Refutation of recursive enumerability and elementary frame definability in predicate modal logic

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Abstract: We evaluate two equations as not tautologous to show the standard translation of a predicate model into a logic language fails. We evaluate five equations as not tautologous, with the last result as contradictory, to show a class defined to contain frames fails to satisfy a classical first-order formula or to claim logic system equivalency. The conjecture of recursive enumerability and elementary frame definability in predicate modal logic is refuted. These results form a non tautologous fragment of the universal logic $VŁ4$.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, $F$ as contradiction, $N$ as truthity (non-contingency), and $C$ as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)


Abstract We investigate the relationship between recursive enumerability and elementary frame definability in first-order predicate modal logic. … We also present an example of a normal predicate modal logic that is recursively enumerable, Kripke complete, and not complete with respect to an elementary class of rooted frames, but is complete with respect to an elementary class of frames that are not rooted.

3 An example over rooted frames In this section, we exhibit the normal predicate modal logic $L_0$ which is recursively enumerable, Kripke complete, but not complete with respect to an elementary class of rooted frames. … Let $ST_x(\phi)$ be the standard translation (...) of the predicate modal formula $\phi$ into the language of $QCl$-, defined as follows: …

\[
ST_x(\forall y \phi_1) = \forall y (\neg W(y) \land D(x,y) \rightarrow ST_x(\phi_1)), \tag{3.5.1}
\]

\[
\text{LET} \quad p, q, s, w, x, y: \quad D, \phi_1, St_x, W, x, y.
\]

\[
(s\&(#y\&q)) = (((\neg w\&#y)\&(p\&(x\&#y)))\&(x\&q)).
\]

\[
\begin{array}{cccc}
\text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} \quad (32) \\
\text{FFFF} & \text{FFFF} & \text{FFNN} & \text{FFNN} \quad (16) \\
\text{FNFN} & \text{FNFN} & \text{FNNN} & \text{FNNN} \quad (8) \\
\text{FFFF} & \text{FFFF} & \text{FFNN} & \text{FFNN} \quad (8) \quad (3.5.2)
\end{array}
\]

[The formula $M$ describes general properties of predicate Kripke frames; it says that the set of worlds is non-empty, that the domain of every world is non-empty, and that, if world $y$ is accessible from world $x$, the domain of $x$ is included in the domain of $y$.]

\[
\text{LET} \quad \sim \text{Not}, \neg; \quad + \text{Or}, \lor, \cup; \quad - \text{Not Or}; \quad \& \text{And}, \land, \cap, \cdot, \Theta; \quad \setminus \text{Not And};
\]

\[
> \text{Imply, greater than}, \rightarrow, \Rightarrow, >; \quad < \text{Not Imply, less than}, \in, \subseteq, \subset, \prec, \preceq, \leq; \quad = \text{Equivalent}, \equiv, \equiv, \leftrightarrow, \iff, \Leftrightarrow, \approx, \asymp; \quad @ \text{Not Equivalent}, \neq, \oplus;
\]

\[
\%	ext{ possibility, for one or some}, \exists, \exists!, \Diamond, M; \quad \# \text{ necessity, for every or all}, \forall, \Box, L;
\]

\[(z=z) \text{ T as tautology, } \\
\text{F as contradiction, } \emptyset, \text{ Null, } \bot; \quad (z\neq z) \text{ F as contradiction, } \emptyset, \text{ Null, } \bot; \quad (z\neq z) \text{ F as contradiction, } \emptyset, \text{ Null, } \bot; \quad (z\neq z) \text{ F as contradiction, } \emptyset, \text{ Null, } \bot;
\]

\[
\Leftrightarrow \text{ Equivalent, } \equiv, \equiv, \leftrightarrow, \iff, \Leftrightarrow, \approx, \asymp;
\]

\[
\text{Note for clarity, we usually distribute quantifiers onto each designated variable.}
\]
Let \( M = \exists x W(x) \land \forall x [W(x) \rightarrow \exists y D(x,y)] \land \forall x \forall y \forall z [W(x) \land W(y) \land \neg W(z) \land R(x,y) \land D(x,z) \rightarrow D(y,z)] \).  

(3.6.1)

\[
(z = z) = (((w \& \%x) \& ((w \& \#x) > (p \& (#x \& \%y)))) + \\
(((w \& \#x) \& (w \& \#y)) \& \neg (w \& \#z)) \& (((r \& (#x \& \#y)) \& ((p \& (#x \& \#y)) > (p \& (#y \& \#z)))));
\]

\[
\text{FFFF FFFF FFFF FFFF (8) } \times 3 \\
\text{CCCC CCC CCCC CCCC (8) }
\]

\[
\text{FFFF FFFF FFFF FFFF (8) } \times 3 \\
\text{CCCC CCC CCC CCC (8) }
\]

\[
\text{FFFF FFFF FFFF FFFF (8) } \\
\text{CTCT CTCT CTCT CTCT (8) (3.6.2)}
\]

Figure 2: Regular construction for \( \mathcal{E}_0 \)-frames
We now describe \( \mathcal{C}_0^* \)-frames with classical first-order formulas (also, see Figure 2).

First, we say that \( \mathcal{C}_0^* \)-frames are irreflexive and do not contain transitive chains with more than two elements:

\[
\Phi_1 = \forall x \neg R(x,x) \land \forall x \forall y \forall z (R(x,y) \land R(y,z) \rightarrow \neg R(x,z)).
\]

Second, we describe the “bottom part” of a \( \mathcal{C}_0^* \)-frame, which looks as follows: the bottom-most world \( w_1 \) sees two worlds, \( w^* \) and \( w_2 \), which is also seen from \( w^*_2 \), all these worlds being distinct (that \( w_2 \) is only seen from \( w_1 \) and \( w^*_2 \) follows from the formula \( \Phi_3 \) below):

\[
\Phi_2(w_1) = \exists w^* \exists w_2 \exists w^*_2 \forall x (R(w_1, x) \leftrightarrow x = w^* \lor x = w_2) \land R(w^*_2, w_2) \land \neg \exists x R(x, w_1) \land \neg \exists x R(x, w^*_2) \land \forall x (R(x, w^*) \rightarrow x = w_1) \land \neg \exists x R(w^*, x) \land w_1 \neq w^*_2.
\]

Third, we say that the bottom-most world \( w_1 \) is the only one that can see more than one world and that no world is seen from more than two worlds:

\[
\Phi_3(w_1) = \forall x (x \neq w_1 \rightarrow \neg \exists y \exists z (y \neq z \land R(x, y) \land R(x, z))) \land \forall w \neg \exists x \exists y \exists z (x \neq y \land y \neq z \land x \neq z \land R(x, w) \land R(y, w) \land R(z, w)).
\]

Lastly, we describe the repetitive procedure of extending an \( \mathcal{C}_0^* \)-frame by appending to the topmost world \( w \) two worlds \( y \) and \( z \), distinct from \( w \) and each other, such that \( wRyRx \) holds, as well as a world \( z \) that lies off the main chain and sees \( x \) (intuitively, \( z \) marks off \( x \) as the topmost world):

\[
\Phi_4 = \forall w [\exists y \exists z (y \neq z \land R(y, w) \land R(z, w)) \rightarrow \neg \exists x R(w, x) \lor \exists x \exists y \exists z (R(w, y) \land R(y, x) \land R(z, x) \land y \neq z \land x \neq w \land \forall u (R(u, y) \rightarrow u = w) \land \neg \exists u R(u, z))].
\]

The procedure of extending a \( \mathcal{C}_0^* \)-frame described by \( \Phi_4 \) can be carried out arbitrarily, including infinitely, many times.

Then, the class \( \mathcal{C}_0^* \) is defined to contain the frames satisfying the following classical first-order formula:

\[
\exists w_1 (\Phi_1 \land \Phi_2(w_1) \land \Phi_3(w_1) \land \Phi_4).
\]

We claim that \( L(\mathcal{C}_0^*) = L(\mathcal{C}_0) \).

\[
\Phi_1: \quad \text{(3.7.1)}
\]

\[
\text{LET} \quad p, \ q, \ r, \ s, \ t, \ u, \ v, \ w, \ x, \ y, \ z:
\]

\[
\neg ((r \&(\#x\&\#x))\&(r \&(\#y\&\#y))\&(r \&(\#z\&\#z))) \land (r \&(\#x\&\#y)) \land (r \&(\#y\&\#z)) \land (r \&(\#x\&\#z)) ;
\]

\[
\text{TTTT TTTT TTTT TTTT (16)}
\]

\[
\text{TTTT CCCC TTTT CCCC (16)}
\]

\[
\Phi_2(w_1): \quad \text{(3.8.1)}
\]

\[
((r\&(p\&\#x))\&(r \&(\#x\&\#s))\&(r \&(\#s\&\#p))\& \neg (r \&(\#x\&\#t)) \land (r \&(\#x\&\#s)) \land (r \&(\#s\&\#x)) \land (r \&(\#p\&\#t))) ;
\]

\[
\text{FFFF FFFF FFFF FFFF} \times 8
\]

\[
\text{FFFF FFFF FFFF FFFF} \times 16
\]

\[
\Phi_3(w_1): \quad \text{(3.8.2)}
\]
Φ₃(ₐ₁):

(\#x@p)=(~\%y@\%z)&(~(r&(\#x&\%y))&(r&(\#x&\%z))))))&(~((\%x@\%y)&

(\%y@\%z))&(\%x@\%z))&(r&(\%x&\#w))&(r&(\%y&\#w))&(r&(\%z&\#w))))) ;

Φ₄:

((\%y@\%z)+(r&(\%y&\#w))+(r&(\%z&\#w))))>((r&(\#w&~\%x))+

(((r&(\#w&\%y))&(r&(\%y&\%x))&(r&(\%z&\%x))&(r&(\%y@\%z)))&

(\%x@\%z)))>(#u=#w)&(r&(\%u&\%z)))) ;

∃(ₐ₁):

((~(r&(\%x&\%y))(r&(\#x&\%z)))))

& (((r&(\%p&\%x))=((-\%x=\%s)+(#x=\%p))&(r&(\%t&\%q)))&(r&(\%x&

\%p))&(r&(\%x&\%t)))) &(((r&(\#x&\%s))>(#x=#p))&(r&(\%s&~\%x))&(p@%

\%t)))) &

(((\%x@\%p)>((-\%y@\%z)&(r&(\%x&\%y))&(r&(\%x&\%z)))))))&((\%x@\%y)&

(\%y@\%z))&(\%x@\%z))&(r&(\%x&\#w))&(r&(\%y&\#w))&(r&(\%z&\#w))))))

& (((\%y@\%z)+(r&(\%y&\#w))+(r&(\%z&\#w))))>((r&(\#w&~\%x))+(((r&(\%w&

\%y))&(r&(\%y&\%x))&(r&(\%z&\%x))&(\%y@\%z)&(\%x@\#w))&(r&(\%u&

\%y))>(#u=#w)))&((r&(~\%u&\%z))))))) ;

The two Eqs. 3.5.2 and 3.6.2 are not tautologous, meaning the standard translation of a predicate model into the language QC1- fails.

The five Eqs. 3.7.2-3.11.2 are not tautologous. Eq. 11.2 is in fact contradictory, meaning the class C*₀ defined to contain frames satisfying a classical first-order formula fails to claim L(C*₀)=L(C₀).

The conjecture of recursive enumerability and elementary frame definability in predicate modal logic is refuted.