Via Geometric Algebra: Intersection of a Plane with a Sphere

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Abstract

As a high-school-level example of solving a problem via Geometric (Clifford) Algebra (GA), we show how to derive equations for the circle formed by the intersection of a plane with a sphere. Among the tasks that we will demonstrate are how to (1) express a plane via GA, (2) calculate the “reject” of a vector from a plane, and (3) express a circle as the rotation of a vector about a point.

“Derive an equation for the circle $C$ formed by the intersection of the sphere $S$ with the plane $P$. ”
1 Problem Statement

“Derive an equation for the circle \( \mathcal{C} \) formed by the intersection of the sphere \( \mathcal{S} \) with the plane \( \mathcal{P} \) (Fig. 1).”

2 Background

Geometric Algebra (GA) is —among other things—a language in which we can express geometric aspects of reality by means of symbols that can be manipulated according to well-defined rules, in order (we hope) to make useful inferences. For example, consider the line \( \mathcal{L} \) in Fig. 2. What aspects of that line can be expressed, with benefit, via GA?

One aspect is that the vector between any two points on \( \mathcal{L} \) is some scalar multiple of the unit vector \( \hat{u} \). That fact enables us to write an equation for \( \mathcal{L} \) in terms of \( \mathcal{L} \)'s direction (\( \hat{u} \)), and any point (e.g. \( q \)) known to be on that line. Specifically, every point \( x \) along \( \mathcal{L} \) fulfills the condition that

\[
x - q = \lambda \hat{u},
\]

where \( \lambda \) is some real number.

Another aspect of \( \mathcal{L} \) enables us to write an equation for that line in terms of \( \hat{u} \) and a point \( d \) that is not on the line (Fig. 3):

\[
(x - d) \wedge \hat{u} = M,
\]

where \( M \) is a constant bivector.
Figure 2: Every point $x$ on the line $\mathcal{L}$ fulfills the condition $x - q = \lambda \hat{u}$, where $\lambda$ is some real number.

Figure 3: Every point $x$ on the line $\mathcal{L}$ fulfills the condition $(x - d) \wedge \hat{u} = M$, where $M$ is a constant bivector.
The problem that we wish to solve in the present document involves a plane. Therefore, we might ask ourselves, “In what way can a plane be defined via GA?” One convenient way, which we employed in Fig. 1, is to use some known point (in our case, \( p \)) in the plane, and a unit bivector (\( \hat{B} \)) parallel to the plane. The vector from \( p \) to any point \( x \) in the plane \( \mathcal{P} \) must be parallel to \( \hat{B} \). Therefore, we may define \( \mathcal{P} \) as the set of the endpoints of all vectors \( x \) such that

\[(x - p) \wedge \hat{B} = 0.\]

### 3 Exploration of the Problem

Which of the other geometric aspects of our present problem might we consider expressing via GA? We already gave the answer away in Fig. 1 by noting that the intersection of \( \mathcal{S} \) and \( \mathcal{P} \) is a circle (\( \mathcal{C} \)). Still, the student is well-advised to confirm that statement by exploring the problem via interactive constructions (e.g. [2]).

### 4 Solution

Our next step might be to ask ourselves, “What are the salient features of a sphere and a circle, and what are the relations between those features that we might express via GA?” One feature is that any given circle is a set of points, each of which is equidistant from another point which we refer to as the circle’s center (e.g. \( c \), Fig. 1). The same is true of any sphere. And now, since those centers are such prominent features, we might attempt to identify some relation between \( \mathcal{S} \) (the center of \( \mathcal{S} \)) and \( c \). We’ll soon find that the line connecting those centers is perpendicular to \( \mathcal{P} \) (Fig. 4).

If we don’t, as yet, have a clear idea of how to proceed from there, we might
Figure 5: To Fig. 4, we’ve added two vectors: one from \(s\) to \(c\), and one from \(p\) to \(s\). Note that the colors of \(S\) and \(P\) have been made lighter in order to show our two new vectors more clearly.

follow one of George Polya’s ([II], p. 73) suggestions: When we don’t know what to do, we might attempt to deduce something potentially useful from the data. In our present problem, one such “something” is the distance from \(s\) to \(c\). Can we calculate that length? Yes, if we first follow one of Mason et al.’s ([3], p. 35), by asking “What can we introduce?” For example, what can we add to our previous diagrams? Two reasonable candidates are (1) a vector from \(s\) to \(c\), and a vector from \(p\) to \(s\) (Fig. 5).

Is there a relationship between those vectors, and the distance of \(s\) from \(P\)? Yes: the vector from \(s\) to \(c\) is the negative of the “reject” of the vector \((s - p)\) from \(\hat{B}\). Thus ([4], p. 119, and [5]),

\[
\mathbf{c} - \mathbf{s} = -\left[\text{Reject of } (\mathbf{s} - \mathbf{p}) \text{ from } \hat{\mathbf{B}}\right]
= -\left[(\mathbf{s} - \mathbf{p}) \land \hat{\mathbf{B}}\right] \hat{\mathbf{B}}^{-1}
= \left[(\mathbf{s} - \mathbf{p}) \land \hat{\mathbf{B}}\right] \hat{\mathbf{B}},
\]

because \(\hat{\mathbf{B}}^{-1} = -\hat{\mathbf{B}}^{-1}\).

Even without proceeding, we can see that that result is useful in two ways. The first is that we can find the point \(c\) via the vector equation

\[
\mathbf{c} = \mathbf{s} + (\mathbf{c} - \mathbf{s})
= \mathbf{s} + \left[(\mathbf{s} - \mathbf{p}) \land \hat{\mathbf{B}}\right] \hat{\mathbf{B}}.
\]

We can also (as we planned) calculate the distance from \(s\) to \(c\): it’s \(\|\mathbf{s} - \mathbf{c}\| = \sqrt{\left[\left[(\mathbf{s} - \mathbf{p}) \land \hat{\mathbf{B}}\right] \hat{\mathbf{B}}\right]^2}\). We expected that distance to be useful to us, and we were right, because we can use it to calculate \(r\) (the radius of \(\mathcal{C}\)) via the Pythagorean Theorem:

\[
r = \sqrt{R^2 - \left\{\left[(\mathbf{s} - \mathbf{p}) \land \hat{\mathbf{B}}\right] \hat{\mathbf{B}}\right\}^2},
\]
Figure 6: Every point in $\mathcal{P}$ is the endpoint of some vector $x$ such that $x = s + (c - s) + \rho e^{\theta \hat{B}}$.

where $R$ is the radius of $S$.

4.1 The First Form

The results we’ve obtained thus far enable us to write a vector equation for $C$. That circle consists of the endpoints of the set of all vectors $x$ that fulfill the condition

$$\|x - c\| = r,$$

with

$$c = s + (s - p) \hat{B}, \text{ and}$$

$$r = \sqrt{R^2 - \left[ (s - p) \wedge \hat{B} \right]^2}.$$

4.2 The Second Form

Another way to express $C$ is as the endpoints of the vectors that result from the rotation of a vector that lies within $\mathcal{P}$. A convenient choice for that vector is one of length $r$, in the direction from $c$ to $p$. We’ll call that vector $\rho$ ($= (c - p)/\|c - p\|$), and call the angle of rotation $\rho$ (Fig. 6). Thus, the equation for $C$ would be

$$x = x = s + (c - s) + \rho e^{\theta \hat{B}}.$$

References


