the *very same statistical argument*, for the statistics involved in defining the action that provides the equation of motion is essentially the same from a general theoretical point of view. Only its type changes, however in a precise manner: *by the character of its basic distribution density*. Indeed, the common observation of a physical nature here, is that the equation of motion of the damped harmonic oscillator can be obtained by making stationary the action related to the Lagrangian:

$$L(q,\dot{q},t)=\frac{1}{2}(M\dot{q}^2-Kq^2)e^{2\lambda t};\quad \omega_0^2\equiv\frac{K}{M},\quad \lambda\equiv\frac{R}{M} \tag{3.7}$$

The first term in paranthesis here is the *kinetic energy* of the particle of mass *M*, while the second one is the *potential energy* of the elastic force acting on it. R is the damping coefficient, assuming that the damping is proportional to velocity. Then, according to its definition, the action corresponding to this Lagrangian, is given by an integral like

$$S_R(t_0,t_1) = \frac{1}{2}\int_{t_0}^{t_1}(M\dot{q}^2 - Kq^2)e^{2\lambda t}\,dt \tag{3.8}$$

The associated variational problem of this action leads to a *Caldirola-Kanai Hamiltonian*, if it is to judge from a purely physical point of view [(Caldirola, 1941, 1983); (Kanai, 1948)]. This Hamiltonian turns out to be no more the sum between kinetic and potential energies as they appear in the Lagrangian from equation (3.7). The participation of physical parameters to time variation is the main thing to be noticed here: depending on the sign of damping coefficient, the inertial mass increases while the elastic stiffness decreases, or the other way around. Therefore, there is still an interdependence between the terms of the Lagrangian, but with *the notable participation of the physical parameters of the oscillator*. The Hamiltonian may not even be a conserved energy, as in the case of undamped harmonic oscillator. However, the physics embodied in equation (3.8) can be, as we said, saved by statistics, for the action integral can still be construed as a time average of the difference between the two well-defined mechanical energies, but on this occasion for different probabilistic measure of the time domain.

Indeed the exponential factor from the integrand of (3.8) can be interpreted as an *exponential distribution density describing the ensemble of time sequences* inside the time interval between the moments t_0 and t_1. In the case of undamped harmonic oscillator, the action is $A_0(t_0,t_1)$ – the index 0 of the action is referring to the value of the damping coefficient – and the exponential factor is *1*. This particular action can be, indeed, interpreted as a mean over a *uniform distribution* of times in a sequence, as stated before. The difference between the two cases – zero and nonzero damping coefficient – rests only upon the exponential factor in expression of the action integral, which, from a statistical point of view, *is thus not an attribute of the oscillator per se*, *but of the time domain*, in its stance as a measured sequence. One can say that an evolution for the damped harmonic oscillator means an ensemble of events characterized by sequences of *equally probable times* in a certain time interval, just like in the undamped case. However, the 'equally probable' attribute in a time sequence is now defined not by a uniform probability distribution, as in the case of undamped oscillator, but by *an exponential distribution proper*.

In this context, the equation (3.2) has a precise statistical meaning, as we anounced before. Indeed, if we write it in the form

$$\omega_0^2 = \lambda^2 + \dot{\phi}^2 + \frac{1}{2}\{\phi, t\} \tag{3.9}$$

then the statistical character of the frequency of undamped harmonic oscillator becomes apparent. At least in the case of the phase like that from equation (3.3), the undamped frequency can be regarded as a standard deviation of a *quadratic variance distribution function* of the type involved in the statistics that led to the initial Planck quantization [see (Morris, 1982) for the statistical theoretical concept]. One can therefore say that the instantaneous frequency of a recorded signal is indeed connected to a statistic of the time sequences revealed with a local harmonic oscillator with time variable physical parameters in the sense of Caldirola-Kanai Hamiltonian theory.

3.2 The Necessity of Space and Time Scale

Thus, at least for the case of damped harmonic oscillator, the physical character of time is, first and foremost, plainly a statistical property. This property is the one that allowed Richard Feynman the construction of his sum over paths, to begin with. This explains why he has placed so much physical emphasis upon harmonic oscillator. In hindsight, this emphasis cannot be explained but only by taking into consideration the results of Berry and Klein regarding the forces of Newtonian type (Berry & Klein, 1984). According to these results, the presence of oscillators at a certain space scale is indicative of the existence of Newtonian forces – the only scale invariant forces in a physical system existing on different space scales. In order to better illustrate the issue at hand, we extract a couple of phrases from the Abstract – giving the customary short presentation of this kind of works – of the Feynman's 1942 famous dissertation. This excerpt contains an observation explaining the importance that the approach of the wave mechanics initiated in that work, bestows upon harmonic oscillator:

> As a special problem, because of its application to electrodynamics, and because the results serve as a confirmation of the proposed generalization, *the interaction of two systems through the agency of an intermediate harmonic oscillator is discussed in detail*. It is shown that in quantum mechanics, just as in classical mechanics, *under certain circumstances the oscillator can be completely eliminated*, its place being taken by a *direct, but, in general, not instantaneous, interaction* between the two systems. [(Brown, 2005); *our Italics*]

There is not too much to say over these words, in order to see in them the future results of Berry and Klein, indeed: the oscillator is present in any conservative Hamiltonian approach whereby the time is specially defined under condition of invariance of Newtonian forces. Involving in its physical structure parameters from 'two worlds', as it were – the far away part of the universe and the closest of its part – the oscillator is the best suited physical structure for describing the interaction between two systems. More importantly though, as we shall see here, this is the property that allows us to turn the special relativity into a universal theory, which thus can stay at the foundations of that special mathematics associated with the scale relativity physics.

It is in order to make this statistical property of time into a physical property, that we need to exhibit the physical reasons for changing the time sequence statistics. This too, will help in a proper understanding of the explanation of the physical parameters of harmonic oscillator, inasmuch as the change of the time statistics seems to be, at least to a certain extent, intrinsic to the *physical properties* of the harmonic oscillator. Indeed, considering the oscillator only, the Caldirola-Kanai Hamiltonian corresponding to the Lagrangian from equation (3.7) indicates, as we mentioned above, the variability with time of the physical parameters reflecting interactions with the *remote* and, respectively, *close* environment of the particle representing the oscillator as a physical structure: the *inertial mass* and the *elastic stiffness*. The case from equation (3.8) is only a particular one among those which led to the *classical idea of gauging*. Before anything should be said here, let us present a first instance of that classical idea of gauging, and its relation whith the ideas of *interpretation* and *memory*.

3.3 A First Gauging and a Definition of the Memory

The equation of motion from the left hand side of (3.1) cannot be obtained quite directly from the variational principle applied to action (3.8), even after adjusting the statistics of time sequences to a genuinely exponential one: one still needs some definite conditions at the ends of the time interval. The first of these conditions, and the most important among them, is that the trajectories of motion must all end in the same position at the time ends, *i.e.* all pass through the same end points, spatially speaking:

$$\delta q\big|_{t_0} = \delta q\big|_{t_1} = 0 \tag{3.10}$$

Further on, the idea of *cycle* connected to the concept of harmonic oscillator, triggers the condition that the evolution starts and ends at the same point:

$$q(t_0) = q(t_1) \tag{3.11}$$

Moreover, if the situation is described in the phase plane of the harmonic oscillator, we need a condition like this for the velocities too. It is therefore a matter of problem setting, to decide which specific conditions we need to take at the ends of time interval, in order to apply them over the variational principle, in order to define it properly. However, conditions like (3.10) and (3.11), involving the ends of the time interval, or some variations thereof, are essential in any formulation of that principle. When we consider them, the Lagrangian proves not to be unique from the point of view of the variational principle: it is defined up to an additive function which represents an exact time derivative, and takes the same values at the ends of time interval. In order to show this, it is better to reason on a general Lagrangian, explicitly dependent on time, like in equation (3.7), but in a more general manner, and then, based on this treatment, to evaluate our case specific case given by equation (3.7).

Let us therefore apply the variational principle in order to obtain the equation of motion for a Lagrangian of the functional form $L(q,dq/dt,t)$. The physical action is given as the definite integral:

$$S_R(t_0,t_1)=\frac{1}{2}\int_{t_0}^{t_1} L(q,\dot{q},t)\,dt$$

The principle of stationary action – the Hamilton principle – shows that for the real motion, the variation of this action, taken into consideration the conditions (3.10) must vanish. Thereby equations of motion are obtained, as Euler-Lagrange equations corresponding to the given Lagrangian.

Notice, however, that in order to get the Euler-Lagrange equation we need the assumption that the Lagrangian has equal values at the end times of the motion, otherwise a redundant term would remain in the variation, which would allow in no condition to extract those equations. As a consequence, in the very same working conditions we can add to the Lagrangian any function of time just as well, provided it has equal values at the ends of time interval: our equations of motion do not change. In other words, within our working conditions, the Lagrangian is defined up to an additive function of time, which is the time derivative of a function having equal values at the ends of the time interval, but otherwise arbitrary.

This is the basis of a well-known, and very instructive, classical *gauging procedure*. However we read it here a little bit differently, having in mind the idea of time sequence as discussed above: *we can reduce the Lagrangian to a perfect square, by gauging it in the manner just described*, and this reduction has a significant meaning. The procedure is well known and largely exploited in the control theory (Zelikin, 2000), so that we can shorten the story. The cycling condition (3.11) now enters the play. All one needs is to add to the Lagrangian from equation (3.7) the term representing an exact derivative:

$$\frac{1}{2}\frac{d}{dt}(w(t)\cdot e^{2\lambda t}\cdot q^2)$$

where $w(t)$ is a continuous function of time, and then ask that the final Lagrangian should be *a perfect square*. In view of condition (3.11), the final equations of motion do not change. Now, the new Lagrangian of the gauged harmonic oscillator, proves to be a *perfect square*, just like the classical kinetic energy that generated idea in the first place:

$$L(q,\dot{q},t)=\frac{1}{2}M\cdot e^{2\lambda t}\cdot(\dot{q}+\frac{w}{M}q)^2 \tag{3.12}$$

provided $w(t)$ satisfies the following Riccati equation:

$$\dot{w}=\frac{1}{M}w^2-2\frac{R}{M}w+K \tag{3.13}$$

Obviously, under this condition, the Lagrangian (3.12) leads to the same equation of motion as Lagrangian from equation (3.7), if we use the condition (3.13) in the results of the corresponding variational problem. However, as we just said, the Lagrangian (3.12) has the property of the classical prototype of the Lagrangians – *the kinetic energy of a free particle* – of being a perfect square. It describes a 'free particle' with its *mass exponentially variable* in time and the *velocity redefined appropriately*.

What is the reason of this reading? Again, the point here is to *physically interpret* – with the interpretation defined in the sense of Charles Galton Darwin – a simple system like the harmonic oscillator. This task usually entails some allegedly fundamental interactions in its physical structure, in order to carry the interpretation over to an ensemble. Naturally, first we need *the constitutive element of this interpretative ensemble*, which is the harmonic oscillator. However, this comes with strings attached, in the form the interactions involved in the explanation of its parameters. The *mass is here inertial* and physics assigns to it an interaction involving the *remote part of the universe*. The elastic stiffness is of a *deformational nature*, and the physics associates with it the *close part of the universe* representing a static environment, like any deformation ever, since Rebert Hooke. The damping term would then represent a *transition* between the physical properties of the oscillator induced by the two parts of the universe. According to our previous analysis, this property is delegated to the *statistical properties of the time sequences*, which is quite natural, inasmuch as the oscillator properties are induced by the universe, and the time sequences in a universe are cosmologically decided.

It is according to this view, that the problem of interpretation needs to be solved, and the Lagrangian from equation (3.12) provides such a solution: it allows us to identify the harmonic oscillator with a free particle. The velocity of this particle is $dq/dt+(w/M)q$, depending linearly on the solutions of Riccati equation (3.13). Then, it is this last equation that needs a sound interpretation, which turns out to be statistical: $\dot{w}$ is the variance function of an exponential family of distributions having quadratic *variance function*, for which w is the *mean* (Morris, 1982). The probability distributions of this ensemble vary in time, but now the time itself represents a parameter indexing the family of probability densities, in much the same manner the temperature marks an ensemble of molecules in thermal equilibrium. Mention should be made that even an undamped oscillator can be nontrivially made this way into a free particle. But there is more to it, mostly along the idea of holography.

3.4 The Holographic Time

As we have shown in Chapter 2 here, the holography must be a natural property of the worls we live in, just like the reflection, refraction and diffraction. Actually this is the spirit of the initial work of Dennis Gabor that started the idea of holography (Gabor, 1948). It was precisely directed to the poor description of the phenomenon of difraction, more to the point, to its inappropriate connection with the other two phenomena relating to the general classical wave theory: reflection and refraction. To wit, the operation of electronic devices is obviously based on diffraction. These devices admit, through their objectives, a certain amount of improvement for the image that goes through them. Optimizing the images means reducing in aberation, and in most of the cases this becomes an impossible task: there is an inherent limit of technological possibilities. Dennis Gabor noticed, however, that this approach is unnecessary. Quoting:

> The new microscopic principle described below offers a way around this difficulty, as it allows one to dispense altogether with electron objectives. Micrographs are obtained in a two-step process, by electronic analysis, followed by optical synthesis, as in Sir Lawrence Bragg's 'X-ray microscope'. But while the 'X-ray microscope' is applicable only in very special cases, where the phases are known beforehand, the new principle provides a complete record of amplitudes *and* phases in one diagram, and is applicable to a general class of objects. (Gabor, 1948)

A few more words are necessary in order to properly understand this excerpt. Notice first that in the association wave-particle here, the electrons of the electronic microscope device are supposed to correspond to electromagnetic waves of X-ray type, for which the difraction pattern is connected with the *presence* of matter – atoms, specifically – in periodic crystals (Gabor, 1949). Remember that in the case of light, such diffraction patterns are obtained when passing the light through pinholes – i.e. to the *absence* of matter – and the question is raised: what is the connection between the two physical situations? For, in view of the wave-corpuscle duality, the two situations must be the same from a conceptual point of view.

Now, Gabor took notice of the fact that the X-ray diffraction pattern for crystals can be explained by the change in phase in the reflected radiation waves, due to the interaction with the electromagnetic structure of the lattice atoms. Therefore, such a pattern can be explained by the presence of matter, which is certainly not the case for a pinhole. Here the case is, obviously, quite contrary: the matter should count as absent. Naturally, if the de Broglie duality is universal, then there should be a universal property of the wave, to reproduce the properties of matter even when this one is absent. Dennis Gabor assumed, and even proved experimentally, that the general case occurs, indeed, when we have a "complete record of amplitudes *and* phases in one diagram", a condition missing for the particular arrays of atom in a crystal. In this last case it is only the amplitude of the signal that counts experimentally and, as Gabor himself observes, the emphasis is "somewhat unlucky". Quoting:

> It is customary to explain this by saying that the diffraction diagrams *contain information on the intensities only, but not on the phases*. The formulation is somewhat unlucky, as it suggests at once that *since the phases are unobservables*, this state of affairs must be accepted. In fact, not only that part of the phase which is unobservable drops out of *conventional diffraction patterns*, but also *the part which corresponds to geometrical and optical properties of the object*, and which in principle could be *determined by comparison with a standard reference wave*. It was this consideration which led me finally to the new method. [(Gabor, 1949), our emphasis]

The fact that phases contain no information in issues related to interpretation, was the hallmark of theoretical physics until the work of Yakir Aharonov and David Bohm, which stirred up the idea of connection between *potential* and *phase* in wave-mechanical problems (Aharonov & Bohm, 1959). In view of our presentation here, one should be entitled to say that the Gabor's principle is actually a proof, *avant la lettre* as it were, of the Aharonov-Bohm effect. As a matter of fact, a kind of general type of Aharonov-Bohm effect, according to which, given a right theoretical approach…

> One might therefore expect wave-optical phenomena to arise which are due to the *presence of a magnetic field* but not due to the *magnetic field itself*, i.e. which arise whilst the rays are in field-free regions only [(Ehrenberg & Siday, 1949); our Italics]

has been voiced, partly based to Gabor's own previous work, just about the time when he introduced the idea of holography. We shall pursue here that difference between the *presence* and the *action* of the field, which was the mark of Maxwellian electrodynamics from its very beginning.

For the rest, we need to notice that the Gabor's target is 'the conventional diffraction pattern', which is incomplete from the point of view of duality wave-corpuscle. The apparent proposal is that the phase should be observable in a hologram, which is just as natural as the diffraction phenomenon itself. The degree of generality of the holographic principle has been noticed by its author from the very beginning:

> *The new principle can be applied in all cases where the coherent monochromatic radiation of sufficient intensity is available to produce a divergent diffraction pattern, with a relatively strong coherent bacground*. While the application to electrom microscopy promises the direct resolution of structures which are outside the range of ordinary electron microscopes, probably the most interesting feature of the new method for light-optical applications is the possibility of recording in one photograph the data of three-dimensional objects. In the reconstruction, one plane after the other can be focused as if the object were in position, though the disturbing effect of the parts of the object outside the sharply focused plane is stronger in coherent light than in incoherent illumination. *But it is very likely that in light optics, where*

beam splitters are available, methods can be found for providing the coherent background which will allow better separation of object planes, and more effective elimination of the effects of the 'twin wave' than simple arrangements which have been investigated. [(Gabor, 1949), our emphasis]

The last italicized part was the main object of technological development in the last times. However, the universality of this principle begs the question: what is the relevant property of waves that makes it work in the world at large? The answer was provided a long time ago by Hugh Christopher Longuet-Higgins, as a property of ensembles of damped harmonic oscillators, involving the equation (3.12) in a particular take (Longuet-Higgins, 1968). However, in order to properly understand such an answer we need a proper view on the idea of scale invariance in the universe (Mazilu, Agop, & Mercheș, 2019).

The previous elaboration reproduces the gist of a statistics involving the transition between kinetic and potential energies at the threshold. As it turns out, this is an old problem involved in the very dynamics of the harmonic oscillator, which was the first to raise doubts on the issue of sufficiency related to the definition of the absolute temperature. Thus, it further turns out that the Planck's quantization is by far not the only lesson we need to learn from this moment of our knowledge. Most importantly, we should say, is the fact that *we need to account for the very structure of the constitutive element of the ensemble serving for interpretation*, as defined for the necessities of the wave mechanics.

4. Carying the Physics into the Brain World

Louis de Broglie has reached the essential idea that there is such a concept as the "wave phenomenon called material point" (de Broglie, 1926c). Of course, when one talks about 'material point' in this context, one understands the classical material point, *i.e.* a position endowed with physical attributes: mass and charge, as this concept appears in the mechanical and electrodynamical equations of motion. However, starting from frequency, de Broglie's theory cannot be but abstract, and this is what perpetuates that strange feeling at the association wave-particle, which generated idea of probabilistic interpretation of the wave. For, the frequency is, in general, a derived concept, from the more fundamental concept, if we may, of phase, taken as a primary concept, as we have shown earlier. To wit, even today the mystery of de Broglie frequency is a subject of debate.

In the physics of brain, though, just as in aphysics of heart, one cannot renounce the concept of material point: apparently, such a physics is entirely based on the motion of charges, and the material point is usually conceived as the support of charge. Therefore, speaking of a classical material point, we still have to maintain the Newtonian point of view in constructing a theory of waves, and this can be done most naturally, and universally we should say, via the experimental *concept of phase*. Indeed, if electrodynamics allowed the introduction of the idea of synchronization as a fundamental idea of special relativity, nowadays it is time to take notice that there are *paradigms* of synchronization that connect the concept of frequency with that of phase (Acebrón, Bonilla, Pérez Vicente, Ritort, & Spigler, 2005), even in a possibly, well-defined statistical way (Martens, Barreto, Strogatz, Ott, So, & Antonsen, 2009). It is this type of paradigm that is usually thought for in the case of research related to living tissues. In such a case, however, we need to take into consideration the electrostatics, and even magnetostatics, rather than electrodynamics; as we said, the cases of heart and brain are well-known examples, but one can easily come up with the general idea that any physical organism on earth functions based on electric phenomena. Now, with a proper consideration based on the idea of scale invariance of the static forces in the universe, a *well defined concept of phase* can be introduced for the general benefit, not only for the benefit of physics *per se*. For, a concept of material point can be defined, at any space location where static forces are in equilibrium. Thus, logically, a phase, therefore a wave, should be related with such a location, making, a 'wave phenomenon called material point' out of it, if it is to use the phrasing of Louis de Broglie. Let us now elaborate on these ideas.

4.1 A Characterization of Luxons

In the Newtonian stand, cosmology is indeed based on forces, and these forces are of a special type. To wit, they should be central forces, but with a magnitude depending exclusively on the distance between bodies, and that in a quite specific way: inversely proportional with the square of that distance. Of course, in reality, such a physical structure of the universe is not possible but only in cases where the structural units of the universe – the 'bodies' of Newtonian natural philosophy – are so far apart from one another, *that their dimensions are negligible with respect to the distance between them*. In discovering the Newtonian forces thus described, one can safely assume that this dimensional condition of their theoretical possibility should have already been a reality in the universe accessible to our knowledge. Indeed, otherwise we would not have the Kepler laws governing the motions of celestial bodies, which led to the invention of such forces. However this dimensional geometrical condition is by no means sufficient for the task of building a cosmology. First of all, a physical structure of the very structural unit of the universe – the 'body' – is still needed. For, in time, mankind became aware of the existence of still *other* Newtonian forces with the very same geometrical properties. The first instance of such forces in the quotidian world of our experience was *the electric force*. And this is how, in fact, we became aware of the necessity of a scale in characterization of the action of Newtonian forces. A phrase like that of Hermann Weyl, justifying the modern approach of cosmology, with a universe where…

> … the electricity, which *obviously does not matter in the economy of cosmos*, *we now completely dismiss*… [(Weyl, 1923), §39, p.290; *our rendering and Italics*]

just reflects such an awareness.

This excerpt from Weyl, shows that we are, at least formally, allowed to think of a certain space scale as being the expression of the dominance of actions of one of the two *different forces* of the same *mathematical nature*, *i.e.* Newtonian forces. It is for these forces, that Michael Berry and David Klein proved their scale transcendence (Berry & Klein, 1984), and these are, in fact, the only forces satisfying the invariance of the dynamical description of the Kepler problem, making it a valid model of physical structures at any space scale. This observation entices us to take notice of what seems to be obvious by itself: the very same property of these forces that allowed Hermann Weyl to emit the argument from the excerpt above, also allowed a long time ago to Charles Coulomb the positive task of experimental description of electric forces. Indeed, in the case of Coulomb, there is just a change in emphasis: in the 'economy' *of daily life*, the gravitation 'does not matter', therefore it can be dismissed, even though not quite by the same reasons the electricity at the cosmic level is dismissed. So, the apparent geometrical rigor, must be supplemented with this rigor of a physical nature, when judging the transition between universes at different space scales.

But there is more to it, and that in quite an essential way: by the presence of Newtonian forces in this outline, the *physical characteristics* of the fundamental units of matter are brought to the fore, in order to regulate the transition of scales. It turns out that they allow a mathematical characterization *independent of the dimensional conditions*, mandatory for a proper mathematical description of the dynamics of Kepler problem. More specifically, the dimensional condition defining this fundamental unit of matter in the physical structure of a certain universe, is capable of accommodating the existence of physical characteristics, and this fact can be formally described by a *geometry of confinement* having, mathematically the characteristic of an *absolute* of *Cayleyan* geometry. One has here, indeed, a concept of confinement which, physically speaking, is completely independent of any spatial condition of the kind historically involved in discovering the forces of Newtonian type. Let us see what is this all about.

First of all, in order to describe the fundamental units of matter in the same way in both cases – cosmos and daily life – the mass should be taken as the *gravitational mass*. Only in this case can one declare, as Weyl did, that for fundamental constitutive units possessing just gravitational mass and charge, in the case of 'cosmos' the Newtonian force of gravitational mass prevails quantitatively over that of electric charge. And only in this case can we add that, in the case of 'daily life', the Newtonian force of electric charge – the Coulombian force, as it is usually called – prevails over the force of their gravitational mass. Therefore, if we are to describe *physically* a certain universe, regardless of scale, the fundamental physical unit of this universe – the 'luxon', as we designated it every now and then, after Bartolomé Coll – should exercise two kinds of Newtonian forces acting simultaneously *in any direction* in space, *at any distance*: gravitational and electric. The universe would be then interpreted as an ensemble of such identical units, like the molecules of an ideal gas or of an ideal fluid, only with Newtonian forces acting between them. The magnitude of the whole Newtonian force, acting between any two such identical fundamental physical units on a certain direction, in a universe described in this way, can be written as

$$G\frac{m^2}{r^2}-\frac{1}{4\pi\varepsilon}\frac{e^2}{r^2} \tag{4.1}$$

with r denoting the distance between them. Here another assumption appears as necessary, and needs to be stressed once more as an essential condition in the description of a universe: the fundamental *physical units of this universe are all identical*, having the mass m and charge e, and the universe is described in a space of gravitational constant G and electric permittivity ε. The equation (4.1) also expresses the fact that, according to our experience, along the same direction, the two Newtonian forces act differently: one is a force of *attraction*, the other is a force of *repulsion*, and the different signs of the monomials in (4.1) represent an algebraic writing with respect to the orientation of the direction along which the action is exerted. Thus, for instance, the universe at the cosmic scale can be characterized by

$$Gm^2-\frac{e^2}{4\pi\varepsilon}\gg 0 \tag{4.2}$$

so that, in the expression of Hermann Weyl, the 'electric force can be dismissed' *at any distance and in any direction*, while the universe at the daily scale can be characterized by

$$Gm^2-\frac{e^2}{4\pi\varepsilon}\ll 0 \tag{4.3}$$

so that the 'gravitational force can be dismissed' *at any distance in any direction*, as proved by Coulomb's experiments. Certainly the 'distance' should have different quantitative meaning in the two cases, and perhaps the 'direction' too. Whence, in our opinion, the possibility of describing *concurrent universes*, even at different space scales, to which we suggest now a positive possibility of approach, offered by the ideas of the absolute – or Cayleyan – geometry, and based on the concept of Newtonian forces.

4.2 Static Definition of Luxons

Start with the observation that the structure of a universe is always a hypothesis, so that the problem occurs: is there an ideal *static structure* of the universe, *formally the same at any scale*, that would be able to describe even the structure of the fundamental physical unit of the universe? Mathematically speaking this is always possible, under the concept of interpretation, with Newtonian forces, insofar as, according to Berry and Klein's theory, the action of these forces transcends the time and space scales. Thus, for instance, a *static universe* dominated by the two Newtonian forces above, can be interpreted as an *ensemble of luxons* at any scale, provided

$$Gm^2 - \frac{e^2}{4\pi\varepsilon} = 0 \tag{4.4}$$

This would mean a structure made of *identical point particles*, each one of them endowed with gravitational mass and electric charge, in an ensemble in static equilibrium: the forces between particles are in equilibrium in any direction for any particle of the ensemble. Such an interpretation naturally restricts the ratio between electric charge and gravitational mass of the luxons.

However there is another significant connotation of this approach of static equilibrium: as we already announced before, we can construct an *absolute geometry* based on the static equilibrium of forces, using equation (4.4) as the equation of *an absolute* in the geometry of *physical attributes* of particles. First of all, contemplating some simplicity of the mathematics involved here, let us arrange a uniform notation based on equation (4.4), in order to simplify the algebra that follows. The terms in equation (4.4) are physically homogeneous, and both have the composed unit ($kg \cdot m^3 \cdot s^{-2}$). So, in order to make the notation uniform, we include the constants in the definition of the physical properties of the material particles, by the following transcriptions:

$$m \;\; for \;\; m\sqrt{G}, \quad e \;\; for \;\; \frac{e}{\sqrt{4\pi\varepsilon}}$$

These notations are intended to suggest that the first term in (4.4) is referring only to gravitational mass (m), while the second is referring only to the electric charge (e). This means that a universe is here interpreted by a static ensemble of identical luxons, each one of them having two physical characteristics: *mass* (gravitational) and *charge* (electric). Therefore, limiting ourselves, for the moment, to just gravitational mass and electric charge, the condition to be satisfied by an ensemble of identical material points serving for the interpretation of a static universe independently of space scale should be written in the form:

$$Q(m,e) \equiv m^2 - e^2 = 0 \tag{4.5}$$

The left hand side of this equation thus symbolizes a homogeneous *quadratic*, *i.e.* a polynomial of second degree in its variables, whence the choice of symbol Q. Taken as absolute of a Cayleyan *geometry of the physical characteristics* – gravitational mass and electric charge – of luxons, it divides the plane of these characteristics in two parts: the 'inside' part, for which we assume that the quadratic form is positive, and the 'outside' part, for which the quadratic form is negative. This is just a convention, adopted so that the 'cosmos', characterized by equation (4.2) should be *inside the absolute*, very close to its center with respect to charge, while the 'microcosmos', described by equation (4.3), should be *outside of absolute*, very far away with respect to gravitational mass. The true measure of these degrees of 'closeness' is, nevertheless, offered by the quantity Q defined in equation (4.5). The positive part of this construction is that, once we give a metric for this geometry, we are always able to find space distributions of the quantities of gravitational mass and charge, using the *harmonic mappings*. The construction of these distributions may not be quite as easy as it sounds, but in some cases it may be indicative of the right path of our knowledge.

For instance, the Barbilian formula, is always fit for such occasions [(Barbilian, 1937); see also (Mazilu, Agop, & Mercheș, 2019), equation (4.28)] and offers, indeed, a Cayleyan metric of the plane of the *two* physical attributes, which we take in the form

$$(ds)^2 = \left(\frac{mdm - ede}{m^2 - e^2} \right)^2 - \frac{(dm)^2 - (de)^2}{m^2 - e^2}$$

The reason for this choice of sign for the Cayleyan metric becomes obvious by noticing that, after due calculations are done, this metric becomes a perfect square:

$$(ds)^2 = \left(\frac{edm - mde}{m^2 - e^2}\right)^2 \tag{4.6}$$

and a perfect square of a real quantity should be always positive. Thus the interior of the absolute is characterized by a proper hyperbolic angle, ψ say, whose variation turns out to be our metric. Indeed we have:

$$(ds)^2 = (d\psi)^2; \quad \tanh\psi \equiv \frac{e}{m}$$

The metric of physical characteristics of this universe depends on the ratio between charge and mass, a case well known in the history of physics. Only, we have to observe that here the mass is gravitational, while in the historical case the mass was inertial, as a consequence of dynamics used in describing the electron.

A problem surfaces when the charge 'splits', so to speak, *i.e.* there are Newtonian forces of *electric nature* and also Newtonian forces of *magnetic nature*. This could be the case if the luxons have *a third physical attribute*, *viz*. a *magnetic charge* and, as a result of this, a magnetic force [(Maxwell, 1873), Volume II, Part II; see §§ 371–376]. As our experience shows, the magnetic poles of the same name behave exactly like electric poles of the same charge. Therefore, a static universe interpreted by ensembles of such luxons with three physical characteristics, not just only two, will be described by a quadratic quantity reflecting the presence of both electric and magnetic charges. According to the very same theory of the Newtonian forces, this quantity must be

$$Q(m, q_E, q_M) = m^2 - q_E^2 - q_M^2 \tag{4.7}$$

instead of (4.4). Indeed, the vanishing of this quantity, with the electric charge q_E and the magnetic charge q_M appropriately defined, describes a static ensemble of luxons in equilibrium under the *three* Newtonian forces prompted by the three physical properties.

In this case, according to our metrization procedure of the physical characteristics of luxons in the matter, the same rules apply for calculating the absolute metric in the case of three physical properties of material particles. To wit, instead of (4.6), we have, with (4.7), the absolute metric

$$\left(\frac{mdq_E - q_E dm}{m^2 - q_E^2 - q_M^2}\right)^2 + \left(\frac{mdq_M - q_M dm}{m^2 - q_E^2 - q_M^2}\right)^2 - \left(\frac{q_M dq_E - q_E dq_M}{m^2 - q_E^2 - q_M^2}\right)^2 \tag{4.8}$$

In spite of its appearance, this quadratic form is a surface property, insofar as it can be expressed in two variables: $x \equiv q_E/m$ and $y \equiv q_M/m$ in the form:

$$\left(\frac{dx}{1 - x^2 - y^2}\right)^2 + \left(\frac{dy}{1 - x^2 - y^2}\right)^2 - \left(\frac{ydx - xdy}{1 - x^2 - y^2}\right)^2$$

However, this property can be seen directly through a transformation suggested by the quadratic form from equation (4.7), namely:

$$m = q\cosh\psi, \quad e = q\sinh\psi, \quad \begin{matrix} q_E = q\sinh\psi\cos\theta \\ q_M = q\sinh\psi\sin\theta \end{matrix} \tag{4.9}$$

Inserting these into (4.8) results in a well-known form of the metric in ψ and θ:

$$(ds)^2 = (d\psi)^2 + \sinh^2\psi \cdot (d\theta)^2 \tag{4.10}$$

This is a metric of negative curvature, that can be uncovered for the relativistic velocity space, for instance. Obviously, the quantity q can be calculated from (4.9) and amounts to

$$q^2 = m^2 - q_E^2 - q_M^2 \tag{4.11}$$

but the absolute metric does not depend on it. In other words, the absolute metric, in the space of physical attributes of matter, like everywhere in fact, is a surface quantity, as we said. From the concept of interpretation point of view, the condition $q \neq 0$ is a nonequilibrium condition, referring to the very ensemble serving to interpretation. There are such ensembles of luxons for the interior of the absolute, as well as for the exterior of the absolute. As *per our convention*, the interior of the absolute describes the cosmic scale of matter, and in this case we have a well known case indicating the nature of q. One notices, indeed, that, towards the center of the absolute, q approaches m, and this fact entitles us in considering it as a measure of the *inertial mass*, which in this case is not exactly equal to the gravitational mass. Rather, the equation (4.11) can be rewritten as:

$$\left(\frac{q}{m}\right)^2 = 1 - x^2 - y^2 \tag{4.12}$$

showing *positively* that in building the general relativity, Einstein was, indeed, entitled to 'dismiss' the difference between the gravitational and inertial mass (Einstein, 2004), because, as Hermann Weyl uttered it, we are always entitled to 'dismiss the electricity in the economy of the universe'.

4.3 A Physical Model of Charges

The equation (4.7) suggests an important identity which is, we should say, in the natural order of things we perceive in this universe. To wit, we can assume the relation

$$e^2 = q_E^2 + q_M^2 \tag{4.13}$$

a mathematical expression signifying the idea that we cannot realize physically how much of the perceived charge is electric and how much is magnetic. This equation represents what we believe as one of the most striking pages of the modern natural philosophy. And unique of its kind too! It therefore deserves a little more elaboration, inasmuch as it is quite useful in developing a theory of the universes based not on mechanical inertia, but static universes, based on the electric and magnetic 'inertia', as it were. These universes can model life sustaining organs, like the brain for instance, which is our subject-matter in this work. Only, we have to pay close attention to an important issue in such a modeling: the living organs do not exercise their function by themselves, but in connection with other organs.

The essential physical trait of a universe, as conceived nowadays, is its *uniqueness*. This means that the universe is conceived as the world we live in, and, no matter of their space scale, all things material in this universe are to be described as simultaneously existing in such a world, and having the same fundamental physical structure. For instance all of them have the same property of inertia, all of them have to be described by the rules of dynamics, etc.: there is no discrimination in dominance of the physical attributes, of the kind we have shown above. This philosophy has always promoted the synthetic approach of the whole, as a set of parts put together by external connections, an approach that may serve the social life to a certain extent, but surely does not touch the fundamental laws of existence. The reason is that the connections in question are only those controllable by human means, and these are quite limited in number, while a part, in its capacity as organ, is itself a universe, with an unlimited number of connections, as it were.

Perhaps these ideas are not quite so clear for everybody, but the previous manner of constructing the physical universe, surely can display the essential points as clearly as possible. First, the matter is there present by a density related to its three attributes – the mass and the two charges – concurrently. The *usual* approach in physics is a 'one-by-one', so to speak: the mechanics deals in density of mass, the electrostatics deals in the density of charge, and so on. But this is not all of it: in the construction of a model universe, we assume that the *mass is dominant at any scale*, be it microcosm, daily life of macrocosm. The Einstein conclusion, sustained by the Eötvös experiments, according to which the inertial mass can be safely considered as identical to gravitational mass, is always presented as independent of space scale where the universe is contemplated. In other words, the *universe is unique*, as we said before, and the physics of its parts is independent of scale. With the model universe presented by an absolute geometry of the physical attributes, the things go in an essentially different direction of thinking.

To wit, that model was constructed bassed on the explicit quantitative dominance of the gravitational mass in each and every one of its identical representative constituents, the luxons. The whole metric geometry of the universe is presented based on an inequality:

$$q^2 \equiv m^2 - q_E^2 - q_M^2 > 0 \tag{4.14}$$

assumed to be valid in this universe. We called the quantity q inertial mass, and if we think of it in classical terms, this universe is filled with matter whose constitutive formations all satisfy (4.14). Dealing in inertial mass, the universe 'extends' until the condition $q^2 = 0$ becomes effective, which represents the infinity in its definition: *the absolute*. Physically, the absolute represents a world of particles having null inertial mass, just like de Broglie's photons in the case of light, or the Coll's luxons here. But here the analogy stops: these null mass particles – which we call luxons – are, unlike the old photons, static fictitious particles, serving merely for the wave-mechanical *interpretation* of matter.

Now, this is only what the mathematics say. Physically, the inequality (4.14) is usually taken as characterizing a universe in its entirety, which becomes unique in this way. The inertial mass is thereby controlled by the matter located beyond the spatial possibilities of human accessibility, and the equality between inertial and gravitational mass should mean that such matter prevails quantitatively regardless of scale. Therefore, regardless of how far can we extend the limit of our observations in space, the matter outside that limit is quantitatively dominant. In other words, we cannot access but an infinitesimal amount of matter in any finite space, and regardless of how much we extend our capability of knowing the universe, there will always be 'missing mass', if it is to use the guise of modern theoretical physics. Not so if we properly use the geometrical model in its fullness!

Indeed, the previous physical assumption (4.14) leaves aside the condition

$$q^2 \equiv m^2 - q_E^2 - q_M^2 < 0 \tag{4.15}$$

as being physically intractable. However, from the point of view of an absolute geometry based on (4.14), this condition represents points *outside* the absolute, and these are describable by an angular metric, as it were. In other words, these points represent phases, like de Broglie phases, characterizing the argument of the waves associated with particles. From physical point of view, we have here a world where the charge prevails over the gravitational mass, and this world is certainly real: it the world of daily life or, even better, at another scale, the microcosmos of particles constituting the matter. Incidentally, one can thus have a logical explanation why the wave mechanics made its mark especially in the microcosmos. Therefore, mathematically and physically, a universe proper cannot exist without its two worlds at three different scales, representing the structure of matter contained in it. Taken as such, the universes can be multiple with no problems: they can be described by the position of their absolutes with respect to each other. However, inasmuch as this description can become awfully involved, for the moment we shall describe a *unique universe* of the microscopic world based on the quantitative inequality (4.15), just like we did before for (4.14). This represents a world where *the charge prevails over the inertial mass*. This would be, unconditionally, the world of human brain, or of human heart, but microscopical in character, so that we can apply all the previously developed wave-mechanical theory. What, then, would be its physical basis?

This one can be understood by sheer analogy with the previous 'inertial universe' of physics, as it were. To wit, the charge of any kind, like the inertial mass in the inertial universe, should be *induced by the matter spatially located outside our perception limits*. Only, as there are two kinds of charges, things become a little more involved than in the case of inertial mass: the charge can be randomly induced as electric, as well as magnetic charge. Due to the equation (4.13) a phase proper is involved in this description, therefore a de Broglie-type wave can be associated with the constitutive particles of this universe.

Now, if wee assume the identity (4.13) the charge e splits, 'Euclidean-wise' so to speak, into an electric charge q_E and a magnetic charge q_M. In today's theoretical physics this speaks for a specific invariance of electromagnetic theory: the invariance with respect to what is today generally known as the *duality rotation* in electromagnetics. Expressed simply, this rotation is just a Euclidean rotation that leaves invariant the *experimental* electric charge e of the 'split'suggested in equation (4.9):

$$q_E = e\cos\theta, \quad q_M = e\sin\theta \tag{4.16}$$

Here θ is the angle variable describing the split among the possible experiments with the charge. Some of these experiments involve the 'non-gravitational force' as it were, in its 'electric instance', some in its 'magnetic instance', and that assignment is unrecognizable in experiment, therefore uncontrollable. The argument then goes on to declare that the Maxwellian theory, privileging the electric charge monopoles and eliminating the magnetic charge ones, represents just one possible choice of the split angle – more to the point, $\theta = 0$ – among infinitely many others. The magnetic pole case would be represented here by the choice $\theta = \pi/2$ for the experiments involving charges. Theoretically, the extra degree of freedom, represented concretely by the existence of duality rotation with respect to which the whole charge behaves invariantly, can be allocated to the known possibility of transition between *field description* by electric and magnetic intensities, and *Maxwell stresses* description (Katz, 1965). As to the *natural philosophical* reason of this possibility, it is indeed quite remarkable. Quoting:

> It is frequently pointed out that the crucial difference between electric and magnetic phenomena, which underlies this dissimilarity (*between Maxwell equation for the electric field and Maxwell equation for the magnetic field, n/a*), is that electric charges *occur in nature* as monopoles whereas magnetic charges do not so occur, *but only as dipoles*, and higher poles. This is demonstrated, for example, by breaking a permanent magnet in two. In so doing one *does not obtain free north and south poles*: each piece has again both polarities of equal magnitude. The mathematical formulation of this situation leads then to the equation div$\mathbf{B}$ = 0. On the other hand, electric charges can be obtained free, it is said.
>
> This reasoning is incomplete and deceptive. It is true that a permanent magnet *has equal and opposite magnetic charges near its ends*, and that by breaking the magnet in two and separating the parts and *inserting a chunk of empty space between them* new poles will appear on the new surfaces. But it is equally true that *electric charges occur only in equal pairs of opposite sign at opposite ends of a chunk of vacuum*, for example, by rubbing a rubber rod with a catskin and then separating the two. The vacuum between the rod and the catskin is analogous to the permanent magnet in that it has charges of equal magnitude and opposite sign at opposite ends. If we now *break the vacuum space between the two ends in two*, by *inserting, for example, an isolated conductor between them*, then charges are induced on the metal-vacuum interfaces such that each of the two chunks of vacuum carries again zero total charges at its ends. A well-known variation of this procedure is the so called ice-pail experiment of Faraday. One can pursue this reasoning further. The conclusion is that also electric charges occur only in pairs which can be looked at as the result of polarization. The only difference is that *magnetic poles appear as a result of polarization of a region of space filled with matter* (and so far no region of space filled with vacuum has yielded to polarization of this kind), whereas electric charges appear as *a result of polarization of a region of space filled with vacuum as well as at one filled with matter*.

Logically and formally it is therefore possible to treat electricity and magnetism completely similarly, as long as *one is willing to treat a region of space filled with vacuum on the same footing as a region of space filled with matter*. [(Katz, 1965), *Italics ours a/n*]

This last sentence is, in our opinion, of considerable importance for the natural philosophy: rarely, if ever, is one willing to recognize in physics that while theoretically treating the vacuum *unreservedly* as a material, one has also the obligation to *think of it* as of a material of the daily life, as of a 'chunk' in the expression of Katz. It seems to us that the electromagnetic theory has more to show than it appears at the first sight. For instance, not only it enforced the relativity at the time it did, an obligation of physics that turned the it upside down, as it were, but also tells us how to turn our very intuition into concept, and that in a right way. As far as we are concerned, the message of the previous excerpt is quite clear: the existence or non-existence of the singular magnetic poles is pending on the necessity of *describing the electromagnetic field by Maxwell stresses*. This requires, indeed, more than the wave-mechanical idea of interpretation, and we shall return to it later on, on a special occasion.

4.4 Three-Dimensional Space of Charges

The equation (4.13) unlocks a circumstance in physics similar to that unlocked by Fresnel's mathematical treatment of diffraction. Just like the light in that historical case, here too, but this time the charge, may be treated via harmonic oscillators. Fact is though, that a *dynamical* problem based on the equation of harmonic oscillator just does not make sense for the charge case. However, just like in the old case of the Fresnel's theory of light, the second order differential equation jumps our mathematical reasoning in the case of charge too. Indeed, if we assume (4.13), we should further assume that

$$e^2 = q_E^2 + q_M^2 \quad and \quad \begin{array}{l} q_E^2 = q_1^2 + p_1^2 \\ q_M^2 = q_2^2 + p_2^2 \end{array} \tag{4.17}$$

Then, as the equation (4.16) suggests, the geometry of charge space is *a priori* dictated by a second order differential equation having the phase θ, which represents the split of charge, as the independent continuous variable. The equation (4.17) extends the logic that led to the equation (4.9). It says that what happens in the case of electric charge can just as well happen in the case of magnetic charge: when physically perceiving the magnetic charge, there is an uncontrollable magnetic part of it, and an uncontrollable magnetic part of it.

Now, as we said, just like in the case of light, the second order differential equation, mathematically valid here by the nature of equation (4.17), may be physically doomed to a nonsense if the inertial mass effects are negligible. Just like in the old historical case of light. However, for the benefit of future developments here, we need to reveal that the split of charge is equivalent to an extremum property of the equation (4.13). Indeed take the case of one of the quadratic equations from among those of (4.17), to wit, let it be the equation from (4.13). Assume an experiment that reveals the value *e* for the charge. Then, naturally, we need to know how much of this charge is assigned to electric property and how much is assigned to magnetic property. We need a constraint in order to do that and, historically speaking at least, physics has assigned a linear connection between the two charges. This represents the natural condition that a real charge can always be represented as located on a straight line in the charge plane. So, if we denote the alleged experimental charges by q_1 and q_2, we are entitled to find the extremum of the function of two variables

$$e^2 = q_1^2 + q_2^2 \tag{4.18}$$

provided the linear expression, conveniently defined by the electric and magnetic properties as a bilinear form, suggested by the differentials in the metric (4.8):

$$q_2 q_E - q_1 q_M \tag{4.19}$$

has a given value. The extremum of charge is then assured for

$$q_1 = -e\cos\Phi, \quad q_2 = e\sin\Phi, \quad \tan\Phi \equiv q_M / q_E \tag{4.20}$$

In other words, the *split angle* from equation (4.16) is the supplement of phase Φ due to the indecision of experimental charge. This means that we have a theoretical reason for considering the idea of charge plane, pending on the physical meaning of the linear connection (4.19) between the two kinds of charge. As it turns out, there is a reason, not only for that, but also for the very quadratic forms from the equation (4.17).

In §2.4, we have presented an analog of the coordinate along the physical ray, necessary, according to Bartolomé Coll, for the construction of a coordinate system. The ray surface is always transversal to this coordinate line, and we have to describe this situation in terms of the coordinate line alone. Assume that it is represented as a vector in a certain reference reference frame, that vary only across the ray, more to the point perpendicularly on the coordinate line. In the case of gravitatiomal field of the Earth – which, more generally, is also the case of a Schwarzschild field in general relativity – this represents the motion of charges lifted from the crust on vortices, in order to create the electricity in the atmosphere. One needs to describe this situation exclusively with respect to the coordinale line as a *primary concept*: any external inclusion in the description would involve the coordinate system to which the coordinate line belongs, and therefore the coordinate line would not be a primary conceot anymore.

Denote $\boldsymbol{m}$ the vector long the coordinate line represented by a certain physical ray. In their capacity as coordinates along the ray, the components of the vector $\boldsymbol{m}$ reveal the *transversal* velocities:

$$\dot{m}_1 = \nu(m_2 - m_3), \quad \dot{m}_2 = \nu(m_3 - m_1), \quad \dot{m}_3 = \nu(m_1 - m_2) \tag{4.21}$$

where ν is a constant frequency and an overdot means time derivative as usual. These equations represent a motion perpendicular to the ray, as in the case of light. Now, the equations of this motion can be 'decoupled', so to speak, by successive differentiation on time. The conclusion is that each component of the motion described by (4.21) is solution of the very same third order differential equation, *i.e.* the vector $\boldsymbol{m}$ is solution of the same equation:

$$\dddot{\boldsymbol{m}} + 3\nu^2 \dot{\boldsymbol{m}} = \boldsymbol{0} \tag{4.22}$$

We are searching for constant integrals of this motion, starting from the differential system (4.21) and trying to find exact differentials that could offer us a physical interpretation to the parameter of continuity. The most obvious method involves linear forms in coordinates. For instance, we can derive the exact differential equivalent to the system (4.21):

$$\frac{adm_1 + bdm_2 + cdm_3}{(c-b)m_1 + (a-c)m_2 + (b-a)m_3} = \nu dt \tag{4.23}$$

with constant (a, b, c). Such exact differentials can exist under the conditions

$$c - b = \lambda a, \quad a - c = \lambda b, \quad b - a = \lambda c \tag{4.24}$$

with λ a parameter. This means that the left hand side of the equation (4.23) is an exact differential only for the cases in which λ has as values the roots of the cubic equation:

$$\lambda(\lambda^2 + 3) = 0$$

representing the condition of compatibility of the system (4.24). This is also the characteristic equation for the differential equation (4.22). In terms of the three roots of this equation, to wit, 0 and $\pm i\sqrt{3}$, the following three complex integrals can be constructed with respect to some initial conditions at the time $t = 0$:

$$
\begin{aligned}
m_1 + m_2 + m_3 &= m_1^0 + m_2^0 + m_3^0 \\
m_1 + j^2 m_2 + j m_3 &= e^{i\nu t\sqrt{3}}(m_1^0 + j^2 m_2^0 + j m_3^0) \\
m_1 + j m_2 + j^2 m_3 &= e^{-i\nu t\sqrt{3}}(m_1^0 + j m_2^0 + j^2 m_3^0)
\end{aligned}
\tag{4.25}
$$

Here j is the *cubic root* of unity, as a counterpart of i, which is the *square root* of negative unityin the case of usual complex numbers. The three complex variables from the left hand side of this equation are related, and in quite a few ways at that, to the name of Paul Appell, and they have a tremendous importance of principle, both from physical [(Appell, 1893), p. 351] as well as from mathematical point of view (Appell, 1877).

Start by noticing that the first of the integrals (4.25) is a constant of motion. Another constant of motion is quadratic, and can be obtained from the product of the last two of them, *i.e.*:

$$
(m_2 - m_3)^2 + (m_3 - m_1)^2 + (m_1 - m_2)^2 = const \tag{4.26}
$$

The trajectory is then to be found in the intersection of this quadric with the real plane given by the first equality from equation (4.25), therefore it also belongs to the quadric

$$
m_2 m_3 + m_3 m_1 + m_1 m_2 = const \tag{4.27}
$$

and thus on a sphere.

Indeed, according to equation (4.22), the charge can be itself represented as a genuine periodical process having as components the solutions of the differential equation

$$
\ddot{\boldsymbol{m}} + 3\nu^2 \boldsymbol{m} = \boldsymbol{c} \tag{4.28}
$$

where $\boldsymbol{c}$ is an arbitrary constant vector introduced by integration. Such a phase space obviously generalizes, by dimension at least, the phase plane of a regular harmonic oscillator: as one can easily see, if we settle for a plane of coordinates in (4.21), we get a two-dimensional harmonic oscillator. The solution of (4.28), on the other hand, is offered by the velocity vector:

$$
3\nu^2 \boldsymbol{m} = \boldsymbol{c} + \boldsymbol{a}\cos(\nu t\sqrt{3}) + \boldsymbol{b}\sin(\nu t\sqrt{3})
$$

with $\boldsymbol{a}$ and $\boldsymbol{b}$ some initial conditions. It is located on the homogeneous quadratic cone, having the equation

$$
[(\boldsymbol{b}\times\boldsymbol{c})\cdot\boldsymbol{m}]^2 + [(\boldsymbol{c}\times\boldsymbol{a})\cdot\boldsymbol{m}]^2 - [(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{m}]^2 = 0 \tag{4.29}
$$

The coefficients are here decided by the initial conditions.

Let us continue along this line for a moment. From the system (4.25) we can get the one-parameter group equations of the very process thus described by equations (4.21). Namely, after taking the inverse of the appropriate matrix in (4.25) we end up with

$$
\boldsymbol{m} = [\boldsymbol{1} + (2/3)\boldsymbol{F}(t)]\cdot \boldsymbol{m}_0 \tag{4.30}
$$

where $\boldsymbol{1}$ is the 3×3 identity matrix, and $\boldsymbol{F}(t)$ is the matrix:

$$
\boldsymbol{F}(t) = \begin{pmatrix}
\cos\phi & \cos(\phi - 2\pi/3) & \cos(\phi + 2\pi/3) \\
\cos(\phi + 2\pi/3) & \cos\phi & \cos(\phi - 2\pi/3) \\
\cos(\phi - 2\pi/3) & \cos(\phi + 2\pi/3) & \cos\phi
\end{pmatrix}
$$

with $\phi(t) \equiv t\cdot(\nu\sqrt{3})$. This matrix is singular: its determinant is

$$
\begin{aligned}
\det \boldsymbol{F}(t) = \cos^3\phi + \cos^3(\phi + 2\pi/3) + \cos^3(\phi - 2\pi/3) \\
-3\cos\phi\cdot\cos(\phi + 2\pi/3)\cdot\cos(\phi - 2\pi/3)
\end{aligned}
$$

and it vanishes in view of the algebraic identity

$$
a^3 + b^3 + c^3 - 3abc \equiv (a+b+c)(a^2 + b^2 + c^2 - bc - ca - ab)
$$

and the trigonometric identity

$$\cos\phi + \cos(\phi + 2\pi/3) + \cos(\phi - 2\pi/3) = 0$$

which can be proved by direct calculation.

We can even complicate a little the equations of motion (4.21), admitting a gauging where the velocity has also a component of speed along the ray, which would correspond to propagation. This is in the spirit of a unitary description of the light phenomenon, which would include both the propagation – measured always along the ray – as well as the light motion proper – measured orthogonally to the ray – in describing the light. Mention should be made that the situation would correspond to the motion of an electric charge in the field of a magnetic pole (Poincaré, 1896). Then the equations of motion corresponding to those from (4.21) are:

$$\frac{dm_1}{lm_1 + m(m_2 - m_3)} = \frac{dm_2}{lm_2 + m(m_3 - m_1)} = \frac{dm_3}{lm_3 + m(m_1 - m_2)} \tag{4.31}$$

where l and m are two parameters representing the 'amounts' in which the motion is decomposed *along the ray* and *perpendicular* to it, respectively. The integration procedure described above, leads to a differential form a little more complicated

$$\frac{adm_1 + bdm_2 + cdm_3}{[la + m(c-b)]m_1 + [lb + m(a-c)]m_2 + [lc + m(b-a)]m_3} = \nu dt$$

which can be considered an exact differential:

$$\frac{adm_1 + bdm_2 + cdm_3}{n(am_1 + bm_2 + cm_3)} = \nu dt \quad \therefore \quad am_1 + bm_2 + cm_3 = Ae^{n\nu t} \tag{4.32}$$

if, and only if, a, b, c are solution of the linear algebraic system given by:

$$(l-n)a + m(c-b) = 0$$

and its positive permutations. This system has nontrivial solutions only if the constants l, m and n satisfy the algebraic equation:

$$(l-n)[(l-n)^2 + 3m^2] = 0$$

which offers three possibilities of construction of the differentials representing the corresponding kinematics. They are given by the system of values:

$$l = n \quad \therefore \quad a = b = c$$

$$l - n = im\sqrt{3} \quad \therefore \quad a = jc; b = j^2 c$$

$$l - n = -im\sqrt{3} \quad \therefore \quad a = j^2 c; b = jc$$

Formally, then, nothing changes with respect to the preceding simpler case: it is just that we have here to do with a *harmonic of the frequency v*, rather than with the frequency itself.

There is a 'hidden' dynamics involved here, and this is, we think, the right place to bring about the name pf Paul Appell. This dynamics appeared for the first time in 1893, in the known *Traité de Mécanique Rationnelle* of Appell, Tome I, on page 351, but only as an exercise. Quoting:

> A point is moving in space, under the action of a force whose components X, Y, Z are functions of x, y, z, which verify the relations
>
> $$\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y} = \frac{\partial Z}{\partial z}, \quad \frac{\partial X}{\partial z} = \frac{\partial Y}{\partial x} = \frac{\partial Z}{\partial y}, \quad \frac{\partial X}{\partial y} = \frac{\partial Y}{\partial z} = \frac{\partial Z}{\partial x} \tag{4.33}$$
>
> Prove that the integration of the equations of motion is reduced to quadratures. [(Appell, 1893), *Exercise 16, p. 351, our translation*]

The proof is simple: first, one has to define a *complex position vector*, having as components the three complex coordinates as in equation (4.25). Then we need to define a *complex force vector*, having as components three corresponding complex quantities, constructed from the real components of force in the same manner the coordinates are constructed. Obviously, the principles of analysis allow us to infer that, if the real forces are functions of real position, the complex forces must be functions of complex positions. Therefore, using our notations for the coordinates along the ray, in the following table constructed by the rules just mentioned:

$$\begin{gathered} x^1 = m_1 + m_2 + m_3, \quad x^2 = m_1 + jm_2 + j^2 m_3, \quad x^3 = m_1 + j^2 m_2 + jm_3 \\ X^1 = X + Y + Z, \quad X^2 = X + jY + j^2 Z, \quad X^3 = X + j^2 Y + jZ \end{gathered} \tag{4.34}$$

every variable of the second line should be a function of the variables of the first line. Then notice that, under the conditions (4.33), each of the components of complex force – assumed conservative, of course – is a function only of the corresponding complex coordinate from the first line. Therefore the differential equations of motion can be written as

$$\frac{d^2 x^1}{dt^2} = X^1(x^1), \quad \frac{d^2 x^2}{dt^2} = X^2(x^2), \quad \frac{d^2 x^3}{dt^2} = X^3(x^3) \tag{4.35}$$

and can be solved by integrating twice, indeed. The property is transmitted as such over to the real corresponding quantities, because the transformations (4.34) are always nonsingular. So the Appell's result is proved.

4.5 A Special Optical Signal

In §3.4 we reproduced an excerpt from Ehrenberg's and Siday's work (Ehrenberg & Siday, 1949), containing a definition of the Aharonov-Bohm effect *avant la lettre*, as we called it there. In fact, taking it a little out of its very words, that excerpt contains a general definition of the action at distance, in the form: *the presence of a charge or a mass, but not the charge or the mass itself, is liable to arise physical phenomena*. This idea is so overrated in physics, that sometimes one can witness *gedanken* experiments whereby an empty universe is invented, with a single particle in it. But the truth is that this is just about the essence of action at a distance as described in physics today: a charge or a mass, present anywhere in the universe, raises phenomena in space anywhere else, and is aroused by such phenomena. This is how physics works, anyway, and the last part – an induction phenomenon, as it were – is what interest us most here.

The practice of EEG suggests what we think is a long due analogy between the brain activity and Earth activity. To wit, the Earth's crust is simply analogous to a skull, perhaps even with respect to their electrical properties. For, in view of the practice of EEG, there should be no doubt regarding the fact that the skull has definite electrical properties. The correspondent of these properties in the case of Earth's crust is a little obscure, but, again no doubt, so it is the very physics of the skull. However, what interests us most here, is a certain philosophy of handling a seismogram, that reminds us of a corresponding philosophy of handling an EEG signal. Further on, the brain electric activity as reflected in the EEG correlations between the different locations on the skull (Pribram, 1998) reminds us of those seismographic correlation used in locating the epicenter of the earthquakes in the case of Earth. Thus, in our opinion, it becomes a necessary chore from our part, to try and extend, along this path, the analogy between mechanics and electrodynamics, beyond its usual limits that led to the classical Maxwellian electrodynamics. A few theoretical facts, both geophysical and electrical in nature, to be presented right away, allow us to describe such an possibility of extension with a promising impact in the theoretical physics of brain itself.

The relevant mathematical problem comes down here to the *construction of a correlation between positions along different rays from the space occupied by matter, at different times*. Classically speaking, this is why we need a reference frame, in order to locate a position, and a clock in order to mark the moments of time when we locate that position. Specifically, the reference frame is usually Euclidean, and *the representation of the coordinates as lengths along any three reciprocally orthogonal directions* is just implicit, so to speak. However, while the correlation in general is thought as a kind of controlled exchange between places, here it must be constructed based upon an equation of propagation, which is actually considered as the infinitesimal form of that correlation. In classical electrodynamics as well as in the continuum mechanics of isotropic and homogeneous media, the equation of propagation is the D'Alembert equation (2.1). In view of the results of Louis de Broglie presented earlier in this work, we limit our discussion to this equation of propagation.

It is in these circumstances that G. L. Shpilker took notice of the fact that a certain treatment of a real seismogram encompasses a set of procedures which, in our opinion, should be universal from theoretical point of view (Shpilker, 1982, 1984). He takes note of the fact that a recorded seismographic signal complicates the act of interpretation beyond the harmonic oscillator structure. Then a conclusion comes out, worth considering in general: *the position of the point of recording must be defined not by lengths, but by three numbers having certain algebraical properties, necessary in order to comply with the definition of the recorded signal*! Taken as the components of a position vector, defined in the sense of Bartolomé Coll, these three numbers actually define a class of phenomena that may be related as well to an electrical activity as we presented it above. The theory can of course, be applied as such to the physics of brain, and this is quite an important conclusion. Let us see what is this all about.

Shpilker's theory starts, as we said, from the observation that a seismogram can never be represented just by the simple harmonic oscillation which, in the geometry of the D'Alembert's equation, would locally represent a plane wave. As we have shown before, the whole physics associated with an arbitrary time record is contained in the parameters of that harmonic oscillation. A general form of the recorded signal, having any realistic appearance at all, would be as a complex-valued function of a locally devised time sequence that serves to ordering the elongations recorded in the seismogram, like, for instance in equation (3.4). In the particular case of linear phase – $\gamma = 0$ in equation (3.4) – and no time dependence of the amplitude of signal – $a = 0$ in equation (3.4) – such a realistic appearance would be given through a function of time having, in Shpilker's notations, the form:

$$v_1 = |A| e^{i\alpha_0} e^{(i\omega+\beta)(t-t_0)} \tag{4.36}$$

with A, t_0, α_0, β and ω – five real constants. They are to be extracted from the seismogram itself. This is, in Shpilker's view, the most general model form of the recording. By comparison with equation (3.4) it is quite a particular model, but this is not the point here: it is the *general philosophy of handling this signal*, as developed by Shpilker, that should concern us. It starts from the observation that *the form of this signal as a function of time* is *all we are able to know* positively with a certain degree of confidence: the rest, starting from the very idea of propagation, the equation representing it, *etc.*, is just a series of hypotheses, educated assumptions, is true, but, still, assumptions. However, we need to emphasize once again from the very beginning, and subsequently certify by a genuine development, that it is the natural philosophy beyond this procedure which is universal. Let us expound a little more on this statement.

First, the equation of D'Alembert, (2.1) say – or any other equation of propagation for that matter – incorporates only a part of these hypotheses. Admiting, as Louis de Broglie did, that the propagation is described by such an equation, the problem of correlating two points in space is usually solved in physics by the corresponding Green function. However, within Shpilker's philosophy, *i.e.* from the point of view of professional needs, we might say, the emphasis is significantly changed: *it falls upon the correlation of the recorded signal with the equation of propagation*, which is a step of knowledge generally bypassed in the regular usage of an equation of propagation. For, it is quite clear that the equation (4.36), which is to be taken as a product of experiment, bestows a *physical content upon the space position where the signal is recorded*. And this physical content is described by equation (4.36) through the intermediary of a local time sequence, in exactly the same manner in which a uniform motion of a classical material point, for instance, bestows a physical content upon a certain time sequence, obtained locally with an arbitrary clock. Let us analyze the way in which, according to G. L. Shpilker, such a physical content should be brought to bear on the geometry.

Once we have at our disposal the equations (2.1) and (4.36) – in general, as we said, an *equation of propagation* and a *physical content of a local sequence of time* – the Shpilker's argument follows quite a simple logic, customary we might say: one must accept that any signal, even a recorded signal, is a solution of the D'Alembert equation, for *this equation defines the concept of signal within matter*, and the recorded signal itself is, obviously, such a signal. The only condition is that the *surface of earthquakes should be a matter surface*, a quality that makes out of it a surface of separation of the matter from space. Notice now that the representation (4.36) of the recorded signal *does not contain explicitly any space position*, be it that of the source of the seism or of the position of seismograph, but just some parameters to be read out of the seismogram. It would be therefore virtually impossible to set this physical content of the signal in connection with the equation of propagation, if one does not assume that the position of the recording point in space is somehow *contained implicitly* among the parameters representing this physical content: α_0, ω, β. Shpilker writes the solution of the equation (2.1) in the form:

$$u(\boldsymbol{x},t) = Ae^{\langle \xi | x-y \rangle + c(t-t_0)\sqrt{\langle \xi | \xi \rangle}}; \quad A \equiv |A| e^{i\alpha_0} \tag{4.37}$$

where $\boldsymbol{\xi}$ is an arbitrary complex vector. Obviously, this solution satisfies D'Alembert equation, both in the variables $(\boldsymbol{x},t)$, and in the variables $(\boldsymbol{y},t_0)$, *no matter of the vector* $\boldsymbol{\xi}$ *and amplitude A*, which are here taken as complex quantities. Consequently, the equation (4.37) actually *represents a correlator analogous to the classical Green function*, of two 'legal signals', whose 'legality' is defined according to a precise criterion: every *admissible signal* must be a solution of D'Alembert's equation. From theoretical point of view this signal must be found all over the places within matter, therefore both at the location of emission and the location of recording. Except that now the functional form of the signal at the emission position, or in fact during propagation, is somewhat more realistic, inasmuch as it is not *a priori* defined – for instance by analogy with a mechanical model, as we did in Chapter 3 above – but empirically, with a physical content defined *in the manner we define the recorded signal*.

Now the solution of our problem comes down to matching the theoretical representation (4.37) with the recorded signal from equation (4.36). In order to do this, Shpilker uses the freedom offered by the arbitrariness of the vector $\boldsymbol{\xi}$: in the surface delimiting the Earth seismically – the *surface of quakes*, as we would like to call it – he takes the signal as being of the form

$$v(\boldsymbol{x},t) = Ae^{\langle z\otimes(k+il) | x-y \rangle + (i\omega+\beta)(t-t_0)} \tag{4.38}$$

This signal reduces to that from equation (4.36) for $|x-y\rangle = |0\rangle$, which means that $|y\rangle$ *may be taken as the position of the point of recording*. Then, again, the function (4.38) is a solution of the D'Alembert equation. This time, however, special conditions must be satisfied, whereby over the recorded signal one overlays another signal, which needs to be conveniently described in order to account for the conditions in which the measurement is performed.

Before any explanation on these conditions, a word about the notations from equation (4.38): the vector $\boldsymbol{z}$, as well as $\boldsymbol{\xi}$ for that matter, is unknown. The vector $\boldsymbol{k}+i\boldsymbol{l}$ is an arbitrary complex vector, submitted by Shpilker to the constraints:

$$k_1^2 + l_1^2 = k_2^2 + l_2^2 = k_3^2 + l_3^2 = \tau^2 \tag{4.39}$$

where τ is an arbitrary real number. Further on, one denotes

$$\left|z\otimes(k+il)\right\rangle \equiv \begin{pmatrix} z_1(k_1+il_1) \\ z_2(k_2+il_2) \\ z_3(k_3+il_3) \end{pmatrix} \tag{4.40}$$

so that this is just *a complex vector with its components given by the diagonal entries of the complex matrix* $\boldsymbol{z}\otimes(\boldsymbol{k}+i\boldsymbol{l}) \equiv |z\rangle\langle k+il|$.

Now, coming back on the track of our discussion, Shpilker claims that in order to get a correct 'reconstruction' of the field from the recorded signal, as defined by equation (4.36), the coordinates of the position of recording must be expressed by the ratios:

$$y_j = l_j / k_j; \quad j = 1,2,3 \tag{4.41}$$

submitted to the conditions

$$y_2 \neq y_3, \quad y_3 \neq y_1, \quad y_1 \neq y_2 \tag{4.42}$$

which are thus necessary and sufficient for a reconstruction of the field from recorded data. Therefore Shpilker's local coordinates *are not regular coordinates*. It is more proper to say that *they define in fact a reference frame*: that reference frame in which the coordinates of the position of the recording point with respect to the point where the signal is originating, are given by the vector $\boldsymbol{y}$ having the components (4.41), submitted to the conditions from equation (4.42). The question is, what are these coordinates, from a physical point of view, and this depends entirely on the physical magnitude we are measuring locally. This is the whole point of Shpilker's natural philosophy: the location is pinpointed in space by the very quantity to be perceived at that location!

Before anything else along this line, let us finish the job we started here: to show that in order to have a solution of the problem in the form from equation (4.38), the Shpilker's demands have to be met, indeed. In order to do this, notice that from the equations (4.36) and (4.37) one gets:

$$u(\boldsymbol{y},t) = v_1 \quad \therefore \quad \begin{cases} c^2(\xi_1^2+\xi_2^2+\xi_3^2) = \beta^2 - \omega^2 + 2i\beta\omega \\ c^2(\left|\xi_1\right|^2+\left|\xi_2\right|^2+\left|\xi_3\right|^2) = \beta^2 + \omega^2 \end{cases} \tag{4.43}$$

by the virtue of the fact that $\boldsymbol{\xi}$ has complex components in general. In fact, according to equation (4.38) these components are defined by equation (4.40), so that the arbitrariness of the vector $\boldsymbol{\xi}$, having six real components, is transferred into the ambiguity of the vectors $\boldsymbol{z}$, $\boldsymbol{k}$ and $\boldsymbol{l}$, which involve nine real components. The situation becomes normal if we have three relations connecting these last two vectors, which must be *measurement constraints*, as those given by Shpilker in equation (4.39), which can be taken just naturally as such. Therefore, using the equations (4.41) in (4.43), the components of vector $\boldsymbol{z}$ can be found as solutions of the linear system:which is obviously equivalent to the system:

$$\begin{aligned} &c^2(k_1^2z_1^2+k_2^2z_2^2+k_3^2z_3^2) = \beta^2 \\ &c^2(k_1l_1z_1^2+k_2l_2z_2^2+k_3l_3z_3^2) = \beta\omega \\ &c^2(l_1^2z_1^2+l_2^2z_2^2+l_3^2z_3^2) = \omega^2 \end{aligned} \tag{4.44}$$

This system is in turn compatible, and has unique solution if, and only if, its principal matrix is nonsingular. The determinant of this matrix can be easily calculated, and gives:

$$\begin{vmatrix} k_1^2 & k_2^2 & k_3^2 \\ k_1l_1 & k_2l_2 & k_3l_3 \\ l_1^2 & l_2^2 & l_3^2 \end{vmatrix} \equiv (k_2l_3 - k_3l_2)(k_3l_1 - k_1l_3)(k_1l_2 - k_2l_1) \tag{4.45}$$

Therefore the compatibility of the system (4.44) comes to the fact that none of the projections of the real vector ***k*** on the planes of coordinate should be collinear with the corresponding projection of the real vector ***l***. Solving the system (4.44) results in

$$z_1^2 = \frac{1}{c^2}\frac{l_2l_3\beta^2 - (k_2l_3 + k_3l_2)\beta\omega + k_2k_3\omega^2}{(k_3l_1 - k_1l_3)(k_1l_2 - k_2l_1)}$$

and its even permutations over indices 1, 2, 3. Considering now the definitions from equation (4.41), we will have right away:

$$z_1^2 = \frac{1}{c^2k_1^2}\frac{y_2y_3\beta^2 - (y_2 + y_3)\beta\omega + \omega^2}{(y_3 - y_1)(y_1 - y_2)} \tag{4.46}$$

and two more relations, given by the even permutations over indices 1,2,3. This proves the necessity and sufficiency of the Shpilker's conditions (4.42), showing moreover that the vector ***k*** must have all its components nonnull for a reconstruction of the field in finite terms. Consequently, the triple $|y\rangle$ represents here the position of recording point of the earthquake, according to its definition contained in the particular relation between D'Alembert equation – describing the signal propagation – and the functional form of the recorded signal.

For G. L. Shpilker – as well as for the whole geophysicists' community, in fact – such a resolution of the problem of quakes is essential. Indeed, the seismogram is actually a *single* closed-form expresssion of a limiting condition in space and time for an equation of propagation – the case in point is D'Alembert equation (2.1) – and, for instance, a conceivable Cauchy problem of this case cannot be solved. Such a solution needs boundary conditions in multiple points on a surface. Usually, for solving such a problem in the case in point, one would need conditions over *the entire surface of the Earth,* defined by the existence of quakes. First of all, such a surface cannot be defined itself, even if we disregard the idea of seismogram, to say nothing of the fact that one cannot place seismographs all over the places where an earthquake is felt, in order to make the necessary measurements. It is therefore instrumental, indeed, to built a signal as the solution of the equation of propagation, *starting from data recorded sporadically*, insular data at best. Which is what Shpilker's theory accomplishes in a brilliant way. This approach has, however, much more general connotations, even fundamental we should say, from the point of view of the theoretical physics. These connotations can be extracted even limiting ourselves to the classical differential geometrical idea of adaptation of a reference frame to a surface embedded in space (Delphenich, 2013).

There is not too much to add over what we just said above, in order to see in this approach of handling the seismic signal, an opportunity of extension to the brain EEG case. The coordinates ***y***, in particular, can be useful in the theory of *evoked potentials*. In seismology they give the coordinates of recording point with respect to the point of initiation of the quake. In the case of EEG, they might as well represent the coordinates of the point of initiation of the neuronal activity having as result the skull recording. Only, in this last case we need to be assured that the vector ***y*** is somehow related to an electric or magnetic activity, which is indeed the general case, if we connect them with the split angle of the charge induced by a recorded signal in a place on the skull.

Indeed, in the previous development we have followed as much as possible the Shpilker's notations. However, even in these notations, it is not too much to say what has already been said, in order to see that in equation (4.39) the parameter τ can be identified with the charge (4.16), and the three Shpilker coordinates (4.41) can be, in fact, three arbitrary charge splits of the very same charge. This interpretation corresponds to a quite natural philosophy of the EEG: the electric signal, initiated somewhere in the brain is perceived on the skull, just like the seismic signal. It creates a local charge, and this is actually measured, Any three splits of this charge define, as in equation (4.20), three coordinates of the recording point with respect to the point of initiation of the signal. The coordinates of such a location are by no means unique, but this arbitrariness can be tied up with the arbitrariness of the reference frame with respect to which we locate the position on the skull.

In order to carry on physically and mathematically such a natural philosophy, a little inventory of the minimal necessities may be in order. First, in the order of things necessary, comes the *definition of the general signal* to replace that from equation (4.36). If the mechanical analogy is to be maintained – and everything recommends it – then the signal given in equation (3.4) is our best candidate, and we shall pursue this idea, under the guidance of Shpilker example, of course. Second in the order of things necessary, comes an *appropriate equation of propagation*, in order to replace, more realistically the idealistic D'Alembert equation. Just as in the case of Schrödinger equation for the case of interpretation, our best candidate is the *heat equation*, inasmuch as it satisfies to the same invariance group as the Schrödinger equation. Besides, its theoretical handling asks for a definite relationship between space and time measures, which is instrumental for a physical theory. Thirdly, the Shpilker's natural philosophy specifically asks for solving an issue raised by the practice of EEG: what is the connection between the *coordinate along the ray* and *any other geometrical coordinate* we may conveniently use in the physics of brain? This can be best understood from the following words of renowned neurologist Karl Lashley, as quoted in the 1998 work of Karl Pribram:

> Here is the dilemma. *Nerve impulses are transmitted over definite, restricted paths* in the sensory and motor nerves and in the central nervous system from cell to cell through definite intercellular connections. Yet *all behavior seems to be determined by masses of excitations*, by the form or relations or proportions of excitation within general fields of activity, without regard to particular nerve cells. It is the pattern, and not the element that counts. *What sort of nervous organization might be capable of responding to a pattern of excitation without limited specialized paths of conduction*? The problem is almost universal in the activities of the nervous system and some *hypothesis* is needed to direct further research. [(Pribram, 1998); our emphasis]

We translate this problem as it is suggested in Pribram's own work, with the benefit of our de Broglie model of ray: through the brain the impulses are transmitted along rays, and from ray to ray when necessary. Yet the connection between the brain locations of specific memory seems to be a direct connection independent of the paths thus formed inside the brain. What is the relationship between the two? We present a natural mathematical hypothesis, in the spirit of Karl Lashley, 'to direct further research', and develop, up to a point, its further consequences. Finally, the fourth of things necessary is the *general meaning of electromagnetics* and its delimitation in the case of brain. It will be shown that a Yang-Mills generalization of electromagnetism will do. This line shall be pursued here, as a gauging procedure, thus encompasssing the idea that the Yang-Mills fields are able to include any properties of the nerve impulses, not only those centered around the electrical properties of these impulses (Drukarch, Holland, Velichkov, Geurts, Voorn, Glas, & de Regt, 2018).

4.6 A Yang-Mills Gauging

Zenaida Uy's scaling procedure, (Uy, 1976) can be taken as a procedure of defining some electromagnetic quantities, indeed, related to the Shpilker coordinates of the point of recording. In view of the fact that in EEG the recording point is on the skull, the electromagnetic quantities in question should be thus connected with the electric properties of the skull itself. But there is more to it: as it turns out, the very same coordinates must be also connected with some magnetic properties. And thus, they can be made responsible for the shape of the magnetic signals recorded in a EMG around the skull.

In order to show this, let us describe the Uy's scaling procedure. It is connected with a geometry of the quadratic quantities (y_2y_3, y_3y_1, y_1y_2) which appear in the definition (4.46) above or, in fact in the equations (4.26) and (4.27) in connection with Appell coordinates. To start with, if one identifies Shpilker coordinates with the components of the vector $\boldsymbol{m}$, the equations (4.21) can be written in the compact form

$$\dot{y}_k + \nu\langle 1|\boldsymbol{h}_k|y\rangle = 0; \quad \langle 1| \equiv (1,1,1) \tag{4.47}$$

Here the 3×3 matrices $\boldsymbol{h}_k$ are constructed from the structure constants of the rotation group according to the rule: $(\boldsymbol{h}_k)_{ij} = \varepsilon_{kij}$, where the total skew-symmetric Levi-Civita tensor $\boldsymbol{\varepsilon}$. This definition results in the following three matrices forming a closed algebraic system, indeed, which generates the exponential part of the three-dimensional rotation group in the Euclidean space, the part connected with the identity matrix:

$$\boldsymbol{h}_1 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}; \boldsymbol{h}_2 \equiv \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \boldsymbol{h}_3 \equiv \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.48}$$

These matrices are, indeed, a three-dimensional linear basis in the space of skew-symmetric 3×3 matrices. Choose now the vector having components of second degree mentioned above, as the column matrix:

$$\boldsymbol{b}_0 \equiv \begin{pmatrix} y_2y_3 \\ y_3y_1 \\ y_1y_2 \end{pmatrix} \tag{4.49}$$

The action of the basis matrices (4.48) on this vector defines three new vectors

$$\boldsymbol{h}_1 \cdot \boldsymbol{b}_0 \equiv y_1 \begin{pmatrix} 0 \\ y_2 \\ -y_3 \end{pmatrix}; \boldsymbol{h}_2 \cdot \boldsymbol{b}_0 \equiv y_2 \begin{pmatrix} -y_1 \\ 0 \\ y_3 \end{pmatrix}; \boldsymbol{h}_3 \cdot \boldsymbol{b}_0 \equiv y_3 \begin{pmatrix} y_1 \\ -y_2 \\ 0 \end{pmatrix} \tag{4.50}$$

Zenaida Uy builds then the following *four* vectors

$$\begin{gathered} \boldsymbol{b}_1 \equiv f(\boldsymbol{y})\cdot(\boldsymbol{h}_1\cdot\boldsymbol{b}_0); \qquad \boldsymbol{b}_2 \equiv f(\boldsymbol{y})\cdot(\boldsymbol{h}_2\cdot\boldsymbol{b}_0); \qquad \boldsymbol{b}_3 \equiv f(\boldsymbol{y})\cdot(\boldsymbol{h}_3\cdot\boldsymbol{b}_0) \\ \boldsymbol{b}_4 \equiv g(\boldsymbol{y})\cdot(\boldsymbol{1}\cdot\boldsymbol{b}_0) \end{gathered} \tag{4.51}$$

where f and g are two functions, arbitrary for the moment. Then she takes notice of the important fact that *the tetrad* ($\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$, $\boldsymbol{b}_4$) can be so chosen as to represent *static Yang-Mills* SU(2) fields. As usually acknowledged, the Yang-Mills fields, in general, are to be considered as the natural generalization of the classical electromagnetic fields, to which one adds the properties brought about by the relativistic and wave-mechanical concepts. However we take them here as they were considered at the moment they were discovered: it is this take that can have any meaning for the physics of brain, as we shall see in the present work. Quoting the abstract of the original work of Chen-Ning Yang and Robert Mills:

> It is pointed out that the usual principle of invariance under isotopic spin rotation *is not consistent with the concept of localized fields*. The possibility is explored of having *invariance under local isotopic spin rotations*. This leads to formulating *a principle of isotopic gauge invariance* and the existence of a ***b*** field which *has the same relation to the isotopic spin* that *the electromagnetic field has to the electric charge*. The ***b*** field satisfies nonlinear differential equations. The quanta of the ***b*** field are *particles with spin unity*, and *electric charge* ±*e* or zero [(Yang & Mills, 1954); our italics].

In connection with this agenda, it is best to explain *our incentives* in considering the Yang-Mills fields here.

For the moment being it is sufficient to declare that these incentives issue all from the idea of memory as defined holographically. A hologram represents a 'deposit' as it were, of a space figure on a certain surface. However, rarely, if ever, is it noticed that this 'deposit' shares with the space original its dimensionality. Specifically, it is three-dimensional just like the space figure it describes, and this fact can be revealed algebraically (Mazilu, Agop, & Mercheș, 2019), Chapter 7]. The physical properties are quite specifically deposited on a surface: in the variation of its curvature parameters. Now, the matrices describing the physical variations of curvature share with the isotopic classical fields the property of three-dimensionality, and this shall be further detailed in the present work. The physics itself can be connected to the space and time only via a reference frame, and this is the whole morale of Zenaida Uy's gauging procedure. In short, it solves one of the important issue of the definition of a reference frame: it has to be defined by a statical condition, to which any dynamics should be referred.

The gauge field intensities related to $\boldsymbol{b}$ fields (4.51) are given by a generalization of the classical electrodynamics relations. These are modified according to Yang-Mills nonabelian prescription [(Yang & Mills, 1954); see also (Wu & Yang, 1969)]

$$\boldsymbol{f}_{\mu\nu} \equiv \partial_\mu \boldsymbol{b}_\nu - \partial_\nu \boldsymbol{b}_\mu - \boldsymbol{b}_\mu \times \boldsymbol{b}_\nu \tag{4.52}$$

under the following 'equations of motion', as it were:

$$\partial_\nu \boldsymbol{f}_{\mu\nu} + \boldsymbol{b}_\nu \times \boldsymbol{f}_{\mu\nu} = 0; \quad \partial_\nu \boldsymbol{b}_\nu = 0 \tag{4.53}$$

Here, the usual summation over repeated indices is assumed. The static feature of the field is explicitly recognized in the fact that the tetrad $|b\rangle$, and the corresponding field intensities (4.52), do not depend explicitly on any 'time' coordinate, y_4 say, that might incidentally complete the position $\boldsymbol{y}$ to an event. Zenaida Uy specifically assumes that the functions f and g depend on coordinates via the 'volume' $\omega = y_1 \cdot y_2 \cdot y_3$ of the cuboid whose diagonal is the vector $\boldsymbol{y}$, and, additionally, that $g = \pm f$. One can then calculate the 'electric' and 'magnetic' gauge field intensities associated to the tetrad $|b\rangle$, by the following prescriptions, replicating the well known classical definition of electromagnetic field intensities (Wu & Yang, 1969):

$$\boldsymbol{E}_k \equiv i\boldsymbol{f}_{k4}; \quad \boldsymbol{H}_k \equiv \frac{1}{2}\varepsilon_{kij}\boldsymbol{f}_{ij} \tag{4.54}$$

Calculating effectively the electric field here, with (4.51) and the definition (4.52) of field intensities, we have the result:

$$\boldsymbol{f}_{k4} = -\partial_k \boldsymbol{b}_4 - \boldsymbol{b}_k \times \boldsymbol{b}_4 \quad \therefore \quad \boldsymbol{E}_k = -i\left[(\Sigma y_2 y_3)(g^{-1}g')\boldsymbol{b}_4 + g\boldsymbol{e}_k\right] \tag{4.55}$$

where the summation runs over the positive permutations of the numerical indices, and a prime means derivative with respect to the unique variable – in this case ω – as usual. It is quite important to take notice here of the form of the vectors that we denoted in equation (4.55) by $\boldsymbol{e}_k$. These are given by the columns of a quasi-orthogonal matrix that plays a crucial part in the *space inversions*. This matrix is, indeed, of the form involved in the Maxwell's construction of the electromagnetic stresses, and we called 'equivalent to a vector' [see (Mazilu, Agop & Merches, 2019), equations (4.37) ff.]

$$\boldsymbol{e}_{ij} = y_i y_j - \frac{1}{2}\boldsymbol{y}^2\delta_{ij} \tag{4.56}$$

We shall return later to this important issue. For now, let us calculate the magnetic fields: using again (4.51), (4.52), and the second of (4.54), we further have

$$f_{ij} = \omega f^2(\omega)\varepsilon_{ijk} y_k \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \quad \therefore \quad \boldsymbol{H}_k = \omega f^2(\omega) y_k \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \tag{4.57}$$

Here we have to notice that the second condition from equation (4.53) is an identity, for we have

$$\partial_\nu \boldsymbol{b}_\nu \equiv \partial_k \boldsymbol{b}_k = [\omega f'(\omega) + f](\Sigma \boldsymbol{u}_k)$$

and in view of the fact that $\Sigma \boldsymbol{u}_k = \boldsymbol{0}$ the result is automatically the null vector.

The *geometric* interpretation of this situation is as follows. Take the elementary vector surface 1-forms:

$$ds^k = \varepsilon^{klm} y_l dy_m \tag{4.58}$$

They can be written as the bilinear forms, using the $\boldsymbol{h}$-matrices from equation (4.48):

$$ds^k \equiv \langle y | \boldsymbol{h}^k | dy \rangle \tag{4.59}$$

so that the column matrices used in the definitions of the vectors from (4.50) are, in fact, defined by identities $\boldsymbol{u}_k \equiv \boldsymbol{h}_k \cdot |y\rangle$. The problem now, is the presence in our theory of the vector from equation (4.49). As one can see, however, a justification is not altogether out of hand: it is the esssential vector for the absolute geometry based upon classical idea of volume. For instance, we get the important gauging relation mentioned above, by noticing that

$$\boldsymbol{y} \cdot \boldsymbol{b}_0 = 3 y_1 y_2 y_3 \tag{4.60}$$

which allows for a statistical discussion of stresses as fluxes, as we shall show here pretty soon.

(to be continued)

Conclusions

The living brain should be physically modeled a universe. Its constitutive unit – the neuron – handles the charges: electric as well as magnetic. By analogy with the existing physical model of the universe, where the gravitation prevails, in the brain universe though, where the charge prevails, the charge is created just like the inertial mass, under the influence of the matter at infinity. This may explain, for instance, the concentration of life in the places of high seismic activity, a circumstance helping in defining the concept of infinity for this model. This preliminary part develops the mathematics by analogy with the mathematics of seisms in the physics of Earth’s crust. This can be taken as the analog of the skull.

REFERENCES

Acebrón, J. A, Bonilla, L. L., Pérez Vicente, C. J., Ritort, F., Spigler, R. (2005): *The Kuramoto Model: A Simple Paradigm for Synchronization Phenomena*, Reviews of Modern Physics, Volume **77**(1), pp. 137 – 185

Aharonov, Y., Bohm, D. (1959): *Significance of Electromagnetic Potentials în the Quantum Theory*, Physical Review, Volume **115**, pp. 485–491

Alfaro, V. de, Fubini, S., Furlan, G. (1976): *Conformal Invariance in Quantum Mechanics*, Il Nuovo Cimento A, Volume **34**, pp. 569–611

Albeverio, S., Høegh-Krohn, R. (1974): *A Remark on the Connection Between Stochastic Mechanics and the Heat Equation*, Journal of Mathematical Physics, Volume **15**, pp. 1745–1747

Appell, P. (1877): *Propositions d'Algèbre et de Géométrie Déduites de la Considération des Racines Cubiques de l'Unité*, Comptes Rendus de l'Académie des Sciences Paris, Tome **84**, pp. 540 – 543 & 1378 – 1380

Appell, P. (1893): *Traité de Mécanique Rationnelle*, Tome **I**, Gauthier-Villars, Paris

Barbilian, D. (1937): *Die von Einer Quantik Induzierte Riemannsche Metrik*, Comptes Rendus de l'Académie Roumaine des Sciences, Volumul **2**, p. 198

Bellman, R. (1997): *Introduction to Matrix Analysis*, Society for Industrial and Applied Mathematics, Philadelphia PA

Berry, M. V., Klein, G. (1984): *Newtonian Trajectories and Quantum Waves in Expanding Force Fields*, Journal of Physics A: Mathematical and General, Volume **17**(8), pp. 1805 – 1815

Bianchi, L. (2001): *On Three-Dimensional Spaces Which Admit a Continuous Group of Motions*, General Relativity and Gravitation, Volume **33**, pp. 2171 – 2253 (This is a translation of the original Italian from year 1897, by R. T. Jantzen under *Golden Oldies* initiative of the periodical)

Boyer, T. H. (1969): *Derivation of the Blackbody Radiation Spectrum without Quantum Assumptions*, Physical Review, Volume **182**(4), pp. 1374 – 1383

Boyer, T. H. (1975): *Random Electrodynamics: the Theory of Classical Electrodynamics with Classical Electromagnetic Zero-Point Radiation*, Physical Review D, Volume **11**, pp. 790 – 808

Broglie, L. de (1923): *Ondes et Quanta*, Comptes Rendus de l'Académie des Sciences, Paris, Tome **177**, pp. 507 – 510

Broglie, L. de (1926): (a) *Sur le Parallélisme entre la Dynamique du Point Matériel et l'Optique Géométrique,* Le Journal de Physique et le Radium, Série VI, Tome **7**(1), pp. 1 – 6; (b) *Sur la Possibilité de Relier les Phénomènes d'Interférence et de Diffraction à la Théorie des Quanta de Lumière,* Comptes Rendus de l'Académie des Sciences de Paris, Tome **183**, pp. 447–448; (c) *Interference and Corpuscular Light*, Nature, Volume **118**, pp. 441 – 442; (d) *Les Principes de la Nouvelle Mécanique Ondulatoire,* Le Journal de Physique et le Radium, Tome **7**(10), pp. 321 – 337; (e) *Ondes et Mouvements*, Gauthier-Villars, Paris

Broglie, L. de (1961): *Sur la Thermodynamique du Corpuscule Isolé,* Comptes Rendus de l'Académie des Sciences de Paris, Tome **253**, pp. 1078 – 1081

Broglie, L. de (1962): (a) *Nouvelle présentation de la Thermodynamique du Particule Isolé,* Comptes Rendus de l'Académie des Sciences de Paris, Tome **255**, pp. 807 – 810; (b) *Quelques conséquences de la Thermodynamique du Particule Isolé*, Ibidem, pp. 1052 – 1054

Broglie, L. de (1966): *Thermodynamique de la Particule Isolée ou Thermodynamique Cachée de la Particule*, Gauthièr-Villars, Paris

Brown, L. M. (Editor) (2005): *Feynman's Thesis – a New Approach to Quantum Mechanics*, World Scientific, Singapore

Caldirola, P. (1941): *Forze Non Conservative nella Meccanica Quantistica*, Il Nuovo Cimento, Volume **18**(9), pp. 393 – 400

Caldirola, P. (1983): *Quantum Theory of Nonconservative Systems*, Il Nuovo Cimento B, Volume **77**(2), pp. 241 – 262

Cartan, E. (1951): *La Théorie des Groupes Finis et Continus et la Géométrie Différentielle Traitées par la Méthode du Repère Mobile*, Gauthier-Villars, Paris

Cohen, L. (1995): *Time-Frequency Analysis*, Prentice Hall, Englewood Cliffs, New Jersey

Coll, B. (1985): *Coordenadas Luz en Relatividad*, In Proceedings of the 1985 Spanish Relativity Meeting, A. Molina Editor, Barcelona, Spain, pp. 29 – 38. There is an English translation as *Light Coordinates in Relativity*, available on the homepage of Bartolomé Coll at Observatoire de Paris/CNRS

Coll, B. (2001): *Elements for a Theory of Relativistic Coordinate Systems. Formal and Physical Aspects*, In *Reference Frames and Gravitomagnetism*, Proceedings of the EREs 2000 Spanish Relativity Meeting, J. F. Pascual-Sánchez, L. Floría, A. San Miguel, and F. Vicente, Editors, Valladolid, Spain, pp. 53 – 65.

Coll, B., Morales, J. A. (1988): *Sur les RepèresSymmétriques Lorentziens*, Comptes Rendus de l'Académie des Sciences Paris, Tome **306**, Série I, pp. 791 – 794

Coll, B., Morales, J. A. (1991): *Symmetric Frames on Lorentzian Spaces*, Journal of Mathematical Physics, Volume **32**(9), pp. 2450 – 2455

Darwin, C. G. (1927): *Free Motion in the Wave Mechanics*, Proceedings of the Royal Society of London Series A, Volume **117**, pp. 258 – 293

Debbasch, F. (1993): *Classical Mechanics and Gauge Theories*, American Journal of Physics, Volume **61**(12), pp. 1131 – 1133

Delphenich, D. H. (2013): (a) *A Strain Tensor that Couples to the Madelung Stress Tensor*, arXiv:1303.3582v1 [quant-ph]; (b) *The Use of Teleparallelism Connection in Continuum Mechanics*, arXiv: 1304.16477v1 [gr-qc]

Dirac, P. A. M. (1931): *Quantised Singularities in the Electromagnetic Field*, Proceedings of the Royal Society of London, Series A, Volume **133**, pp. 60 – 72

Dirac, P. A. M. (1933): *The Lagrangian in Quantum Mechanics*, Physikalische Zeitschrift der Sowietunion, Volume **3**(1), pp. 64 – 72; reproduced in (Brown, 2005)

Dirac, P. A. M. (1948): *The Theory of Magnetic Poles*, Physical Review, Volume **74**(6), pp. 807 – 830

Dirac, P. A. M. (1962): *An Extensible Model for the Electron*, Proceedings of the Royal Society of London, Series A, Volume **268**, pp. 57 – 67

Drukarch, B., Holland, H. A., Velichkov, M., Geurts, J. J.G., Voorn, P., Glas, G., de Regt, H. W. (2018): *Thinking about the Nerve Impulse: a Critical Analysis of the Electricity-Centered Conception of the Nerve Excitability*, Progress in Neurobiology, Volume **169**, pp. 172 – 185

Duval, Ch., Ovsienko, V. (2000): *Lorentzian Worldlines and the Schwarzian Derivative*, Functional Analysis and its Applications, Volume **34**(2), pp. 135 – 137

Eells, J., Sampson, J. H. (1964): *Harmonic Mappings of Riemannian Manifolds*, American Journal of Mathematics, Volume **86**, pp. 109–160

Ehrenberg, W., Siday, R. E. (1949): *The Refractive Index in Electron Optics and the Principles of Dynamics*, Proceedings of the London Physical Society, Section B, Volume **62**(1), pp. 8 – 21

Einstein, A. (1905): *On the Electrodynamics of Moving Bodies*, in *The Principle of Relativity, a Collection of Original Memoirs on the Special and General Theory of Relativity*, Dover Publications, 1923, 1952, 2003. English translation of the original from Annalen der Physik, Volume **17** (1905)

Einstein, A. (1909): *Planck's Theory of Radiation and the Theory of Specific Heat*, The Collected Papers of Albert Einstein, Volume 2, Princeton University Press; English translation of the original from Deutsche Physikalische Gesellschaft, Verhandlungen **7**, pp. 482 – 500; also published in Physikalische Zeitschrift, Volume **10**, pp. 817 – 825.

Einstein, A. (1956): *Investigations on the Theory of the Brownian Movement*, edited with notes by R. Fürth, Dover Publications, Inc., New York

Einstein, A. (1965): *Concerning a Heuristic Point of View toward the Emission and Transformation of Light*, American Journal of Physics, Volume **33**(4), pp. 367 – 374; English translation of the original from Annalen der Physik, Volume **17** (1905)

Einstein, A. (2004): *The Meaning of Relativity,* Routledge Classics, London and New York

Feynman, R. P. (1949): *The Theory of Positrons*, Physical Review, Volume **76**, 749 – 759

Flanders, H. (1970): *The Schwarzian as a Curvature*, Journal of Differential Geometry, Volume **4**(4), pp. 515 – 519

Fresnel, A. (1827): *Mémoire sur la Double Réfraction*, Mémoirs de l'Académie des Sciences de l'Institute de France, Tome **7**, pp. 45–176; reproduced in Oeuvres Complètes, Imprimerie Impériale, Paris 1866

Gabor, D., (1948): *A New Microscopic Principle*, Nature, Volume **161**, pp. 777 – 778

Gabor, D. (1949): *Microscopy by Reconstructed Wave-Fronts*, Proceedings of the Royal Society of London, Series A, Volume **197**, pp. 454 – 487

Gradshteyn, I. S., Ryzhik, I. M. (2007): *Table of Integrals, Series and Products*, Seventh Edition, A. Jeffrey & D. Zwillinger Editors, Academic Press-Elsevier

Hagmann, P., Grant, P. E., Fair, D. A. (2012): *MR Connectomics: a Conceptual Framework for Studying the Developing Brain*, Frontiers in Systems Neuroscience, Volume **5**, Article 43

Harvey, R. J. (1966): *Navier-Stokes Analog of Quantum Mechanics*, Physical Review, Volume **152**, p. 1115

Heisenberg, W. (1925): *Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen*, Zeitschrift für Physik, Volume **33**, pp. 879 – 893

Herrero, A., Morales, J. A. (1999): *Radial Conformal Motions in Minkowski Space-Time*, Journal of Mathematical Physics, Volume **40**(6), pp. 3499 – 3508

Herrero, A., Morales, J. A. (2010): *Painlevé-Gullstrand Synchronizations in Spherical Symmetry*, Classical and Quantum Gravity, Volume **27**, 175007

Hertz, H. (2003): *The Principles of Mechanics, Presented in a New Form*, Dover Phoenix Editions

Hobbes, T. (1644): *Tractatus Opticus*, în *Thomae Hobbes Malmesburiensis Opera Philosophica Quae Latine Scripsit Omnia*, Volume **5**, pp. 215 – 248, J. Bohn, London, 1839

Hooke, R. (1665): *Micrographia, or Some Physiological Descriptions of Minute Bodies Made by Magnifying Glasses*, Martyn & Allestry, London

Horwitz, B. (2003): *The Elusive Concept of Brain Connectivity*, NeuroImage, Volume **19**, pp. 466 – 470

James, D. F. V., Agarwal, G. S. (1996): *The Generalized Fresnel Transform and its Applications to Optics*, Optics Communications, Volume **126**, pp. 207 – 212

Jeans, J. H. (1913): *Discussions on Radiations, with the participation of H. A. Lorentz, E. Pringsheim, A. E. H. Love, and J. Larmor*; in British Association Report on Birmingham Meeting from, September 1913, pp. 376 – 386

Kanai, E. (1948): *On the Quantization of Dissipative Systems*, Progress of Theoretical Physics (Japan), Volume **3**(4), pp. 440 – 442

Katz, E. (1965): *Concerning the Number of Independent Variables of the Classical Electromagnetic Field*, American Journal of Physics, Volume **33**, pp. 306 – 312

Larmor, J. W. (1900): *On the Statistical Dynamics of Gas Theory as Illustrated by Meteor Swarms and Optical Rays*, Nature, Volume **63**(1624), pp. 168 – 169; British Association Report, September 1900, pp. 632 – 634

Lavenda, B. H. (2012): *A New Perspective on Relativity – An Odyssey in Non-Euclidean Geometries*, World Scientific, New Jersey

Lévy, P. (1943): *Un Théorème d'Invariance Projective Relatif au Mouvement Brownien*, Commentarii Mathematici Helvetici, Tome **16**, pp. 242 – 248

Lévy, P. (1944): *Une Propriété d'Invariance Projective dans le Mouvement Brownien*, Comptes Rendus de l'Académie des Sciences Paris, Tome **219**, pp. 377 – 379

Lévy, P. (1954): *Le Mouvement Brownien*, Gauthier-Villars, Paris

Lévy, P. (1965): *Processus Stochastiques et Mouvement Brownien*, Gauthier-Villars, Paris

Lewis, G. N., (1926): *The Conservation of Photons*, Nature, Volume **118**, pp. 874 – 875

Longuet-Higgins, H. C., (1968): (a) *Holographic Model of Temporal Recall*, Nature, Volume **217**, p. 104; (b) *The Non-Local Storage of Temporal Information*, Proceedings of the Royal Society of London, Series B, Volume **171**, pp. 327 – 334

Lorentz, H. A. (1904): *Electromagnetic Phenomena in a System Moving with any Velocity Smaller than that of Light*, Proceedings of the Royal Netherlands Academy of Arts and Sciences at Amsterdam, Volume **6**, pp. 809 – 831

Lorentz, H. A. (1916): *Theory of Electrons*, Teubner, Leipzig, New York

Madelung, E. (1927): *Quantentheorie in hydrodynamischer Form*, Zeitschrift für Physik, Volume **40**, pp. 322 – 326; *Quantum Theory in Hydrodynamical Form*, English translation by D. H. Delphenich; www.neo-classical-physics.info

Mandel, L. (1974): *Interpretation of Instantaneous Frequencies*, American Journal of Physics, Volume **42**, pp. 840–848

Martens, E. A, Barreto, E., Strogatz, S. H., Ott, E., So, P., Antonsen, T. M. (2009): *Exact Results for the Kuramoto Model with a Bimodal Frequency Distribution*, Physical Review E, Volume **79**(2), Article 026204

Maxwell, J. C. (1873): *A Treatise on Electricity and Magnetism*, Clarendon Press, London

Mazilu, N. (2010): *Black-Body Radiation Once More*, Bulletin of the Polytechnic Institute of Jassy, Volume **54**, pp. 69 – 97

Mazilu, N., Agop, M., Mercheș, I. (2019): *The Mathematical Principles of Scale Relativity Physics*, Taylor & Francis Group, London, New York

Mazilu, N., Porumbreanu, M. (2018): *Devenirea Mecanicii Ondulatorii*, Editura Limes, Cluj-Napoca (soon to appear in English as *The Coming to Being of Wave Mechanics*)

Morris, C. N. (1982): *Natural Exponential Families with Quadratic Variance Functions*, Annals of Statistics, Volume **10,** pp. 65–80

Needham, T. (2001): *Visual Complex Analysis*, Clarendon Press, Oxford

Nelson, E. (1966): *Derivation of the Schrödinger Equation from Newtonian Mechanics*, Physical Review, Volume **150,** pp. 1079–1085

Newton, Sir Isaac (1952): *Opticks, or a Treatise of the Reflections, Refractions, Inflections & Colours of Light*, Dover Publications, Inc., New York

Niederer, U. (1972): *The Maximal Kinematical Invariance Group of the Free Schrödinger Equation*, Helvetica Physica Acta, Volume **45**, pp. 802 – 810

Nottale, L. (2011): *Scale Relativity and Fractal Space-Time*, Imperial College Press, London, UK

Poincaré, H. (1896): *Remarques sur une Expérience de M. Birkeland*, Comptes Rendus de l'Académie des Sciences de Paris, Tome **123**, pp. 530 – 533

Pribram, K. H. (1998): *Thoughts on the Meaning of Brain Electrical Activity*, Internat. Journal of Psychology, Volume **33**(4.15), pp.213 – 225

Pribram, K. H. (2007): *Holonomic Brain Theory*, Scholarpedia, 2(5): 2735

Römer, O. (1676): *Demonstration Touchant le Mouvement de la Lumière*, Le Journal des Sçavans, Lundi 7 Décembre 1676, pp. 233 – 236

Römer, O. (1677): *A Demonstration Concerning the Motion of Light*, Philosophical Transactions of the Royal Society of London, Volume **12**, pp. 893 – 894

Rosen, N. (1945): *On Waves and Particles*, Journal of Elisha Mitchell Scientific Society, Volume **61**, pp. 67–73

Rosen, N. (1947): *Statistical Geometry and Fundamental Particles*, Physical Review, Volume **72**, pp. 298 – 303

Schleich, W. P., Greenberger, D. M., Kobe, D. H., Scully, M. O. (2013): *Schrödinger Equation Revisited*, Proceedings of the National Academy of Sciences USA, Volume **110**(14), pp. 5374 – 5379

Шпилькер, Г. Л. (1982): *Корректное Восстановление Полей Удовлетворяющих Уравнению Даламбера*; ДАН СССР, том **264**, pp. 1355–1358; translated in English as:

Shpilker, G. L. (1984): *The Correct Reconstruction of Fields Satisfying d'Alembert Equation*, Doklady, Earth Sciences Chapter, American Geological Institute

Schrödinger, E. (1933): *Mémoires sur la Mécanique Ondulatoire*, Librairie Félix Alcan, Paris

Shapiro, A. E. (1973): *Kinematic Optics: A Study of the Wave Theory of Light in the Seventeenth Century*, Archives for the History of Exact Sciences, Volume **11**, pp.134 – 266

Shapiro, A. E. (1975): *Newton's Definition of a Light Ray and the Diffusion Theories of Chromatic Dispersion*, Isis, Volume **66**, pp. 194 – 210

Skinner, D. (2016): *Mathematical Methods*, Course Notes at the University of Cambridge, UK

Smith, J. F., Pillai, A., Chen, K., Horwitz, B. (2012): *Effective Connectivity Modeling for fMRI: Six Issues and Possible Solutions Using Linear Dynamic Systems*, Frontiers in Systems Neuroscience, Volume **5**, Article 104

Takabayasi, T (1952): *On the Formulation of Quantum Mechanics associated with Classical Pictures*, Progress of Theoretical Physics, Volume **8**(2), pp. 143–182

Uy, Z. E. S. (1976): *A Solution to the SU(2) Classical Sourceless Gauge Field Equation*, Nuclear Physics B, Volume **110**, pp. 389 – 396

Weyl, H. (1923): *Raum. Zeit. Materie, Vorlesungen über allgemeine Relativitätstheorie*, Sechste unveränderte Auflage, Springer, Berlin

Wien, W., Lummer, O. (1895): *Methode zur Prüfung des Strahlüngsgesetzes absolut schwarzer Körper*, Annalen der Physik (Wiedemann), Band **56**, pp. 451 – 456

Wilhelm, H. E. (1971): *Hydrodynamic Model of Quantum Mechanics*, Physical Review D, Volume **1**, pp. 2278 – 2285

Wu, T. T., Yang, C. N. (1969): *Some Solutions of the Classical Isotopic Gauge Field Equations*, in 'Properties of Matter Under Unusual Conditions', H. Mark & S. Fernbach Editors, Wiley-Interscience, New York, pp. 349–354; reprinted in 'Selected Papers of C. N. Yang, with Commentaries', World Scientific, Singapore, 2005, pp. 400 – 405

Yang, C. N., Mills, R. L. (1954): *Conservation of Isotopic Spin and Isotopic Gauge Invariance*, Physical Review, Volume **96**, pp. 191–195

Zanette, D. E. (2006): *The Kuramoto's Model on Networks*, Presentation at the Conference '*Dynamics on Complex Networks and Applications*', Dresden, 2006

Zelikin, M. I. (2000): *Control Theory and Optimization*, Encyclopaedia of Mathematical Sciences, Volume **86**, Springer