“Well, Papa, can you multiply triplets?” – “Yes, I can.”

Clumsy Foo[1]

Abstract
It is possible to model rotations in three-dimensional space with triplets.

Introduction
It is recorded that Hamilton went down to breakfast every morning and forced his darling son to ask: “Well, Papa, can you multiply triplets?” Whereto Hamilton was always obliged to reply: “No, I can only add and subtract them.”

Hamilton wanted to model rotations in three-dimensional space with three-dimensional quantities

$r = 1 + i_1 + i_2$

But he failed and instead misused quaternions

$q = 1 + i_1 + i_2 + i_3$

to model three-dimensional rotations. This was a severe and unhappy misuse because quaternions of course are made for modeling rotations in four-dimensional space [1, chap. 4].

Misconceptions about quaternions
Hamilton destroyed Brougham Bridge in Dublin by carving his formulas

$i_1^2 = i_2^2 = i_3^2 = i_1\, i_2\, i_3 = -1$

$$i_1\, i_2 = i_3 = -\, i_2\, i_1$$

$$i_2\, i_3 = i_1 = -\, i_3\, i_2$$

$$i_3\, i_1 = i_2 = -\, i_1\, i_3$$

into its walls. This was a dramatic mathematical crime because he led mankind astray with these equations. It is not possible to understand

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the geometry of rotations in an algebraic convincing way if products of vectors are considered again as vectors.

Therefore we will be thrifty and apply these equations in an economical way by cancelling the middle parts of these equations:

$$i_1^2 = i_2^2 = i_3^2 = \cancel{i_1\, i_2\, i_3} = -1$$

$$\begin{aligned} i_1\, i_2 &= \cancel{i_3} = -\, i_2\, i_1 \\ i_2\, i_3 &= \cancel{i_1} = -\, i_3\, i_2 \\ i_3\, i_1 &= \cancel{i_2} = -\, i_1\, i_3 \end{aligned}$$

Shorter equations will result in better mathematics, and we will use only the reduced version

$$i_1^2 = i_2^2 = i_3^2 = -1$$

$$\begin{aligned} i_1\, i_2 &= -\, i_2\, i_1 \\ i_2\, i_3 &= -\, i_3\, i_2 \\ i_3\, i_1 &= -\, i_1\, i_3 \end{aligned}$$

of Hamilton’s equations in the following.

Then a rotation of a vector

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

in a plane which is spanned by two unit vectors

$$\mathbf{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \qquad \text{with} \qquad n_x^2 + n_y^2 + n_z^2 = 1$$

and $$\mathbf{m} = \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} \qquad \text{with} \qquad m_x^2 + m_y^2 + m_z^2 = 1$$

about an angle α which equals twice the angle between the two vectors **n** and **m**

$$\alpha = 2\,\sphericalangle(\mathbf{n};\,\mathbf{m}) = 2\,\arccos\,(n_x\,m_x + n_y\,m_y + n_z\,m_z)$$

can be modeled by the simple sandwich product equation

$$\mathbf{r}_{rot} = \mathbf{m}\,(\mathbf{n}\,\mathbf{r}^*\,\mathbf{n})^*\,\mathbf{m} = \mathbf{m}\,\mathbf{n}^*\,\mathbf{r}\,\mathbf{n}^*\,\mathbf{m}$$

if the column vectors are identified with pure quaternions and its conjugates by

$r = x\,i_1 + y\,i_2 + z\,i_3$	and	$r^* = -\,x\,i_1 - y\,i_2 - z\,i_3$
$n = n_x\,i_1 + n_y\,i_2 + n_z\,i_3$	and	$n^* = -\,n_x\,i_1 - n_y\,i_2 - n_z\,i_3$
$m = m_x\,i_1 + m_y\,i_2 + m_z\,i_3$	and	$m^* = -\,m_x\,i_1 - m_y\,i_2 - m_z\,i_3$

First simple example

A rotation will take place in the xy-plane in positive direction about an angle of 90°. Thus it can be modeled by the two unit vectors

$$n = i_1$$

$$m = \frac{1}{\sqrt{2}}\,(i_1 + i_2)$$

Then the original value x of the i_1-coordinate will become the new value of the i_2-coordinate, the original value y of the i_2-coordinate will become the new negative value –y of the i_1-coordinate, and the value z of the i_3-coordinate will remain unchanged:

$$\mathbf{r}_{rot} = \mathbf{m}\,\mathbf{n}^*\,\mathbf{r}\,\mathbf{n}^*\,\mathbf{m}$$

$$= \frac{1}{\sqrt{2}}\,(i_1 + i_2)\,i_1^*\,(x\,i_1 + y\,i_2 + z\,i_3)\,i_1^*\,\frac{1}{\sqrt{2}}\,(i_1 + i_2)$$

$$= \frac{1}{2}\,(i_1 + i_2)\,(-\,i_1)\,(x\,i_1 + y\,i_2 + z\,i_3)\,(-\,i_1)\,(i_1 + i_2)$$

$$= \frac{1}{2}\,(i_1 + i_2)\,i_1\,(x\,i_1 + y\,i_2 + z\,i_3)\,i_1\,(i_1 + i_2)$$

$$= \frac{1}{2}\,(i_1 + i_2)\,(-\,x\,i_1 + y\,i_2 + z\,i_3)\,(i_1 + i_2)$$

$= \frac{1}{2}\ (x + x\, i_1 i_2 + y\, i_1 i_2 - y - z\, i_3 i_1 + z\, i_2 i_3)\ (i_1 + i_2)$

$= \frac{1}{2}\ (x\, i_1 + x\, i_2 + x\, i_2 - x\, i_1 + y\, i_2 - y\, i_1 - y\, i_1 - y\, i_2 + z\, i_3 - z\, i_1 i_2 i_3 + z\, i_1 i_2\, i_3 + z\, i_3)$

$= \frac{1}{2}\ (-2\, y\ i_1 + 2\, x\, i_2 + 2\, z\, i_3)$

$= -\, y\, i_1 + x\ i_2 + z\ i_3$

As this quaternion represents the column vector,

$$\mathbf{r}_{\text{rot}} = \begin{pmatrix} -\,y \\ x \\ z \end{pmatrix}$$

it is the expected result.

Triplet multiplications
The simple sandwich product equation

$\mathbf{r}_{\text{rot}} = \mathbf{m}\ (\mathbf{n}\ \mathbf{r}^*\ \mathbf{n})^*\ \mathbf{m} = \mathbf{m}\ \mathbf{n}^*\ \mathbf{r}\ \mathbf{n}^*\ \mathbf{m}$

is valid for an arbitrary number of base units. Thus we can use this sandwich product equation to model rotations of vectors in three-dimensioanl space by using the three base units 1, i_1, and i_2 only.

The column vectors are now identified with triplets according to

$r = x + y\, i_1 + z\, i_2$

$n = n_x + n_y\, i_1 + n_z\, i_2$

$m = m_x + m_y\, i_1 + m_z\, i_2$

There is no i_3. We do not need it. We will multiply triplets now!

First example revisited
A rotation will take place in the xy-plane in positive direction about an angle of 90°. Thus it can be modeled by the two unit vectors

$n = 1$

$$m = \frac{1}{\sqrt{2}}\ (1 + i_1)$$

because the real coordinate axis will represent the x-direction, the first imaginary i_1-coordinate axis will represent the y-direction, and the second imaginary i_2-coordinate axis will represent the z-direction now.

Then the original value x of the real coordinate will become the new value of the imaginary i_1-coordinate, the original value y of the imaginary i_1-coordinate will become the new negative value –y of the real coordinate, and the value z of the second imaginary i_2-coordinate will remain unchanged:

$$\begin{aligned}
\mathbf{r}_{rot} &= \mathbf{m\ n^*\ r\ n^*\ m} \\
&= \frac{1}{\sqrt{2}}\ (1 + i_1)\ 1^*\ (x + y\,i_1 + z\,i_2)\ 1^*\ \frac{1}{\sqrt{2}}\ (1 + i_1) \\
&= \frac{1}{2}\ (1 + i_1)\ 1\ (x + y\,i_1 + z\,i_2)\ 1\ (1 + i_1) \\
&= \frac{1}{2}\ (1 + i_1)\ (x + y\,i_1 + z\,i_2)\ (1 + i_1) \\
&= \frac{1}{2}\ (x + y\,i_1 + z\,i_2 + x\,i_1 - y + z\,i_1\,i_2)\ (1 + i_1) \\
&= \frac{1}{2}\ (x + y\,i_1 + z\,i_2 + x\,i_1 - y + z\,i_1\,i_2 + x\,i_1 - y - z\,i_1\,i_2 - x - y\,i_1 + z\,i_2) \\
&= \frac{1}{2}\ (-\,2\ y + 2\ x\,i_1 + 2\ z\,i_2) \\
&= -\ y + x\ i_1 + z\ i_2
\end{aligned}$$

As this triplet represents the column vector

$$\mathbf{r}_{rot} = \begin{pmatrix} -\,y \\ x \\ z \end{pmatrix}$$

we have again reached the correct and expected result.

This result shows two simple facts:

1. It is possible to multiply triplets in a reasonable and effect-tive way.

2. Hamilton and many mathematicians after him have read Grassmann's *Ausdehnungslehre*, but they have not under-stood the central message of this book

Second example
Now a more complicated situation will be modeled: A rotation will take place in a plane which is spanned by the following two unit vec-tors:

$$\mathbf{n} = \frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix} \qquad \text{and} \qquad \mathbf{p} = \frac{1}{\sqrt{77}}\begin{pmatrix}4\\5\\6\end{pmatrix}$$

about an angle of $\alpha = 60°$, starting from **n** and rotating about

$$\operatorname{arc\,cos}(\mathbf{n} \bullet \mathbf{p}) = \operatorname{arc\,cos} \frac{32}{\sqrt{14 \cdot 77}} \approx 12.9332°$$

into the direction of **p** and then going on another 47.0668° into this direction.

Find the resulting vector $\mathbf{r}_{\text{rot}}$, if vector

$$\mathbf{r} = \begin{pmatrix}7\\8\\9\end{pmatrix}$$

is rotated. It is easy to see that

$$\mathbf{r} = 2\sqrt{77}\ \mathbf{p} - \sqrt{14}\ \mathbf{n}$$

Now the column vectors are identified with pure quaternions by

$$\mathbf{n} = \frac{1}{\sqrt{14}}\,(i_1 + 2\,i_2 + 3\,i_3)$$

$$\mathbf{p} = \frac{1}{\sqrt{77}}\,(4\,i_1 + 5\,i_2 + 6\,i_3)$$

$$\mathbf{r} = (7\,i_1 + 8\,i_2 + 9\,i_3)$$

To solve this problem, we first have to find the second reflection vector **m**. As the **n p**-plane is represented by the non-scalar part **A** of the product

$$\mathbf{n}^*\,\mathbf{p} = \frac{1}{\sqrt{14}}\,(i_1 + 2\,i_2 + 3\,i_3)^*\,\frac{1}{\sqrt{77}}\,(4\,i_1 + 5\,i_2 + 6\,i_3)$$

$$= \frac{1}{\sqrt{14\cdot 77}}\,(-\,i_1 - 2\,i_2 - 3\,i_3)\,(4\,i_1 + 5\,i_2 + 6\,i_3)$$

$$= \frac{1}{7\sqrt{22}}\,(32 + 3\,i_1\,i_2 + 3\,i_2\,i_3 - 6\,i_3\,i_1)$$

we get the unit area element

$$\mathbf{A} = \frac{1}{\sqrt{54}}\,(3\,i_1\,i_2 + 3\,i_2\,i_3 - 6\,i_3\,i_1)$$

$$= \frac{1}{\sqrt{6}}\,(i_1\,i_2 + i_2\,i_3 - 2\,i_3\,i_1)$$

This area element can alternatively by spanned by the two orthogonal unit vectors **n** and **q**:

$$\mathbf{n}^*\,\mathbf{q} = \mathbf{A}$$

Therefore we can find the unit vector **q** which is perpendicular to **n** and lies in the **n p**-plane by pre-dividing the unit area element **A** by vector **n** from the left:

$$\mathbf{q} = \mathbf{n}^{-1}\,\mathbf{A} = \mathbf{n}\,\mathbf{A}$$

$$= \frac{1}{\sqrt{14}}\,(i_1 + 2\,i_2 + 3\,i_3)\,\frac{1}{\sqrt{6}}\,(i_1\,i_2 + i_2\,i_3 - 2\,i_3\,i_1)$$

$$= \frac{1}{2\sqrt{21}} \; (8\,i_1 + 2\,i_2 - 4\,i_3)$$

$$= \frac{1}{\sqrt{21}} \; (4\,i_1 + i_2 - 2\,i_3)$$

Thus the **np**-plane (which can be renamed as **nq**-plane) is spanned by the two orthogonal unit vectors **n** and **q**.

This can be shown by a short check. We simple reproduce unit vector **p** as linear combination of **n** and **q** by using the trigonometric values of the angle calculated above:

$$\mathbf{p} = \frac{32}{\sqrt{14 \cdot 77}} \; \mathbf{n} + \frac{\sqrt{(14 \cdot 77)^2 - 32^2}}{\sqrt{14 \cdot 77}} \; \mathbf{q}$$

$$= \frac{32}{\sqrt{14 \cdot 77}} \cdot \frac{1}{\sqrt{14}} \; (i_1 + 2\,i_2 + 3\,i_3) + \frac{\sqrt{14 \cdot 77 - 32^2}}{\sqrt{14 \cdot 77}} \; \cdot \frac{1}{\sqrt{21}} \; (4\,i_1 + i_2 - 2\,i_3)$$

$$= \frac{16}{7\sqrt{77}} \; (i_1 + 2\,i_2 + 3\,i_3) + \frac{3}{7\sqrt{77}} \; (4\,i_1 + i_2 - 2\,i_3)$$

$$= \frac{1}{\sqrt{77}} \; (4\,i_1 + 5\,i_2 + 6\,i_3)$$

Now we are able to find the second reflection vector **m** in the same way:

$$\mathbf{m} = \cos \frac{60°}{2} \; \mathbf{n} + \sin \frac{60°}{2} \; \mathbf{q}$$

$$= \frac{1}{2}\sqrt{3} \cdot \frac{1}{\sqrt{14}} \; (i_1 + 2\,i_2 + 3\,i_3) + \frac{1}{2} \cdot \frac{1}{\sqrt{21}} \; (4\,i_1 + i_2 - 2\,i_3)$$

$$= \frac{1}{2\sqrt{42}} \; \big((3\,i_1 + 6\,i_2 + 9\,i_3 + \sqrt{2} \; (4\,i_1 + i_2 - 2\,i_3)\big)$$

$$= \frac{1}{2\sqrt{42}} \; \big((3 + 4\sqrt{2}\,)\,i_1 + (6 + \sqrt{2}\,)\,i_2 + (9 - 2\sqrt{2}\,)\,i_3\big)$$

Finally, we are able to find the rotated vector:

$\mathbf{r}_{rot} = \mathbf{m}\ \mathbf{n}^*\ \mathbf{r}\ \mathbf{n}^*\ \mathbf{m}$

$$= \frac{1}{2\sqrt{42}}\Big((3+4\sqrt{2})\, i_1 + (6+\sqrt{2})\, i_2 + (9-2\sqrt{2})\, i_3\Big)$$
$$\frac{1}{\sqrt{14}}\,(i_1 + 2\, i_2 + 3\, i_3)^*\,(7\, i_1 + 8\, i_2 + 9\, i_3)\,\frac{1}{\sqrt{14}}\,(i_1 + 2\, i_2 + 3\, i_3)^*$$
$$\frac{1}{2\sqrt{42}}\Big((3+4\sqrt{2})\, i_1 + (6+\sqrt{2})\, i_2 + (9-2\sqrt{2})\, i_3\Big)$$

$$= \frac{1}{2352}\Big((3+4\sqrt{2})\, i_1 + (6+\sqrt{2})\, i_2 + (9-2\sqrt{2})\, i_3\Big)$$
$$(i_1 + 2\, i_2 + 3\, i_3)\,(7\, i_1 + 8\, i_2 + 9\, i_3)\,(i_1 + 2\, i_2 + 3\, i_3)$$
$$\Big((3+4\sqrt{2})\, i_1 + (6+\sqrt{2})\, i_2 + (9-2\sqrt{2})\, i_3\Big)$$

$$= \frac{1}{2352}\Big((3+4\sqrt{2})\, i_1 + (6+\sqrt{2})\, i_2 + (9-2\sqrt{2})\, i_3\Big)$$
$$(-50 - 6\, i_1 i_2 - 6\, i_2 i_3 + 12\, i_3 i_1)\,(i_1 + 2\, i_2 + 3\, i_3)$$
$$\Big((3+4\sqrt{2})\, i_1 + (6+\sqrt{2})\, i_2 + (9-2\sqrt{2})\, i_3\Big)$$

$$= \frac{1}{2352}\Big((3+4\sqrt{2})\, i_1 + (6+\sqrt{2})\, i_2 + (9-2\sqrt{2})\, i_3\Big)$$
$$(-2\, i_1 - 88\, i_2 - 174\, i_3)$$
$$\Big((3+4\sqrt{2})\, i_1 + (6+\sqrt{2})\, i_2 + (9-2\sqrt{2})\, i_3\Big)$$

$$= \frac{-1}{1176}\Big((3+4\sqrt{2})\, i_1 + (6+\sqrt{2})\, i_2 + (9-2\sqrt{2})\, i_3\Big)\,(i_1 + 44\, i_2 + 87\, i_3)$$
$$\Big((3+4\sqrt{2})\, i_1 + (6+\sqrt{2})\, i_2 + (9-2\sqrt{2})\, i_3\Big)$$

$$= \frac{-1}{1176}\Big(-1050 + 126\sqrt{2} + (126 + 175\sqrt{2})\, i_1 i_2 + (126 + 175\sqrt{2})\, i_2 i_3$$
$$+ (-252 - 350\sqrt{2})\, i_3 i_1\Big)$$
$$\Big((3+4\sqrt{2})\, i_1 + (6+\sqrt{2})\, i_2 + (9-2\sqrt{2})\, i_3\Big)$$

$$= \frac{-1}{168}\Big(-150 + 18\sqrt{2} + (18 + 25\sqrt{2})\, i_1 i_2 + (18 + 25\sqrt{2})\, i_2 i_3 + (-36 - 50\sqrt{2})\, i_3 i_1\Big)$$

$$\Big((3 + 4\sqrt{2})\, i_1 + (6 + \sqrt{2})\, i_2 + (9 - 2\sqrt{2})\, i_3\Big)$$

$$= \frac{-1}{168}\Big((-588 - 1092\sqrt{2})\, i_1 + (-672 - 84\sqrt{2})\, i_2 + (-756 + 924\sqrt{2})\, i_3\Big)$$

$$= \frac{1}{2}\Big((7 + 13\sqrt{2})\, i_1 + (8 + \sqrt{2})\, i_2 + (9 - 11\sqrt{2})\, i_3\Big)$$

$$\approx 12.6924\, i_1 + 4.7071\, i_2 - 3.2782\, i_3$$

Thus the rotated column vector can be identified as:

$$\mathbf{r}_{\mathrm{rot}} = \begin{pmatrix} 3.5 + 6.5\sqrt{2} \\ 4 \ + 0.5\sqrt{2} \\ 4.5 - 5.5\sqrt{2} \end{pmatrix}$$

Solving the second example with triplets
After identifying the given vectors with triplets

$$\mathbf{n} = \frac{1}{\sqrt{14}}\,(1 + 2\, i_1 + 3\, i_2)$$

$$\mathbf{p} = \frac{1}{\sqrt{77}}\,(4 + 5\, i_1 + 6\, i_2)$$

$$\mathbf{r} = (7 + 8\, i_1 + 9\, i_2)$$

the second problem can be solved by triplet multiplications (There is no i_3. We do not need it!) in the following way.

Again the **np**-plane is represented by the non-scalar part **A** of the product

$$\mathbf{n}^*\,\mathbf{p} = \frac{1}{\sqrt{14}}\,(1 + 2\, i_1 + 3\, i_2)^*\ \frac{1}{\sqrt{77}}\,(4 + 5\, i_1 + 6\, i_2)$$

$$= \frac{1}{\sqrt{14\cdot 77}}\,(1 - 2\, i_1 - 3\, i_2)\,(4 + 5\, i_1 + 6\, i_2)$$

$$= \frac{1}{7\sqrt{22}} \ (32 - 3\, i_1 - 6\, i_2 + 3\, i_1 i_2)$$

Comparing with the pure quaternion solution we find some different signs now, but this will be repaired by the following triplet multiplications to find vector **q** perpendicular to **n** lying in the **np**-plane.

The unit area element will then be:

$$\mathbf{A} = \frac{1}{\sqrt{54}} \ (-3\, i_1 - 6\, i_2 + 3\, i_1 i_2)$$

$$= \frac{1}{\sqrt{6}} \ (-i_1 - 2\, i_2 + i_1 i_2)$$

The area element can alternatively by spanned by the two orthogonal vectors **n** and **q**:

$\mathbf{n}^* \mathbf{q} = \mathbf{A}$

Therefore we can find the vector **q** which is perpendicular to **n** and lies in the **np**-plane by pre-dividing the unit area element **A** by vector **n** from the left:

$\mathbf{q} = \mathbf{n}^{-1}\, \mathbf{A} = \mathbf{n}\, \mathbf{A}$

$$= \frac{1}{\sqrt{14}} \ (1 + 2\, i_1 + 3\, i_2) \ \frac{1}{\sqrt{6}} \ (-i_1 - 2\, i_2 + i_1 i_2)$$

$$= \frac{1}{2\sqrt{21}} \ (8 + 2\, i_1 - 4\, i_2)$$

$$= \frac{1}{\sqrt{21}} \ (4 + i_1 - 2\, i_2)$$

Thus the **np**-plane (which can be renamed as **nq**-plane) is spanned by the two orthogonal unit vectors **n** and **q**.

This can be shown by a short check. We simple reproduce unit vector **p** as linear combination of **n** and **q** by using the trigonometric values of the angle calculated at the beginning:

$$\mathbf{p} = \frac{32}{\sqrt{14\cdot 77}}\ \mathbf{n} + \frac{\sqrt{(14\cdot 77)^2 - 32^2}}{\sqrt{14\cdot 77}}\ \mathbf{q}$$

$$= \frac{32}{\sqrt{14\cdot 77}}\cdot\frac{1}{\sqrt{14}}\ (1 + 2\,i_1 + 3\,i_2) + \frac{\sqrt{14\cdot 77 - 32^2}}{\sqrt{14\cdot 77}}\ \cdot\frac{1}{\sqrt{21}}\ (4 + i_1 - 2\,i_2)$$

$$= \frac{16}{7\sqrt{77}}\ (1 + 2\,i_1 + 3\,i_2) + \frac{3}{7\sqrt{77}}\ (4 + i_1 - 2\,i_2) \quad \text{hier}$$

$$= \frac{1}{\sqrt{77}}\ (4 + 5\,i_1 + 6\,i_2)$$

Now we are able to find the second reflection vector **m** in the same way:

$$\mathbf{m} = \cos\frac{60°}{2}\ \mathbf{n} + \sin\frac{60°}{2}\ \mathbf{q}$$

$$= \frac{1}{2}\sqrt{3}\cdot\frac{1}{\sqrt{14}}\ (1 + 2\,i_1 + 3\,i_2) + \frac{1}{2}\cdot\frac{1}{\sqrt{21}}\ (4 + i_1 - 2\,i_2)$$

$$= \frac{1}{2\sqrt{42}}\ \big((3 + 6\,i_1 + 9\,i_2 + \sqrt{2}\ (4 + i_1 - 2\,i_2)\big)$$

$$= \frac{1}{2\sqrt{42}}\ \big(3 + 4\sqrt{2} + (6 + \sqrt{2})\ i_1 + (9 - 2\sqrt{2})\ i_2\big)$$

Finally, we are able to find the rotated vector:

$$\mathbf{r}_{rot} = \mathbf{m}\ \mathbf{n}^*\ \mathbf{r}\ \mathbf{n}^*\ \mathbf{m}$$

$$= \frac{1}{2\sqrt{42}}\ \big(3 + 4\sqrt{2} + (6 + \sqrt{2})\ i_1 + (9 - 2\sqrt{2})\ i_2\big)$$
$$\frac{1}{\sqrt{14}}\ (1 + 2\,i_1 + 3\,i_2)^*\ (7 + 8\,i_1 + 9\,i_2)\ \frac{1}{\sqrt{14}}\ (1 + 2\,i_1 + 3\,i_2)^*$$
$$\frac{1}{2\sqrt{42}}\ \big(3 + 4\sqrt{2} + (6 + \sqrt{2})\ i_1 + (9 - 2\sqrt{2})\ i_2\big)$$

$$= \frac{1}{2352}\left(3 + 4\sqrt{2} + (6+\sqrt{2})\,i_1 + (9-2\sqrt{2})\,i_2\right)$$
$$(1-2\,i_1-3\,i_2)\,(7+8\,i_1+9\,i_2)\,(1-2\,i_1-3\,i_2)$$
$$\left(3 + 4\sqrt{2} + (6+\sqrt{2})\,i_1 + (9-2\sqrt{2})\,i_2\right)$$

$$= \frac{1}{2352}\left(3 + 4\sqrt{2} + (6+\sqrt{2})\,i_1 + (9-2\sqrt{2})\,i_2\right)$$
$$(50-6\,i_1-12\,i_2+6\,i_1i_2)\,(1-2\,i_1-3\,i_2)$$
$$\left(3 + 4\sqrt{2} + (6+\sqrt{2})\,i_1 + (9-2\sqrt{2})\,i_2\right)$$

$$= \frac{1}{2352}\left(3 + 4\sqrt{2} + (6+\sqrt{2})\,i_1 + (9-2\sqrt{2})\,i_2\right)$$
$$(2-88\,i_1-174\,i_2)$$
$$\left(3 + 4\sqrt{2} + (6+\sqrt{2})\,i_1 + (9-2\sqrt{2})\,i_2\right)$$

$$= \frac{1}{1176}\left(3 + 4\sqrt{2} + (6+\sqrt{2})\,i_1 + (9-2\sqrt{2})\,i_2\right)(1-44\,i_1-87\,i_2)$$
$$\left(3 + 4\sqrt{2} + (6+\sqrt{2})\,i_1 + (9-2\sqrt{2})\,i_2\right)$$

$$= \frac{1}{1176}\Big(1050-126\sqrt{2} + (-126-175\sqrt{2})\,i_1 + (-252-350\sqrt{2})\,i_2$$
$$+ (-126-175\sqrt{2})\,i_1i_2\Big)$$
$$\left(3 + 4\sqrt{2} + (6+\sqrt{2})\,i_1 + (9-2\sqrt{2})\,i_2\right)$$

$$= \frac{1}{168}\left(150-18\sqrt{2} + (-18-25\sqrt{2})\,i_1 + (-36-50\sqrt{2})\,i_2 + (-18-25\sqrt{2})\,i_1i_2\right)$$
$$\left(3 + 4\sqrt{2} + (6+\sqrt{2})\,i_1 + (9-2\sqrt{2})\,i_2\right)$$

$$= \frac{1}{168}\left(588+1092\sqrt{2} + (672+84\sqrt{2})\,i_1 + (756-924\sqrt{2})\,i_2\right)$$

$$= \frac{1}{2}\left(7+13\sqrt{2} + (8+\sqrt{2})\,i_1 + (9-11\sqrt{2})\,i_2\right)$$

$$\approx 12.6924 + 4.7071\,i_1 - 3.2782\,i_2$$

All the values and signs turned out to be identical to the first solution. Thus the rotated column vector can be identified again as:

$$\mathbf{r}_{rot} = \begin{pmatrix} 3.5 + 6.5\sqrt{2} \\ 4 \ \ + 0.5\sqrt{2} \\ 4.5 - 5.5\sqrt{2} \end{pmatrix}$$

A simple check shows that the result is indeed correct, because the lengths (or their squares) of vector **r**

$$\mathbf{r}^* \, \mathbf{r} = (7 + 8\,i_1 + 9\,i_2)^* \, (7 + 8\,i_1 + 9\,i_2)$$

$$= (7 - 8\,i_1 - 9\,i_2)\,(7 + 8\,i_1 + 9\,i_2)$$

$$= 194$$

and vector $\mathbf{r}_{rot}$

$$\mathbf{r}_{rot}^* \, \mathbf{r}_{rot} = \frac{1}{2}\left(7 + 13\sqrt{2} + (8 + \sqrt{2})\,i_1 + (9 - 11\sqrt{2})\,i_2\right)^* \frac{1}{2}\left(7 + 13\sqrt{2} + (8 + \sqrt{2})\,i_1 + (9 - 11\sqrt{2})\,i_2\right)$$

$$= \frac{1}{4}\left(7 + 13\sqrt{2} + (-8 - \sqrt{2})\,i_1 + (-9 + 11\sqrt{2})\,i_2\right)\left(7 + 13\sqrt{2} + (8 + \sqrt{2})\,i_1 + (9 - 11\sqrt{2})\,i_2\right)$$

$$= \frac{1}{4}\left(49 + 182\sqrt{2} + 338 + 64 + 16\sqrt{2} + 2 + 81 - 198\sqrt{2} + 242\right)$$

$$= \frac{1}{4} \cdot 776$$

$$= 194$$

are indeed identical. And the angle α between vectors of $\mathbf{r}_{rot}$ and **r** can be checked by the unit vector multiplication

$$\frac{1}{194}\ \mathbf{r}^*\,\mathbf{r}_{rot} = \frac{1}{194}\,(7 + 8\,i_1 + 9\,i_2)^*\ \frac{1}{2}\,\big(7 + 13\sqrt{2}\ + (8 + \sqrt{2}\,)\,i_1 + (9 - 11\sqrt{2}\,)\,i_2\big)$$

$$= \frac{1}{388}\,(7 - 8\,i_1 - 9\,i_2)\,\big(7 + 13\sqrt{2}\ + (8 + \sqrt{2}\,)\,i_1 + (9 - 11\sqrt{2}\,)\,i_2\big)$$

$$= \frac{1}{388}\,\big(194 - 97\sqrt{2}\ i_1 - 194\sqrt{2}\ i_2 + 97\sqrt{2}\ i_1\,i_2\big)$$

$$= \frac{1}{4}\,\big(2 - \sqrt{2}\ i_1 - 2\sqrt{2}\ i_2 + \sqrt{2}\ i_1\,i_2\big)$$

Therefore the angle correctly results in

$$\alpha = \text{arc cos}\ \frac{2}{4} = \text{arc cos}\ 0.5 = 60°$$

And the non-scalar part

$$\frac{1}{4}\,\big(-\sqrt{2}\ i_1 - 2\sqrt{2}\ i_2 + \sqrt{2}\ i_1\,i_2\big) = \frac{1}{2\sqrt{2}}\,(-\,i_1 - 2\,i_2 + i_1\,i_2)$$

$$= \frac{1}{2}\sqrt{3}\ \frac{1}{\sqrt{6}}\,(-\,i_1 - 2\,i_2 + i_1\,i_2)$$

$$= \frac{1}{2}\sqrt{3}\ \mathbf{A}$$

clearly shows that both vectors are situated in the **np**-plane.

Conclusion

Triplet multiplication works!

Literature

[1] Anonymous & Clumsy Foo: Raumzeit und Quaternionen. Anmerkungen zur Mathematik der Außerirdischen. BoD – Books on Demand, Norderstedt 2019, ISBN: 978-3-7504-1787-8.

www.lehmanns.de/shop/mathematik-informatik/50800225-9783750417878-raumzeit-und-quaternionen