The Navier–Stokes problem

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A proposed solution to the millennium problem on the existence and smoothness of the Navier–Stokes equations.

1. Introduction

The Navier–Stokes equations are thought to govern the motion of a fluid in $\mathbb{R}^d$ where $d \in \mathbb{N}$, see [1,3,7]. Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^d$ be the velocity and let $p = p(\mathbf{x}, t) \in \mathbb{R}$ be the pressure, each dependent on position $\mathbf{x} \in \mathbb{R}^d$ and time $t \geqslant 0$. I take the externally applied force to be identically zero. The fluid is assumed to be incompressible with constant viscosity $\nu \geqslant 0$ and to fill all of $\mathbb{R}^d$. The Navier–Stokes equations can then be written as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \nabla^2 \mathbf{u} - \nabla p, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

with initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0 \tag{3}$$

where $\mathbf{u}_0 = \mathbf{u}_0(\mathbf{x}) \in \mathbb{R}^d$. In these equations

$$\nabla = (\frac{\partial}{\partial \mathbf{x}_1}, \frac{\partial}{\partial \mathbf{x}_2}, \ldots, \frac{\partial}{\partial \mathbf{x}_d}) \tag{4}$$

is the gradient operator and

$$\nabla^2 = \sum_{i=1}^{d} \frac{\partial^2}{\partial {\mathbf{x}_i}^2} \tag{5}$$

is the Laplacian operator. When $\nu = 0$, equations (1), (2), (3) are called the Euler equations. Solutions of (1), (2), (3) are to be found with

$$\mathbf{u}_0(\mathbf{x} + Le_i) = \mathbf{u}_0(\mathbf{x}) \tag{6}$$

for $1 \leqslant i \leqslant d$ where e_i is the i^{th} unit vector in $\mathbb{R}^d$ and $L > 0$ is a constant [7]. The initial condition $\mathbf{u}_0$ is a given C^∞ divergence-free vector field on $\mathbb{R}^d$. A solution of (1), (2), (3) is then accepted to be physically reasonable [3] if

$$\mathbf{u}(\mathbf{x} + Le_i, t) = \mathbf{u}(\mathbf{x}, t), \quad p(\mathbf{x} + Le_i, t) = p(\mathbf{x}, t) \tag{7}$$

on $\mathbb{R}^d \times [0, \infty)$ for $1 \leqslant i \leqslant d$ and

$$\mathbf{u}, p \in C^\infty(\mathbb{R}^d \times [0, \infty)). \tag{8}$$

2. Solution to the Navier–Stokes problem

I provide a proof of the following theorem [2,3,6,7].
Theorem. Take $\nu > 0$. Let $\mathbf{u}_0$ be any smooth, divergence-free vector field satisfying (6). Then there exist smooth functions $\mathbf{u}, p$ on $\mathbb{R}^d \times [0, \infty)$ that satisfy (1), (2), (3), (7), (8).
Proof. It is sufficient to rule out the possibility that there is a smooth, divergence-free $\mathbf{u}_0$ for which (1), (2), (3) have a solution with a finite blowup time [3].
Let the Fourier series of $\mathbf{u}, p$ be

$$\tilde{\mathbf{u}} = \sum_{\mathbf{L}=-\infty}^{\infty} \mathbf{u}_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}}, \tag{9}$$

$$\tilde{p} = \sum_{\mathbf{L}=-\infty}^{\infty} p_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{10}$$

respectively. Here $\mathbf{u}_{\mathbf{L}} = \mathbf{u}_{\mathbf{L}}(t) \in \mathbb{C}^d$, $p_{\mathbf{L}} = p_{\mathbf{L}}(t) \in \mathbb{C}$, $\mathrm{i} = \sqrt{-1}$, $k = 2\pi/L$, and $\sum_{\mathbf{L}=-\infty}^{\infty}$ denotes the sum over all $\mathbf{L} \in \mathbb{Z}^d$. The initial condition $\mathbf{u}_0$ is a Fourier series [2] of which is convergent for all $\mathbf{x} \in \mathbb{R}^d$. Substituting $\mathbf{u} = \tilde{\mathbf{u}}$, $p = \tilde{p}$ into (1) gives

$$\begin{aligned}\sum_{\mathbf{L}=-\infty}^{\infty} \frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} e^{ik\mathbf{L}\cdot\mathbf{x}} + \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u}_{\mathbf{L}} \cdot \mathrm{i}k\mathbf{M}) \mathbf{u}_{\mathbf{M}} e^{ik(\mathbf{L}+\mathbf{M})\cdot\mathbf{x}} \\ = -\sum_{\mathbf{L}=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 \mathbf{u}_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}} - \sum_{\mathbf{L}=-\infty}^{\infty} \mathrm{i}k\mathbf{L} p_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}}.\end{aligned} \tag{11}$$

Equating like powers of the exponentials in (11) yields

$$\frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \mathrm{i}k\mathbf{M}) \mathbf{u}_{\mathbf{M}} = -\nu k^2 |\mathbf{L}|^2 \mathbf{u}_{\mathbf{L}} - \mathrm{i}k\mathbf{L} p_{\mathbf{L}} \tag{12}$$

on using the Cauchy product type formula [4]

$$\sum_{l=-\infty}^{\infty} a_l x^l \sum_{m=-\infty}^{\infty} b_m x^m = \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{l-m} b_m x^l. \tag{13}$$

Substituting $\mathbf{u} = \tilde{\mathbf{u}}$ into (2) gives

$$\sum_{\mathbf{L}=-\infty}^{\infty} \mathrm{i}k\mathbf{L} \cdot \mathbf{u}_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}} = 0. \tag{14}$$

Equating like powers of the exponentials in (14) yields

$$\mathbf{L} \cdot \mathbf{u}_{\mathbf{L}} = 0. \tag{15}$$

Applying $\mathbf{L}\cdot$ to (12) and noting (15) leads to

$$p_{\mathbf{L}} = -\sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \hat{\mathbf{L}})(\mathbf{u}_{\mathbf{M}} \cdot \hat{\mathbf{L}}) \tag{16}$$

where $p_{\mathbf{0}}$ is arbitrary and $\hat{\mathbf{L}} = \mathbf{L}/|\mathbf{L}|$ is the unit vector in the direction of $\mathbf{L}$. Then substituting (16) into (12) gives

$$\frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} = -\sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \mathrm{i}k\mathbf{M})\mathbf{u}_{\mathbf{M}} - \nu k^2|\mathbf{L}|^2\mathbf{u}_{\mathbf{L}} + \sum_{\mathbf{M}=-\infty}^{\infty} \mathrm{i}k\mathbf{L}(\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \hat{\mathbf{L}})(\mathbf{u}_{\mathbf{M}} \cdot \hat{\mathbf{L}}) \tag{17}$$

where $\mathbf{u}_{\mathbf{0}} = \mathbf{u}_{\mathbf{0}}(0)$. Without loss of generality [2], I take $\mathbf{u}_{\mathbf{0}} = \mathbf{0}$. The equations for $\mathbf{u}_{\mathbf{L}}$ are to be solved for all $\mathbf{L} \in \mathbb{Z}^d$.

Let

$$\mathbf{u}_{\mathbf{L}} = \mathbf{a}_{\mathbf{L}} + \mathrm{i}\mathbf{b}_{\mathbf{L}}, \tag{18}$$

$$p_{\mathbf{L}} = c_{\mathbf{L}} + \mathrm{i}d_{\mathbf{L}} \tag{19}$$

where $\mathbf{a}_{\mathbf{L}} = \mathbf{a}_{\mathbf{L}}(t) \in \mathbb{R}^d$, $\mathbf{b}_{\mathbf{L}} = \mathbf{b}_{\mathbf{L}}(t) \in \mathbb{R}^d$, $c_{\mathbf{L}} = c_{\mathbf{L}}(t) \in \mathbb{R}$, and $d_{\mathbf{L}} = d_{\mathbf{L}}(t) \in \mathbb{R}$. Substituting (18), (19) into (12) gives

$$\begin{aligned} &\frac{\partial \mathbf{a}_{\mathbf{L}}}{\partial t} + \mathrm{i}\frac{\partial \mathbf{b}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} ((\mathbf{a}_{\mathbf{L}-\mathbf{M}} + \mathrm{i}\mathbf{b}_{\mathbf{L}-\mathbf{M}}) \cdot \mathrm{i}k\mathbf{M})(\mathbf{a}_{\mathbf{M}} + \mathrm{i}\mathbf{b}_{\mathbf{M}}) \\ &= -\nu k^2|\mathbf{L}|^2(\mathbf{a}_{\mathbf{L}} + \mathrm{i}\mathbf{b}_{\mathbf{L}}) - \mathrm{i}k\mathbf{L}(c_{\mathbf{L}} + \mathrm{i}d_{\mathbf{L}}). \end{aligned} \tag{20}$$

Equating real and imaginary parts in (20) gives

$$\frac{\partial \mathbf{a}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} (-(\mathbf{a}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{b}_{\mathbf{M}} - (\mathbf{b}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}}) = -\nu k^2|\mathbf{L}|^2\mathbf{a}_{\mathbf{L}} + k\mathbf{L}d_{\mathbf{L}}, \tag{21}$$

$$\frac{\partial \mathbf{b}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} ((\mathbf{a}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}} - (\mathbf{b}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{b}_{\mathbf{M}}) = -\nu k^2|\mathbf{L}|^2\mathbf{b}_{\mathbf{L}} - k\mathbf{L}c_{\mathbf{L}}. \tag{22}$$

Substituting (18) into (15) gives

$$\mathbf{L} \cdot (\mathbf{a}_{\mathbf{L}} + \mathrm{i}\mathbf{b}_{\mathbf{L}}) = 0. \tag{23}$$

Equating real and imaginary parts in (23) gives

$$\mathbf{L} \cdot \mathbf{a}_{\mathbf{L}} = 0, \tag{24}$$

$$\mathbf{L} \cdot \mathbf{b}_{\mathbf{L}} = 0. \tag{25}$$

From (21) and in light of (24) it is possible to write

$$\frac{\partial \mathbf{a_L}}{\partial t}\cdot\hat{\mathbf{a}}_\mathbf{L}+\sum_{\mathbf{M}=-\infty}^{\infty}(-(\mathbf{a_{L-M}}\cdot k\mathbf{M})\mathbf{b_M}-(\mathbf{b_{L-M}}\cdot k\mathbf{M})\mathbf{a_M})\cdot\hat{\mathbf{a}}_\mathbf{L}=-\nu k^2|\mathbf{L}|^2\mathbf{a_L}\cdot\hat{\mathbf{a}}_\mathbf{L} \quad (26)$$

where $\hat{\mathbf{a}}_\mathbf{L}=\mathbf{a_L}/|\mathbf{a_L}|$ is the unit vector in the direction of $\mathbf{a_L}$. Then (26) implies

$$\frac{\partial |\mathbf{a_L}|}{\partial t}+\sum_{\mathbf{M}=-\infty}^{\infty}(-(\mathbf{a_{L-M}}\cdot k\mathbf{M})\mathbf{b_M}-(\mathbf{b_{L-M}}\cdot k\mathbf{M})\mathbf{a_M})\cdot\hat{\mathbf{a}}_\mathbf{L}=-\nu k^2|\mathbf{L}|^2|\mathbf{a_L}|. \quad (27)$$

From (27) it is possible to write

$$\frac{\partial |\mathbf{a_L}|}{\partial t}\leqslant\sum_{\mathbf{M}=-\infty}^{\infty}(|\mathbf{a_{L-M}}|k|\mathbf{M}||\mathbf{b_M}|+|\mathbf{b_{L-M}}|k|\mathbf{M}||\mathbf{a_M}|)-\nu k^2|\mathbf{L}|^2|\mathbf{a_L}| \quad (28)$$

on using the Cauchy–Schwarz inequality [5]

$$|\mathbf{a}\cdot\mathbf{b}|\leqslant|\mathbf{a}||\mathbf{b}|. \quad (29)$$

It then follows from (28) that

$$\begin{aligned}\sum_{\mathbf{L}=-\infty}^{\infty}\frac{\partial |\mathbf{a_L}|}{\partial t}e^{k|\mathbf{L}||\mathbf{x}|}&\leqslant\sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{a_{L-M}}|k|\mathbf{M}||\mathbf{b_M}|e^{k|\mathbf{L}||\mathbf{x}|}\\&+\sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{b_{L-M}}|k|\mathbf{M}||\mathbf{a_M}|e^{k|\mathbf{L}||\mathbf{x}|}-\sum_{\mathbf{L}=-\infty}^{\infty}\nu k^2|\mathbf{L}|^2|\mathbf{a_L}|e^{k|\mathbf{L}||\mathbf{x}|}\end{aligned} \quad (30)$$

implying that

$$\begin{aligned}\sum_{\mathbf{L}=-\infty}^{\infty}\frac{\partial |\mathbf{a_L}|}{\partial t}e^{k|\mathbf{L}||\mathbf{x}|}&\leqslant\sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{a_L}|k|\mathbf{M}||\mathbf{b_M}|e^{k|\mathbf{L+M}||\mathbf{x}|}\\&+\sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{b_L}|k|\mathbf{M}||\mathbf{a_M}|e^{k|\mathbf{L+M}||\mathbf{x}|}-\sum_{\mathbf{L}=-\infty}^{\infty}\nu k^2|\mathbf{L}|^2|\mathbf{a_L}|e^{k|\mathbf{L}||\mathbf{x}|}\end{aligned} \quad (31)$$

in light of (13), which yields

$$\begin{aligned}\sum_{\mathbf{L}=-\infty}^{\infty}\frac{\partial |\mathbf{a_L}|}{\partial t}e^{k|\mathbf{L}||\mathbf{x}|}&\leqslant\sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{a_L}|k|\mathbf{M}||\mathbf{b_M}|e^{k(|\mathbf{L}|+|\mathbf{M}|)|\mathbf{x}|}\\&+\sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{b_L}|k|\mathbf{M}||\mathbf{a_M}|e^{k(|\mathbf{L}|+|\mathbf{M}|)|\mathbf{x}|}-\sum_{\mathbf{L}=-\infty}^{\infty}\nu k^2|\mathbf{L}|^2|\mathbf{a_L}|e^{k|\mathbf{L}||\mathbf{x}|}\end{aligned} \quad (32)$$

on using the triangle inequality [5]

$$|\mathbf{a}+\mathbf{b}|\leqslant|\mathbf{a}|+|\mathbf{b}|. \quad (33)$$

From (22) and in light of (25) it is possible to write

$$\frac{\partial \mathbf{b}_{\mathbf{L}}}{\partial t}\cdot\hat{\mathbf{b}}_{\mathbf{L}}+\sum_{\mathbf{M}=-\infty}^{\infty}((\mathbf{a}_{\mathbf{L}-\mathbf{M}}\cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}}-(\mathbf{b}_{\mathbf{L}-\mathbf{M}}\cdot k\mathbf{M})\mathbf{b}_{\mathbf{M}})\cdot\hat{\mathbf{b}}_{\mathbf{L}}=-\nu k^2|\mathbf{L}|^2\mathbf{b}_{\mathbf{L}}\cdot\hat{\mathbf{b}}_{\mathbf{L}} \quad (34)$$

where $\hat{\mathbf{b}}_{\mathbf{L}}=\mathbf{b}_{\mathbf{L}}/|\mathbf{b}_{\mathbf{L}}|$ is the unit vector in the direction of $\mathbf{b}_{\mathbf{L}}$. Then (34) implies

$$\frac{\partial|\mathbf{b}_{\mathbf{L}}|}{\partial t}+\sum_{\mathbf{M}=-\infty}^{\infty}((\mathbf{a}_{\mathbf{L}-\mathbf{M}}\cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}}-(\mathbf{b}_{\mathbf{L}-\mathbf{M}}\cdot k\mathbf{M})\mathbf{b}_{\mathbf{M}})\cdot\hat{\mathbf{b}}_{\mathbf{L}}=-\nu k^2|\mathbf{L}|^2|\mathbf{b}_{\mathbf{L}}|. \quad (35)$$

From (35) it is possible to write

$$\frac{\partial|\mathbf{b}_{\mathbf{L}}|}{\partial t}\leqslant\sum_{\mathbf{M}=-\infty}^{\infty}(|\mathbf{a}_{\mathbf{L}-\mathbf{M}}|k|\mathbf{M}||\mathbf{a}_{\mathbf{M}}|+|\mathbf{b}_{\mathbf{L}-\mathbf{M}}|k|\mathbf{M}||\mathbf{b}_{\mathbf{M}}|)-\nu k^2|\mathbf{L}|^2|\mathbf{b}_{\mathbf{L}}| \quad (36)$$

on using the Cauchy–Schwarz inequality. It then follows from (36) that

$$\begin{aligned}\sum_{\mathbf{L}=-\infty}^{\infty}\frac{\partial|\mathbf{b}_{\mathbf{L}}|}{\partial t}e^{k|\mathbf{L}||\mathbf{x}|}&\leqslant\sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{a}_{\mathbf{L}-\mathbf{M}}|k|\mathbf{M}||\mathbf{a}_{\mathbf{M}}|e^{k|\mathbf{L}||\mathbf{x}|}\\&+\sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{b}_{\mathbf{L}-\mathbf{M}}|k|\mathbf{M}||\mathbf{b}_{\mathbf{M}}|e^{k|\mathbf{L}||\mathbf{x}|}-\sum_{\mathbf{L}=-\infty}^{\infty}\nu k^2|\mathbf{L}|^2|\mathbf{b}_{\mathbf{L}}|e^{k|\mathbf{L}||\mathbf{x}|}\end{aligned} \quad (37)$$

implying that

$$\begin{aligned}\sum_{\mathbf{L}=-\infty}^{\infty}\frac{\partial|\mathbf{b}_{\mathbf{L}}|}{\partial t}e^{k|\mathbf{L}||\mathbf{x}|}&\leqslant\sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{a}_{\mathbf{L}}|k|\mathbf{M}||\mathbf{a}_{\mathbf{M}}|e^{k|\mathbf{L}+\mathbf{M}||\mathbf{x}|}\\&+\sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{b}_{\mathbf{L}}|k|\mathbf{M}||\mathbf{b}_{\mathbf{M}}|e^{k|\mathbf{L}+\mathbf{M}||\mathbf{x}|}-\sum_{\mathbf{L}=-\infty}^{\infty}\nu k^2|\mathbf{L}|^2|\mathbf{b}_{\mathbf{L}}|e^{k|\mathbf{L}||\mathbf{x}|}\end{aligned} \quad (38)$$

in light of (13), which yields

$$\begin{aligned}\sum_{\mathbf{L}=-\infty}^{\infty}\frac{\partial|\mathbf{b}_{\mathbf{L}}|}{\partial t}e^{k|\mathbf{L}||\mathbf{x}|}&\leqslant\sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{a}_{\mathbf{L}}|k|\mathbf{M}||\mathbf{a}_{\mathbf{M}}|e^{k(|\mathbf{L}|+|\mathbf{M}|)|\mathbf{x}|}\\&+\sum_{\mathbf{L}=-\infty}^{\infty}\sum_{\mathbf{M}=-\infty}^{\infty}|\mathbf{b}_{\mathbf{L}}|k|\mathbf{M}||\mathbf{b}_{\mathbf{M}}|e^{k(|\mathbf{L}|+|\mathbf{M}|)|\mathbf{x}|}-\sum_{\mathbf{L}=-\infty}^{\infty}\nu k^2|\mathbf{L}|^2|\mathbf{b}_{\mathbf{L}}|e^{k|\mathbf{L}||\mathbf{x}|}\end{aligned} \quad (39)$$

on using the triangle inequality.
Let

$$\psi=\sum_{\mathbf{L}=-\infty}^{\infty}|\mathbf{a}_{\mathbf{L}}|e^{k|\mathbf{L}|X}, \quad (40)$$

$$\phi = \sum_{\mathbf{L}=-\infty}^{\infty} |\mathbf{b}_{\mathbf{L}}| \mathrm{e}^{k|\mathbf{L}|X} \tag{41}$$

where $X = |\mathbf{x}|$ and note that $|\tilde{\mathbf{u}}| \leqslant Q$ where $Q = \psi + \phi$. Then (32) can be written as

$$\frac{\partial \psi}{\partial t} \leqslant \psi \frac{\partial \phi}{\partial X} + \phi \frac{\partial \psi}{\partial X} - \nu \frac{\partial^2 \psi}{\partial X^2} \tag{42}$$

and (39) can be written as

$$\frac{\partial \phi}{\partial t} \leqslant \psi \frac{\partial \psi}{\partial X} + \phi \frac{\partial \phi}{\partial X} - \nu \frac{\partial^2 \phi}{\partial X^2}. \tag{43}$$

Adding (42) and (43) yields

$$\frac{\partial Q}{\partial t} \leqslant Q \frac{\partial Q}{\partial X} - \nu \frac{\partial^2 Q}{\partial X^2}. \tag{44}$$

Equation (44) can be written as

$$\frac{\partial Q}{\partial t} - Q \frac{\partial Q}{\partial X} + \nu \frac{\partial^2 Q}{\partial X^2} = H \tag{45}$$

where $H = H(X, t) \leqslant 0$. Here $Q|_{t=0}$ converges for all $X \geqslant 0$ since $\tilde{\mathbf{u}}|_{t=0}$ converges for all $\mathbf{x} \in \mathbb{R}^d$. Note also that

$$\frac{\partial^s Q}{\partial X^s} \geqslant 0 \text{ for } s \geqslant 0. \tag{46}$$

At points where Q is a maximum,

$$\frac{\partial Q}{\partial t} \geqslant 0. \tag{47}$$

The extreme case is then $Q = \Omega$ where

$$\frac{\partial \Omega}{\partial t} = \Omega \frac{\partial \Omega}{\partial X} - \nu \frac{\partial^2 \Omega}{\partial X^2}. \tag{48}$$

Let

$$\Omega = \lambda \frac{\partial A}{\partial X} / A = \lambda \frac{\partial}{\partial X} \log_{\mathrm{e}} A \tag{49}$$

where λ is a constant. Substituting (49) into (48) gives

$$\lambda \frac{\partial}{\partial X} (\frac{\partial A}{\partial t} / A) = \lambda^2 \frac{1}{2} \frac{\partial}{\partial X} ((\frac{\partial A}{\partial X} / A)^2) - \lambda \nu \frac{\partial}{\partial X} ((\frac{\partial^2 A}{\partial X^2} A - (\frac{\partial A}{\partial X})^2) / A^2). \tag{50}$$

Then with $\lambda = -2\nu$, equation (50) gives

$$\frac{\partial}{\partial X}(\frac{\partial A}{\partial t}/A) = -\nu\frac{\partial}{\partial X}(\frac{\partial^2 A}{\partial X^2}/A) \tag{51}$$

which leads to

$$\frac{\partial A}{\partial t} = -\nu\frac{\partial^2 A}{\partial X^2} + hA \tag{52}$$

where $h = h(t)$ is arbitrary.
Let

$$A = \sum_{\mathbf{L}=-\infty}^{\infty} A_{\mathbf{L}} \mathrm{e}^{k|\mathbf{L}|X} \tag{53}$$

where $A_{\mathbf{L}} = A_{\mathbf{L}}(t)$. Substituting (53) into (52) gives

$$\sum_{\mathbf{L}=-\infty}^{\infty} \frac{\partial A_{\mathbf{L}}}{\partial t}\mathrm{e}^{k|\mathbf{L}|X} = -\nu \sum_{\mathbf{L}=-\infty}^{\infty} k^2|\mathbf{L}|^2 A_{\mathbf{L}}\mathrm{e}^{k|\mathbf{L}|X} + h\sum_{\mathbf{L}=-\infty}^{\infty} A_{\mathbf{L}}\mathrm{e}^{k|\mathbf{L}|X}. \tag{54}$$

Equating like powers of the exponentials in (54) leads to

$$\frac{\partial A_{\mathbf{L}}}{\partial t} = -\nu k^2|\mathbf{L}|^2 A_{\mathbf{L}} + A_{\mathbf{L}} h. \tag{55}$$

Equation (55) is easily solved to find

$$A_{\mathbf{L}} = A_{\mathbf{L}}(0)\mathrm{e}^{-\nu k^2|\mathbf{L}|^2 t + \int_0^t h(\tau)\,d\tau}. \tag{56}$$

It then follows that

$$\Omega = \frac{\partial}{\partial X}\log_{\mathrm{e}}((\sum_{\mathbf{L}=-\infty}^{\infty} A_{\mathbf{L}}(0)\mathrm{e}^{-\nu k^2|\mathbf{L}|^2 t}\mathrm{e}^{k|\mathbf{L}|X})^{-2\nu}). \tag{57}$$

Now with

$$\Omega = \sum_{\mathbf{L}=-\infty}^{\infty} \Omega_{\mathbf{L}}\mathrm{e}^{k|\mathbf{L}|X},\ \ \Omega_{\mathbf{0}} = 0 \tag{58}$$

where $\Omega_{\mathbf{L}} = \Omega_{\mathbf{L}}(t) \geqslant 0$ it follows that

$$\begin{aligned} A &= \mathrm{e}^{\int^X \frac{\Omega}{\lambda}\,dX} \\ &= \mathrm{e}^{\frac{1}{\lambda}\sum_{\mathbf{L}\neq\mathbf{0}} \frac{\Omega_{\mathbf{L}}\mathrm{e}^{k|\mathbf{L}|X}}{k|\mathbf{L}|}} \\ &= 1 + \frac{1}{\lambda}\sum_{\mathbf{L}\neq\mathbf{0}}\frac{\Omega_{\mathbf{L}}\mathrm{e}^{k|\mathbf{L}|X}}{k|\mathbf{L}|} + \frac{1}{2}(\frac{1}{\lambda}\sum_{\mathbf{L}\neq\mathbf{0}}\frac{\Omega_{\mathbf{L}}\mathrm{e}^{k|\mathbf{L}|X}}{k|\mathbf{L}|})^2 + \dots. \end{aligned} \tag{59}$$

For consistency, matching (53) with (59) yields

$$A_{\mathbf{0}} = 1,\ \ A_{\mathbf{L}} = \frac{\Omega_{\mathbf{L}}}{\lambda k|\mathbf{L}|} + O(\frac{1}{\lambda^2 k^2})\ \text{ for }\ \mathbf{L} \neq \mathbf{0}. \tag{60}$$

Then (57) becomes

$$\Omega = \frac{\partial}{\partial X} \log_{\mathrm{e}}(A^{\lambda}) \tag{61}$$

where

$$A = 1 + \sum_{\mathbf{L}\neq\mathbf{0}} \left(\frac{\Omega_{\mathbf{L}}(0)}{\lambda k|\mathbf{L}|} + O(\frac{1}{\lambda^2 k^2})\right)\mathrm{e}^{-\nu k^2|\mathbf{L}|^2 t}\mathrm{e}^{k|\mathbf{L}|X}. \tag{62}$$

Note that it is sufficient to take X to be in a finite domain due to the spatially periodic boundary conditions. Without loss of generality let $\Omega_{\mathbf{L}}(t) = \alpha_{|\mathbf{L}|}(t)$. Then

$$\begin{aligned} A \;=\; & 1 + \{\frac{1}{2}(\frac{1}{\lambda k})^2 \sum_{\mathbf{L}\neq\mathbf{0}}\sum_{\mathbf{M}\neq\mathbf{0}} \frac{\alpha_{|\mathbf{L}|}(0)\alpha_{|\mathbf{M}|}(0)}{|\mathbf{L}||\mathbf{M}|}\mathrm{e}^{k(|\mathbf{L}|+|\mathbf{M}|)X}\mathrm{e}^{-\nu k^2(|\mathbf{L}|+|\mathbf{M}|)^2 t} \\ & + \frac{1}{24}(\frac{1}{\lambda k})^4 \sum_{\mathbf{L}\neq\mathbf{0}}\sum_{\mathbf{M}\neq\mathbf{0}}\sum_{\mathbf{N}\neq\mathbf{0}}\sum_{\mathbf{P}\neq\mathbf{0}} \frac{\alpha_{|\mathbf{L}|}(0)\alpha_{|\mathbf{M}|}(0)\alpha_{|\mathbf{N}|}(0)\alpha_{|\mathbf{P}|}(0)}{|\mathbf{L}||\mathbf{M}||\mathbf{N}||\mathbf{P}|} \\ & \times \mathrm{e}^{k(|\mathbf{L}|+|\mathbf{M}|+|\mathbf{N}|+|\mathbf{P}|)X}\mathrm{e}^{-\nu k^2(|\mathbf{L}|+|\mathbf{M}|+|\mathbf{N}|+|\mathbf{P}|)^2 t} + \ldots\} + \{(\frac{1}{\lambda k})\sum_{\mathbf{L}\neq\mathbf{0}} \frac{\alpha_{|\mathbf{L}|}(0)}{|\mathbf{L}|}\mathrm{e}^{k|\mathbf{L}|X}\mathrm{e}^{-\nu k^2|\mathbf{L}|^2 t} \\ & + \frac{1}{6}(\frac{1}{\lambda k})^3 \sum_{\mathbf{L}\neq\mathbf{0}}\sum_{\mathbf{M}\neq\mathbf{0}}\sum_{\mathbf{N}\neq\mathbf{0}} \frac{\alpha_{|\mathbf{L}|}(0)\alpha_{|\mathbf{M}|}(0)\alpha_{|\mathbf{N}|}(0)}{|\mathbf{L}||\mathbf{M}||\mathbf{N}|}\mathrm{e}^{k(|\mathbf{L}|+|\mathbf{M}|+|\mathbf{N}|)X}\mathrm{e}^{-\nu k^2(|\mathbf{L}|+|\mathbf{M}|+|\mathbf{N}|)^2 t} \\ & + \ldots\}. \end{aligned} \tag{63}$$

In light of (63) and due to $A \in [0, 1]$ from (59) it is then clear that A increases with increasing $t \geqslant 0$. It then follows that Ω has no finite-time singularity and $|\tilde{\mathbf{u}}| \leqslant \Omega$. $\therefore$ blowup is ruled out. □

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