

# The Navier–Stokes problem

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A proposed solution to the millennium problem on the existence and smoothness of the Navier–Stokes equations.

## 1. Introduction

The Navier–Stokes equations are thought to govern the motion of a fluid in  $\mathbb{R}^d$  where  $d \in \mathbb{N}$ , see [1,3]. Let  $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^d$  be the velocity and let  $p = p(\mathbf{x}, t) \in \mathbb{R}$  be the pressure, each dependent on position  $\mathbf{x} \in \mathbb{R}^d$  and time  $t \geq 0$ . We take the externally applied force to be identically zero. The fluid is assumed to be incompressible with constant viscosity  $\nu \geq 0$  and to fill all of  $\mathbb{R}^d$ . The Navier–Stokes equations can then be written as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \nu \nabla^2 \mathbf{u} - \nabla p, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (2)$$

with initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0 \quad (3)$$

where  $\mathbf{u}_0 = \mathbf{u}_0(\mathbf{x}) \in \mathbb{R}^d$ . In these equations

$$\nabla = \left( \frac{\partial}{\partial \mathbf{x}_1}, \frac{\partial}{\partial \mathbf{x}_2}, \dots, \frac{\partial}{\partial \mathbf{x}_d} \right) \quad (4)$$

is the gradient operator and

$$\nabla^2 = \sum_{i=1}^d \frac{\partial^2}{\partial \mathbf{x}_i^2} \quad (5)$$

is the Laplacian operator. When  $\nu = 0$ , equations (1), (2), (3) are called the Euler equations. Solutions of (1), (2), (3) are to be found with

$$\mathbf{u}_0(\mathbf{x} + e_i) = \mathbf{u}_0(\mathbf{x}) \quad (6)$$

for  $1 \leq i \leq d$  where  $e_i$  is the  $i^{\text{th}}$  unit vector in  $\mathbb{R}^d$ . The initial condition  $\mathbf{u}_0$  is a given  $C^\infty$  divergence-free vector field on  $\mathbb{R}^d$ . A solution of (1), (2), (3) is then accepted to be physically reasonable [3] if

$$\mathbf{u}(\mathbf{x} + e_i, t) = \mathbf{u}(\mathbf{x}, t), \quad p(\mathbf{x} + e_i, t) = p(\mathbf{x}, t) \quad (7)$$

on  $\mathbb{R}^d \times [0, \infty)$  for  $1 \leq i \leq d$  and

$$\mathbf{u}, p \in C^\infty(\mathbb{R}^d \times [0, \infty)). \quad (8)$$

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## 2. Solution to the Navier–Stokes problem

I provide a proof of the following theorem [2,3,6].

**Theorem.** Take  $\nu > 0$ . Let  $\mathbf{u}_0$  be any smooth, divergence-free vector field satisfying (6). Then there exist smooth functions  $\mathbf{u}, p$  on  $\mathbb{R}^d \times [0, \infty)$  that satisfy (1), (2), (3), (7), (8).

**Proof.** It is sufficient to rule out the possibility that there is a smooth, divergence-free  $\mathbf{u}_0$  for which (1), (2), (3) have a solution with a finite blowup time [3].

Let the Fourier series of  $\mathbf{u}, p$  be

$$\tilde{\mathbf{u}} = \sum_{\mathbf{L}=-\infty}^{\infty} \mathbf{u}_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}}, \quad (9)$$

$$\tilde{p} = \sum_{\mathbf{L}=-\infty}^{\infty} p_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}} \quad (10)$$

respectively. Here  $\mathbf{u}_{\mathbf{L}} = \mathbf{u}_{\mathbf{L}}(t) \in \mathbb{C}^d$ ,  $p_{\mathbf{L}} = p_{\mathbf{L}}(t) \in \mathbb{C}$ ,  $i = \sqrt{-1}$ ,  $k = 2\pi$ , and  $\sum_{\mathbf{L}=-\infty}^{\infty}$  denotes the sum over all  $\mathbf{L} \in \mathbb{Z}^d$ . The initial condition  $\mathbf{u}_0$  is a Fourier series [2] of which is convergent for all  $\mathbf{x} \in \mathbb{R}^d$ . Substituting  $\mathbf{u} = \tilde{\mathbf{u}}, p = \tilde{p}$  into (1) gives

$$\begin{aligned} & \sum_{\mathbf{L}=-\infty}^{\infty} \frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} e^{ik\mathbf{L}\cdot\mathbf{x}} + \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u}_{\mathbf{L}} \cdot ik\mathbf{M}) \mathbf{u}_{\mathbf{M}} e^{ik(\mathbf{L}+\mathbf{M})\cdot\mathbf{x}} \\ &= - \sum_{\mathbf{L}=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 \mathbf{u}_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}} - \sum_{\mathbf{L}=-\infty}^{\infty} ik\mathbf{L} p_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}}. \end{aligned} \quad (11)$$

Equating like powers of the exponentials in (11) yields

$$\frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot ik\mathbf{M}) \mathbf{u}_{\mathbf{M}} = -\nu k^2 |\mathbf{L}|^2 \mathbf{u}_{\mathbf{L}} - ik\mathbf{L} p_{\mathbf{L}} \quad (12)$$

on using the Cauchy product type formula [4]

$$\sum_{l=-\infty}^{\infty} a_l x^l \sum_{m=-\infty}^{\infty} b_m x^m = \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{l-m} b_m x^l. \quad (13)$$

Substituting  $\mathbf{u} = \tilde{\mathbf{u}}$  into (2) gives

$$\sum_{\mathbf{L}=-\infty}^{\infty} ik\mathbf{L} \cdot \mathbf{u}_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}} = 0. \quad (14)$$

Equating like powers of the exponentials in (14) yields

$$\mathbf{L} \cdot \mathbf{u}_{\mathbf{L}} = 0. \quad (15)$$

Applying  $\mathbf{L} \cdot$  to (12) and noting (15) leads to

$$p_{\mathbf{L}} = - \sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \hat{\mathbf{L}})(\mathbf{u}_{\mathbf{M}} \cdot \hat{\mathbf{L}}) \quad (16)$$

where  $p_0$  is arbitrary and  $\hat{\mathbf{L}} = \mathbf{L}/|\mathbf{L}|$  is the unit vector in the direction of  $\mathbf{L}$ . Then substituting (16) into (12) gives

$$\frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} = - \sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot ik\mathbf{M})\mathbf{u}_{\mathbf{M}} - \nu k^2 |\mathbf{L}|^2 \mathbf{u}_{\mathbf{L}} + \sum_{\mathbf{M}=-\infty}^{\infty} ik\mathbf{L}(\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \hat{\mathbf{L}})(\mathbf{u}_{\mathbf{M}} \cdot \hat{\mathbf{L}}) \quad (17)$$

where  $\mathbf{u}_0 = \mathbf{u}_0(0)$ . The equations for  $\mathbf{u}_{\mathbf{L}}$  are to be solved for all  $\mathbf{L} \in \mathbb{Z}^d$ .

Let

$$\mathbf{u}_{\mathbf{L}} = \mathbf{a}_{\mathbf{L}} + i\mathbf{b}_{\mathbf{L}}, \quad (18)$$

$$p_{\mathbf{L}} = c_{\mathbf{L}} + id_{\mathbf{L}} \quad (19)$$

where  $\mathbf{a}_{\mathbf{L}} \in \mathbb{R}^d$ ,  $\mathbf{b}_{\mathbf{L}} \in \mathbb{R}^d$ ,  $c_{\mathbf{L}} \in \mathbb{R}$ , and  $d_{\mathbf{L}} \in \mathbb{R}$ . Substituting (18), (19) into (12) gives

$$\begin{aligned} \frac{\partial \mathbf{a}_{\mathbf{L}}}{\partial t} + i \frac{\partial \mathbf{b}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} ((\mathbf{a}_{\mathbf{L}-\mathbf{M}} + i\mathbf{b}_{\mathbf{L}-\mathbf{M}}) \cdot ik\mathbf{M})(\mathbf{a}_{\mathbf{M}} + i\mathbf{b}_{\mathbf{M}}) \\ = -\nu k^2 |\mathbf{L}|^2 (\mathbf{a}_{\mathbf{L}} + i\mathbf{b}_{\mathbf{L}}) - ik\mathbf{L}(c_{\mathbf{L}} + id_{\mathbf{L}}). \end{aligned} \quad (20)$$

Equating real and imaginary parts in (20) gives

$$\frac{\partial \mathbf{a}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} (-(\mathbf{a}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{b}_{\mathbf{M}} - (\mathbf{b}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}}) = -\nu k^2 |\mathbf{L}|^2 \mathbf{a}_{\mathbf{L}} + k\mathbf{L}d_{\mathbf{L}}, \quad (21)$$

$$\frac{\partial \mathbf{b}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} ((\mathbf{a}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}} - (\mathbf{b}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{b}_{\mathbf{M}}) = -\nu k^2 |\mathbf{L}|^2 \mathbf{b}_{\mathbf{L}} - k\mathbf{L}c_{\mathbf{L}}. \quad (22)$$

Substituting (18) into (15) gives

$$\mathbf{L} \cdot (\mathbf{a}_{\mathbf{L}} + i\mathbf{b}_{\mathbf{L}}) = 0. \quad (23)$$

Equating real and imaginary parts in (23) gives

$$\mathbf{L} \cdot \mathbf{a}_{\mathbf{L}} = 0, \quad (24)$$

$$\mathbf{L} \cdot \mathbf{b}_{\mathbf{L}} = 0. \quad (25)$$

From (21) and in light of (24) it is possible to write

$$\frac{\partial \mathbf{a}_{\mathbf{L}}}{\partial t} \cdot \hat{\mathbf{a}}_{\mathbf{L}} + \sum_{\mathbf{M}=-\infty}^{\infty} (-(\mathbf{a}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{b}_{\mathbf{M}} - (\mathbf{b}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}}) \cdot \hat{\mathbf{a}}_{\mathbf{L}} = -\nu k^2 |\mathbf{L}|^2 \mathbf{a}_{\mathbf{L}} \cdot \hat{\mathbf{a}}_{\mathbf{L}} \quad (26)$$

where  $\hat{\mathbf{a}}_{\mathbf{L}} = \mathbf{a}_{\mathbf{L}}/|\mathbf{a}_{\mathbf{L}}|$  is the unit vector in the direction of  $\mathbf{a}_{\mathbf{L}}$ . Then (26) implies

$$\frac{\partial |\mathbf{a}_{\mathbf{L}}|}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} (-(\mathbf{a}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{b}_{\mathbf{M}} - (\mathbf{b}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}}) \cdot \hat{\mathbf{a}}_{\mathbf{L}} = -\nu k^2 |\mathbf{L}|^2 |\mathbf{a}_{\mathbf{L}}|. \quad (27)$$

From (27) it is possible to write

$$\frac{\partial |\mathbf{a}_{\mathbf{L}}|}{\partial t} \leq \sum_{\mathbf{M}=-\infty}^{\infty} (|\mathbf{a}_{\mathbf{L}-\mathbf{M}}|k|\mathbf{M}||\mathbf{b}_{\mathbf{M}}| + |\mathbf{b}_{\mathbf{L}-\mathbf{M}}|k|\mathbf{M}||\mathbf{a}_{\mathbf{M}}|) - \nu k^2 |\mathbf{L}|^2 |\mathbf{a}_{\mathbf{L}}| \quad (28)$$

on using the Cauchy–Schwarz inequality [5]

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|. \quad (29)$$

It then follows from (28) that

$$\begin{aligned} \sum_{\mathbf{L}=-\infty}^{\infty} \frac{\partial |\mathbf{a}_{\mathbf{L}}|}{\partial t} e^{k|\mathbf{L}||\mathbf{x}|} &\leq \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{a}_{\mathbf{L}-\mathbf{M}}|k|\mathbf{M}||\mathbf{b}_{\mathbf{M}}| e^{k|\mathbf{L}||\mathbf{x}|} \\ &+ \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{b}_{\mathbf{L}-\mathbf{M}}|k|\mathbf{M}||\mathbf{a}_{\mathbf{M}}| e^{k|\mathbf{L}||\mathbf{x}|} - \sum_{\mathbf{L}=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 |\mathbf{a}_{\mathbf{L}}| e^{k|\mathbf{L}||\mathbf{x}|} \end{aligned} \quad (30)$$

implying that

$$\begin{aligned} \sum_{\mathbf{L}=-\infty}^{\infty} \frac{\partial |\mathbf{a}_{\mathbf{L}}|}{\partial t} e^{k|\mathbf{L}||\mathbf{x}|} &\leq \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{a}_{\mathbf{L}}|k|\mathbf{M}||\mathbf{b}_{\mathbf{M}}| e^{k|\mathbf{L}+\mathbf{M}||\mathbf{x}|} \\ &+ \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{b}_{\mathbf{L}}|k|\mathbf{M}||\mathbf{a}_{\mathbf{M}}| e^{k|\mathbf{L}+\mathbf{M}||\mathbf{x}|} - \sum_{\mathbf{L}=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 |\mathbf{a}_{\mathbf{L}}| e^{k|\mathbf{L}||\mathbf{x}|} \end{aligned} \quad (31)$$

in light of (13), which yields

$$\begin{aligned} \sum_{\mathbf{L}=-\infty}^{\infty} \frac{\partial |\mathbf{a}_{\mathbf{L}}|}{\partial t} e^{k|\mathbf{L}||\mathbf{x}|} &\leq \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{a}_{\mathbf{L}}|k|\mathbf{M}||\mathbf{b}_{\mathbf{M}}| e^{k(|\mathbf{L}|+|\mathbf{M}|)|\mathbf{x}|} \\ &+ \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{b}_{\mathbf{L}}|k|\mathbf{M}||\mathbf{a}_{\mathbf{M}}| e^{k(|\mathbf{L}|+|\mathbf{M}|)|\mathbf{x}|} - \sum_{\mathbf{L}=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 |\mathbf{a}_{\mathbf{L}}| e^{k|\mathbf{L}||\mathbf{x}|} \end{aligned} \quad (32)$$

on using the triangle inequality [5]

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|. \quad (33)$$

From (22) and in light of (25) it is possible to write

$$\frac{\partial |\mathbf{b}_{\mathbf{L}}|}{\partial t} \cdot \hat{\mathbf{b}}_{\mathbf{L}} + \sum_{\mathbf{M}=-\infty}^{\infty} ((\mathbf{a}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}} - (\mathbf{b}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{b}_{\mathbf{M}}) \cdot \hat{\mathbf{b}}_{\mathbf{L}} = -\nu k^2 |\mathbf{L}|^2 |\mathbf{b}_{\mathbf{L}}| \cdot \hat{\mathbf{b}}_{\mathbf{L}} \quad (34)$$

where  $\hat{\mathbf{b}}_{\mathbf{L}} = \mathbf{b}_{\mathbf{L}}/|\mathbf{b}_{\mathbf{L}}|$  is the unit vector in the direction of  $\mathbf{b}_{\mathbf{L}}$ . Then (34) implies

$$\frac{\partial|\mathbf{b}_{\mathbf{L}}|}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} ((\mathbf{a}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}} - (\mathbf{b}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{b}_{\mathbf{M}}) \cdot \hat{\mathbf{b}}_{\mathbf{L}} = -\nu k^2 |\mathbf{L}|^2 |\mathbf{b}_{\mathbf{L}}|. \quad (35)$$

From (35) it is possible to write

$$\frac{\partial|\mathbf{b}_{\mathbf{L}}|}{\partial t} \leq \sum_{\mathbf{M}=-\infty}^{\infty} (|\mathbf{a}_{\mathbf{L}-\mathbf{M}}|k|\mathbf{M}||\mathbf{a}_{\mathbf{M}}| + |\mathbf{b}_{\mathbf{L}-\mathbf{M}}|k|\mathbf{M}||\mathbf{b}_{\mathbf{M}}|) - \nu k^2 |\mathbf{L}|^2 |\mathbf{b}_{\mathbf{L}}| \quad (36)$$

on using the Cauchy–Schwarz inequality. It then follows from (36) that

$$\begin{aligned} \sum_{\mathbf{L}=-\infty}^{\infty} \frac{\partial|\mathbf{b}_{\mathbf{L}}|}{\partial t} e^{k|\mathbf{L}||\mathbf{x}|} &\leq \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{a}_{\mathbf{L}-\mathbf{M}}|k|\mathbf{M}||\mathbf{a}_{\mathbf{M}}| e^{k|\mathbf{L}||\mathbf{x}|} \\ &+ \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{b}_{\mathbf{L}-\mathbf{M}}|k|\mathbf{M}||\mathbf{b}_{\mathbf{M}}| e^{k|\mathbf{L}||\mathbf{x}|} - \sum_{\mathbf{L}=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 |\mathbf{b}_{\mathbf{L}}| e^{k|\mathbf{L}||\mathbf{x}|} \end{aligned} \quad (37)$$

implying that

$$\begin{aligned} \sum_{\mathbf{L}=-\infty}^{\infty} \frac{\partial|\mathbf{b}_{\mathbf{L}}|}{\partial t} e^{k|\mathbf{L}||\mathbf{x}|} &\leq \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{a}_{\mathbf{L}}|k|\mathbf{M}||\mathbf{a}_{\mathbf{M}}| e^{k|\mathbf{L}+\mathbf{M}||\mathbf{x}|} \\ &+ \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{b}_{\mathbf{L}}|k|\mathbf{M}||\mathbf{b}_{\mathbf{M}}| e^{k|\mathbf{L}+\mathbf{M}||\mathbf{x}|} - \sum_{\mathbf{L}=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 |\mathbf{b}_{\mathbf{L}}| e^{k|\mathbf{L}||\mathbf{x}|} \end{aligned} \quad (38)$$

in light of (13), which yields

$$\begin{aligned} \sum_{\mathbf{L}=-\infty}^{\infty} \frac{\partial|\mathbf{b}_{\mathbf{L}}|}{\partial t} e^{k|\mathbf{L}||\mathbf{x}|} &\leq \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{a}_{\mathbf{L}}|k|\mathbf{M}||\mathbf{a}_{\mathbf{M}}| e^{k(|\mathbf{L}|+|\mathbf{M}|)|\mathbf{x}|} \\ &+ \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{b}_{\mathbf{L}}|k|\mathbf{M}||\mathbf{b}_{\mathbf{M}}| e^{k(|\mathbf{L}|+|\mathbf{M}|)|\mathbf{x}|} - \sum_{\mathbf{L}=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 |\mathbf{b}_{\mathbf{L}}| e^{k|\mathbf{L}||\mathbf{x}|} \end{aligned} \quad (39)$$

on using the triangle inequality.

Let

$$\psi = \sum_{\mathbf{L}=-\infty}^{\infty} |\mathbf{a}_{\mathbf{L}}| e^{k|\mathbf{L}|X}, \quad (40)$$

$$\phi = \sum_{\mathbf{L}=-\infty}^{\infty} |\mathbf{b}_{\mathbf{L}}| e^{k|\mathbf{L}|X} \quad (41)$$

where  $X = |\mathbf{x}|$  and note that

$$|\tilde{\mathbf{u}}| \leq Q \quad (42)$$

where  $Q = \psi + \phi$ . Then (32) can be written as

$$\frac{\partial \psi}{\partial t} \leq \psi \frac{\partial \phi}{\partial X} + \phi \frac{\partial \psi}{\partial X} - \nu \frac{\partial^2 \psi}{\partial X^2} \quad (43)$$

and (39) can be written as

$$\frac{\partial \phi}{\partial t} \leq \psi \frac{\partial \psi}{\partial X} + \phi \frac{\partial \phi}{\partial X} - \nu \frac{\partial^2 \phi}{\partial X^2}. \quad (44)$$

Adding (43) and (44) yields

$$\frac{\partial Q}{\partial t} \leq Q \frac{\partial Q}{\partial X} - \nu \frac{\partial^2 Q}{\partial X^2}. \quad (45)$$

Equation (45) can be written as

$$\frac{\partial Q}{\partial t} - Q \frac{\partial Q}{\partial X} + \nu \frac{\partial^2 Q}{\partial X^2} = H \quad (46)$$

where  $H = H(X, t) \leq 0$  and can be thought of as a force. Here  $Q|_{t=0}$  converges for all  $X \geq 0$  since  $\tilde{\mathbf{u}}|_{t=0}$  converges for all  $\mathbf{x} \in \mathbb{R}^d$ . Note also that

$$\frac{\partial^s Q}{\partial X^s} \geq 0 \text{ for } s \geq 0. \quad (47)$$

At points where  $Q$  is a maximum,

$$\frac{\partial Q}{\partial t} \geq 0. \quad (48)$$

The extreme case is then  $Q = \Omega$  where

$$\frac{\partial \Omega}{\partial t} = \Omega \frac{\partial \Omega}{\partial X} - \nu \frac{\partial^2 \Omega}{\partial X^2}. \quad (49)$$

Let

$$\Omega = \lambda \frac{\partial A}{\partial X} / A = \lambda \frac{\partial}{\partial X} \log_e A \quad (50)$$

where  $\lambda$  is a constant. Substituting (50) into (49) gives

$$\lambda \frac{\partial}{\partial X} \left( \frac{\partial A}{\partial t} / A \right) = \lambda^2 \frac{1}{2} \frac{\partial}{\partial X} \left( \left( \frac{\partial A}{\partial X} / A \right)^2 \right) - \lambda \nu \frac{\partial}{\partial X} \left( \left( \frac{\partial^2 A}{\partial X^2} A - \left( \frac{\partial A}{\partial X} \right)^2 \right) / A^2 \right). \quad (51)$$

Then with  $\lambda = -2\nu$ , equation (51) gives

$$\frac{\partial}{\partial X} \left( \frac{\partial A}{\partial t} / A \right) = -\nu \frac{\partial}{\partial X} \left( \frac{\partial^2 A}{\partial X^2} / A \right) \quad (52)$$

which leads to

$$\frac{\partial A}{\partial t} = -\nu \frac{\partial^2 A}{\partial X^2} + hA \quad (53)$$

where  $h = h(t)$  is arbitrary.

Let

$$A = \sum_{l=0}^{\infty} A_l e^{klX} \quad (54)$$

where  $A_l = A_l(t)$ . Substituting (54) into (53) gives

$$\sum_{l=0}^{\infty} \frac{\partial A_l}{\partial t} e^{klX} = - \sum_{l=0}^{\infty} \nu k^2 l^2 A_l e^{klX} + \sum_{l=0}^{\infty} A_l h e^{klX}. \quad (55)$$

Equating like powers of the exponentials in (55) yields

$$\frac{\partial A_l}{\partial t} = -\nu k^2 l^2 A_l + A_l h. \quad (56)$$

Equation (56) is easily solved to find

$$A_l = c_l e^{-\nu k^2 l^2 t + \int h dt} \quad (57)$$

where  $c_l$  are constants. It then follows that

$$\Omega = \frac{\partial}{\partial X} \log_c \left( \sum_{l=0}^{\infty} c_l e^{-\nu k^2 l^2 t} e^{klX} \right)^{-2\nu}. \quad (58)$$

It is sufficient to take  $X$  to be in a finite domain due to the spatially periodic boundary conditions. Then  $\Omega$  has no finite-time singularity and

$$|\tilde{\mathbf{u}}| \leq \Omega. \quad (59)$$

$\therefore$  blowup is ruled out. □

## References

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