Theorem 1 If the real number \( \alpha \) satisfies an irreducible polynomial over the field of rational numbers of degree \( k \), and if \( k \) is not a power of 2, then \( \alpha \) is not constructible.

*Proof.*

This is Corollary 2 to Theorem 5.4.1 in [1].

Theorem 2 A regular heptagon is not constructible by straightedge and compass.

*Proof.*

A regular heptagon is a seven-sided polygon with sides of equal length. The construction of a regular heptagon requires the constructibility of the real number \( \cos \frac{2\pi}{7} \). Given the formula

\[
\cos 7\theta = 64 \cos^7 \theta - 112 \cos^5 \theta + 56 \cos^3 \theta - 7 \cos \theta,
\]

putting \( \theta = \frac{2\pi}{7} \), \( \alpha = \cos \frac{2\pi}{7} \) and rearranging terms give

\[
64\alpha^7 - 112\alpha^5 + 56\alpha^3 - 7\alpha - 1 = 0.
\]

Thus \( \alpha \) is a root of the polynomial \( 64x^7 - 112x^5 + 56x^3 - 7x - 1 \) over the rational field. Upon factorization,

\[
64x^7 - 112x^5 + 56x^3 - 7x - 1 = (x - 1)(8x^3 + 4x^2 - 4x - 1)^2.
\]

Since \( x = 1 \neq 0 \),

\[
8x^3 + 4x^2 - 4x - 1 = 0.
\]

Since \( 8(\frac{1}{2}x + 1)^3 + 4(\frac{1}{2}x + 1)^2 - 4(\frac{1}{2}x + 1) - 1 = x^3 + 7x^2 + 14x + 7 \), this polynomial is irreducible over the rationals by the Eisenstein criterion with the prime number 7. It follows that the polynomial \( 8x^3 + 4x^2 - 4x - 1 \) is irreducible over the rational field and since its degree is 3, which is not a power of 2, \( \alpha \) is not constructible by Theorem 1.

**Reference**