Proof that there are no odd perfect numbers

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Abstract

If $y$ is an odd perfect number, let $p$ be one of the prime factors of $y$, the exponent of $p$ be an integer $n$ ($n \geq 1$), the prime factors other than $p$ and different from each other be $p_1, p_2, \ldots, p_r \ldots p_s$ and the even exponent of $p_k$ be $q_k$.

\[
y/p^n = (1 + p + p^2 + \cdots + p^n) \prod_{k=1}^{r} (1 + p_k + p_k^2 + \cdots + p_k^{q_k})/(2p^n) = \prod_{k=1}^{r} p_k^{q_k}
\]

must be satisfied. Let $m$ be a non negative integer and $q$ be a positive integer,

\[
n = 4m + 1 \\
p = 4q + 1
\]

Letting $a$ and $b$ be odd integers, satisfying following expressions,

\[
a = \prod_{k=1}^{r} (1 + p_k + p_k^2 + \cdots + p_k^{q_k}) \\
b = \prod_{k=1}^{r} p_k^{q_k}
\]

\[
a(p^n + \cdots + 1) - 2bp^n = 0
\]
is established. This is a known content that has been proven by Euler. Let $s$ ($s \geq r$) be an integer, $v$ be a rational number,

\[
v = \prod_{k=1}^{s} (1 + p_k + p_k^2 + \cdots + p_k^{q_k})/\prod_{k=1}^{s} p_k^{q_k}
\]
holds. By the consideration of this research paper, we proved that when $v$ is not an integer, due to the uniqueness of $a(p^n + \cdots + 1)/(2bp^n)$, etc there are no solutions that satisfies the equation $a(p^n + \cdots + 1) - 2bp^n = 0$ other than inappropriate solution $(a, b, p, n) = (1, 1, 1, 1)$. Then, because we proved that a contradiction arises when $v$ becomes an integer, we have obtained a conclusion that there are no odd perfect numbers.

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1. Introduction

The perfect number is one in which the sum of the divisors other than itself is the same value as itself and the smallest perfect number is

\[ 1 + 2 + 3 = 6 \]

It is 6. Whether odd perfect numbers exist or not is currently an unsolved problem in mathematics.

2. Proof

Let \( y \) be an odd perfect number, one of the prime factors of \( y \) be an odd prime \( p \) and an exponent of \( p \) be an integer \( n (n \geq 1) \). Let the prime factors other than \( p \) and different from each other be \( p_1, p_2, \ldots, p_r \) \( q_k \) be the index of \( p_k \) and an integer \( a \) be the product of series of prime numbers other than prime \( p \).

\[
a = \prod_{k=1}^{r} (1 + p_k + p_k^2 + \cdots + p_k^{q_k}) \ldots (1)
\]

The number of terms \( N \) of variable \( a \) is

\[
N = \prod_{k=1}^{r} (q_k + 1) \ldots (2)
\]

When \( y \) is a perfect number,

\[
y = a(1 + p + p^2 + \cdots + p^n) - y (n > 0)
\]

is established.

\[
a \sum_{k=0}^{n} p^k /2 = y
\]

\[
a \sum_{k=0}^{n} p^k / (2p^n) = y/p^n \ldots (3)
\]

2.1. When \( q_k \) has at least one odd integer

Letting the number of terms where \( q_k \) is odd be a positive integer \( u \), because \( y/p^n = \prod_{k=1}^{r} p_k^{q_k} \) is odd and the denominator on the left side of the expression \( 3 \) has a prime factor 2, from the expression \( 2 \) variable \( a \) has prime factor 2 more than \( u \) and variable \( a \) is even. Therefore, \( n \) is even and \( u \) is 1 since \( \sum_{k=0}^{n} p^k \) must be an odd integer.
2.2. When all $q_k$ are even integers

$y/p^n$ is odd and the denominator on the left side of the expression $\textcircled{2}$ is even. Since $N$ is odd when $q_k$ are all even integers, variable $a$ is odd. Therefore, $\sum_{k=0}^{n} p^k \equiv 0 \pmod{2}$ is established and $n$ must be an odd integer since $\sum_{k=0}^{n} p^k$ is necessary to include one prime factor 2.

From 2.1, 2.2, in order for odd perfect numbers to exist, only one exponent of the prime factor of $y$ must be an odd integer. We consider the case of 2.2 below.

In order for $y$ to be an odd perfect number, the following expression must be established.

\[
y/p^n = (1 + p + p^2 + \cdots + p^n) \prod_{k=1}^{r} (1 + p_k + p_k^2 + \cdots + p_k^{q_k})/(2p^n) = \prod_{k=1}^{r} p_k^{q_k}
\]

However, $q_1, q_2, \cdots, q_r$ are all even integers.

Here, let $b$ be an odd integer.

\[
b = \prod_{k=1}^{r} p_k^{q_k} \quad \text{... (4)}
\]

A following expression is established.

\[
y/p^n = a(1 + p + p^2 + \cdots + p^n)/(2p^n) = b
\]

\[
a(p^{n+1} - 1)/(2(p - 1)p^n) = b
\]

\[
(a - 2b)p^{n+1} + 2bp^n - a = 0 \quad \text{... (5)}
\]

\[
(ap - 2bp + 2b)p^n = a
\]

Since $ap - 2bp + 2b$ is odd, $a/p^n$ is odd. Let $a/p^n$ be an odd integer $c$.

\[
ap - 2bp + 2b = c \quad (c > 0) \quad \text{... (6)}
\]

\[
(2b - a)p = 2b - c
\]

Since variable $a$ is odd, $2b - a$ is odd and $2b - a \neq 0$.

\[
p = (2b - c)/(2b - a)
\]
Since $n \geq 1$, 
\[ a - c = cp^n - c \geq cp - c > 0 \]
a > c is.

From the equation ⑥, 
\[ 2b(p - 1) - (ap - c) = 0 \]
\[ 2b - c(p^{n+1} - 1)/(p - 1) = 0 \]
Since $(p^n + \cdots + 1)/2$ is odd, $n = 4m + 1$ must be hold with $m$ as an integer.
\[ 2b(p - 1) = c(p^{n+1} - 1) \]
\[ 2b = c(p^n + \cdots + 1) \]
\[ 2b = c(p + 1)(p^{n-1} + p^{n-3} + \cdots + 1) \ldots ⑦ \]
Since $b$ is odd when $p + 1$ is not a multiple of 4, $p - 1$ must be a multiple of 4. A positive integer is taken as $q$.
\[ p = 4q + 1 \]
is established. The above conditions have been proved by Euler.

When $p > 1$, 
\[ p^n - 1 < p^n \]
\[ (p^n - 1)/(p - 1) < p^n/(p - 1) \]
\[ p^{n-1} + \cdots + 1 < p^n/(p - 1) \]

Since $p$ is an odd prime number satisfying $p = 4q + 1$ and $p \geq 5$, 
\[ p^{n-1} + \cdots + 1 < p^n/4 \]
\[ 2b - a = c(p^n + \cdots + 1) - cp^n = c(p^{n-1} + \cdots + 1) \]
\[ 2b - a < cp^n/4 = a/4 \]
\[ 2b < 5a/4 \]
a $> 8b/5 \ldots ⑧$
Let $a_k$ and $b_k$ be odd integers and if

- $a_k = 1 + p_k + p_k^2 + \cdots + p_k^{q_k}$, $b_k = p_k^{q_k}$
- $a_k - b_k < b_k/(p_k - 1)$
- $a_k < b_k p_k/(p_k - 1)$

$$a = \prod_{k=1}^{r} a_k < \prod_{k=1}^{r} b_k p_k/(p_k - 1) = b \prod_{k=1}^{r} p_k/(p_k - 1)$$

$$a/b < \prod_{k=1}^{r} p_k/(p_k - 1)$$

When $r = 1$, since $a/b < 3/2$ is established, it becomes inappropriate contrary to inequality (8).

From the expression (7),

$$b = c(p + 1)/2 \times (p^{n-1} + p^{n-3} + \cdots + 1)$$

holds. Since $(p + 1)/2$ is the product of only prime numbers of $b$, let $d_k$ be the index.

$$(p + 1)/2 = \prod_{k=1}^{r} p_k^{d_k}$$

$$p = 2 \prod_{k=1}^{r} p_k^{d_k} - 1 \ldots (9)$$

From $a = cp^n$ and the expression (7),

$$2bp^n = a(p^n + \cdots + 1)$$

$$a(p^n + \cdots + 1)/(2bp^n) = 1 \ldots (A)$$

When $r = 1$,

- $a = (p_1^{q_1+1} - 1)/(p_1 - 1)$
- $b = p_1^{q_1}$

The equation (A) does not hold since there are no odd perfect numbers when $r = 1$. 
Let $A_k$ and $B_k$ to be odd integers.

\[ A_k = \frac{(p_k q_{k+1}) - 1}{p_k - 1} \]
\[ B_k = p_k q_k \]

\[ a = \prod_{k=1}^{i} A_k \text{ and } b = \prod_{k=1}^{i} B_k \] hold. Let $R$ be a rational number.

\[ R = a(p^n + \cdots + 1)/(2bp^n) \]

Define an operation [multiplication] and an operation [division] as follows.

An operation [multiplication] of $B_i$ is performed by multiplying $R$ by $A_i/B_i$ and changing $p$ and $n$ in the expression of $R$ to $p'$ and $n'$. If $R$ is changed to $R'$ by this operation,

\[ R' = R \times A_i/B_i \times p^n/p' \times (p'' + \cdots + 1)/(p^n + + 1) \]
\[ p' = 2 \prod_{k=1}^{r} p_k^{d_k} \times p_i^{d_i} - 1 \]
\[ 0 \leq d_i \leq q_i \]

Here, let $i = r + 1$. When a point $A(a, b, p, n)$ moves to a point $B(aA_i, bB_i, p', n')$, this operation calculates the value of $R'$ at the point $B$ from the value of $R$ at the point $A$.

An operation [division] of $B_j$ is performed by dividing $R$ by $A_j/B_j$ and changing $p$ and $n$ in the expression of $R$ to $p''$ and $n''$. If $R$ is changed to $R''$ by this operation,

\[ R'' = R \times B_j/A_j \times p^n/p'' \times (p'' + \cdots + 1)/(p^n + + 1) \]

If $p_j$ is included in the expression ③, $p_j$ is deleted as $d_j = 0$.

\[ p'' = 2 \prod_{k=1}^{j-1} p_k^{d_k} \times \prod_{k=j+1}^{r} p_k^{d_k} - 1 \]

Here, $j$ is assumed to satisfy $1 \leq j \leq r$. When the point $A(a, b, p, n)$ moves to a point $C(a/A_j, b/B_j, p'', n'')$, this operation calculates the value of $R''$ at the point $C$ from the value of $R$ at the point $A$.

When the operation [multiplication] of $B_{r+1}$ is performed on $R$, there are both cases that $p$ is increased or not changed. When this operation is performed, the rate of change of $R$ is $A_{r+1}p^n(p^n + \cdots + 1)/(B_{r+1}p^n(p^n + \cdots + 1))$, if $p$ after variation is $p'$. If the rate of change of $R$ is 1,

\[ A_{r+1}p^n(p^n + \cdots + 1)/(B_{r+1}p^n(p^n + \cdots + 1)) = 1 \]
\[ A_{r+1}p^n(p^n + \cdots + 1) = B_{r+1}p^n(p^n + \cdots + 1) \]

This expression does not hold since the right side is not a multiple of $p$ when $p' > p$ and $A_{r+1} > B_{r+1}$ holds when $p' = p$. Due to this operation, $R$ becomes larger or smaller than the original value since the rate of change of $R$ does not become 1.
Assuming that \( R = 1 \) in some \( r \), letting \( t (t < r) \) be an integer and assuming that 
\[ A_t A_{t+1} \ldots A_r \] is not a multiple of \( p \), \( R \) is performed the operations [division] of \( B_t, \ldots, B_{t+1}, B_t \). Furthermore, letting \( x \) \((x > t)\) be an integer and \( R \) is performed the operations [multiplication] of \( B'_{t}, B'_{t+1}, \ldots, B_x \) and it is assumed that finally \( R = 1 \). At this time, assuming that \( n \) is changed when the operations [multiplication] or the operations [division] are performed.

\[
1 \times B_t p^n (p^{n+1} + \ldots + 1) / (A_t p^{n+1} (p^{n+1} + \ldots + 1) + 1) \times \ldots \times B_t p^{n+1} (p^{n+1} + \ldots + 1) \times B_t p^{n+1} (p^{n+1} + \ldots + 1) \times \ldots \\
\]

When \( x = p \) and \( n_x < n \), if both sides of the above expression are divided by \( p^{n_x} \) it becomes a contradiction since the right side does not include the prime factor \( p \).

\[
A_1 \ldots A_r = cp^n \\
2B_1 \ldots B_r = c(p^n + \ldots + 1) \\
A_1 \ldots A_x = c'p_x^{n_x} \\
2B_1 \ldots B_x = c'(p_x^{n_x} + \ldots + 1)
\]

Assume that these equations hold. If these equations are substituted into the expression (B), the expression is established, so no contradiction arises by performing the operations [division].

Let \( s (s \geq r) \) be an integer and \( v \) be a rational number, if
\[
v = \prod_{k=1}^{s} (1 + p_k + p_k^2 + \ldots + p_k^{q_k}) / \prod_{k=1}^{s} p_k^{q_k}
\]
holds, we define a condition \( C \) as follows.

\( v \) is not an integer \( \cdots (C) \)

Consider a directed graph whose root is \((a, b, p, n) = (1, 1, 1, 1)\) and when the operations [multiplication] are performed, it becomes a child node. For example, consider a child node connected to a root as follows.

\((a, b, p, n) = (13, 9, 5, 5)\) as \( p_1 = 3, q_1 = 2 \) and \( d_1 = 1 \)
\((a, b, p, n) = (13, 9, 17, 9)\) as \( p_1 = 3, q_1 = 2 \) and \( d_1 = 2 \)
\((a, b, p, n) = (57, 49, 97, 13)\) as \( p_1 = 7, q_1 = 2 \) and \( d_1 = 2 \)
Suppose that a node is connected from a vertex \((1,1,1,1)\) when operations [multiplication] of \(B_k\) \((k = 1,2,\ldots,r)\) are performed \(r\) times. Here, it is assumed that the bases \(p_k\) of \(B_k\) are all different odd prime numbers. When an operation [multiplication] of \(B_i\) is performed from this node, the base \(p_i\) of \(B_i\) can take any odd prime number other than \(p_k\) and its exponent \(q_i\) \((q_i \mod 2 \equiv 0, q_i > 0)\), the variable \(d_i\) \((0 \leq d_i \leq q_i)\) in the expression 5 and the index \(n\) \((n \mod 4 \equiv 1, n > 0)\) of \(p\) can be assumed to take all possible values. The number of operations [multiplication] before connecting to a certain node from the root is defined as a distance of this node from the root. When this graph is composed, even if \(p\) is not a prime in a halfway layer, this case is also called into account since it can be considered that \(p\) becomes a prime in a layer having a greater distance than this node. For example, a case where \(B_1 = 5^2, B_2 = 11^2, p = 2 \times 5 - 1 = 9\) and \(p' = 2 \times 5 \times 11 - 1 = 109\) can be considered. If an operation [multiplication] with different values of \(p_i,q_i,d_i,n\) is performed at a certain node, the graph branches. By repeating this operation, all nodes that can be considered as solutions can be connected from the root. Since this directed graph is connected to the same node even if the order of a plurality of operations [multiplication] is exchanged, it becomes a graph having a cycle.

When performing the same base operation [multiplication] as the last operation [multiplication] of \(B_j\) performed up to a certain node, the operation [division] of \(B_j\) must be performed and then the operation [multiplication] of \(B_j\) must be performed. That is, it returns from the child node to the parent node. By repeating the operations [division], it is possible to finally return to the root, so that it is possible to connect any two points in the four-dimensional space that can be considered as solutions by these operations.

Suppose that the directed graph is created by repeating the operations [multiplication] as described above. Here, a case is considered in which a solution exists at a certain point in this four-dimensional space \((a,b,p,n)\) and another solution exists at other point connected by only the operations [multiplication]. If \(R = 1\) holds again when the operations [multiplication] are performed from one point where \(R = 1\),

\[
1 \times A_{r+1} p^n (p_{r+1}^{n_{r+1}} \cdots + 1) / (B_{r+1} p_{r+1}^{n_{r+1}} (p^n + \cdots + 1)) \times A_{r+2} p_{r+2}^{n_{r+2}} (p_{r+2}^{n_{r+2}} + \cdots + 1) / (B_{r+2} p_{r+2}^{n_{r+2}} (p_{r+1}^{n_{r+1}} + \cdots + 1)) \times \cdots \times A_k p_{k-1}^{n_{k-1}} (p_k^{n_k} + \cdots + 1) / (B_k p_k^{n_k} (p_{k-1}^{n_{k-1}} + \cdots + 1)) = 1
\]

\[
A_{r+1} A_{r+2} \ldots A_k / (B_{r+1} B_{r+2} \ldots B_k) = p_k^{n_k} (p^n + \cdots + 1) / (p^n (p_k^{n_k} + \cdots + 1))
\]

\[
A_1 A_2 \ldots A_k (p_k^{n_k} + \cdots + 1) / (B_1 B_2 \ldots B_k p_k^{n_k}) = A_1 A_2 \ldots A_r (p^n + \cdots + 1) / (B_1 B_2 \ldots B_r p^n) \quad \text{(D)}
\]
Assume that $G_r = A_1A_2...A_r(p^n + \cdots + 1)/(B_1B_2...B_dp^n)$ holds. Assume that $q_k - h_k$ and $n$ becomes $n - h (n - h \geq 0)$ by changing $q_k$ and $n$ with respect to $G_r$. Here, $h_k$ is an even integer and $h$ is an integer to be a multiple of 4 or be $n$. Then, $G_s$ is a function obtained by performing the operations [multiplication] $s - r$ ($s > r$) times. If $h$ equals $n$, since it means that $p$ has been deleted, the operations [multiplication] are performed so that $p$ becomes a new value. $G_s$ represents all points $(a, b, p, n)$ that can be solutions. At this time, if there is no change at $G_s = G_r$,

$$G_s/G_r = p_{r+1}q_{r+1} \times ... \times p_sq_s/((p_{r+1}q_{r+1} + \cdots + 1) \times ... \times (p_sq_s + \cdots + 1)) \times p_1q_1 \times p_2q_2 \times ... \times p_rq_r(p_1q_1-h_1 + \cdots + 1) \times ... \times (p_rq_r-h_r + \cdots + 1)(p^n-h + \cdots + 1)/(p_1q_1-h_1 \times ... \times p_rq_r-h_r)(p_1q_1 + \cdots + 1)(p^n + \cdots + 1) = 1$$

$$p_{r+1}q_{r+1} \times ... \times p_sq_s/((p_{r+1}q_{r+1} + \cdots + 1) \times ... \times (p_sq_s + \cdots + 1)) \times p_1q_1 \times p_2q_2 \times ... \times p_rq_r(p_1q_1-h_1 + \cdots + 1) \times ... \times (p_rq_r-h_r + \cdots + 1)(p^n-h + \cdots + 1) \times ... \times (p_rq_r-h_r + \cdots + 1)(p^n + \cdots + p^h)$$

Since $\Pi_{k=1}^bA_k = cp^n$ holds,

$$p_{r+1}q_{r+1} \times ... \times p_sq_s/((p_{r+1}q_{r+1} + \cdots + 1) \times ... \times (p_sq_s + \cdots + 1)) \times p_1q_1 \times p_2q_2 \times ... \times p_rq_r(p_1q_1-h_1 + \cdots + 1) \times ... \times (p_rq_r-h_r + \cdots + 1)(p^n-h + \cdots + 1)$$

When $h_k < 0 \quad (1 \leq k \leq r)$, being multiplied by $p_k^{-h_k}$ so that both sides become integers. When $\Pi_{k=r+1}^b(A_k/B_k)$ is not an integer, if both sides are divided by the prime numbers from $p_{r+1}$ to $p_s$, at least one prime number among the prime numbers from $p_{r+1}$ to $p_s$ is left on the left side. Since $c$ and $p^n + \cdots + 1$ are products of prime numbers from $p_1$ to $p_r$ and in the case of $s > r + 1$ the left side has some prime numbers that are not on the right side as factors, this expression does not hold.

In the case of $s = r + 1$, letting $w (1 \leq w \leq s)$ be an integer, when $p \neq p_w$, this expression does not hold similarly. From the above, if $s > r + 1$ or $p \neq p_w$, the expression (C) does not hold because $G_r$ must be represented uniquely. When $s = r + 1$ if $p_x = p_w$, in the expression (D) as $x = s$,

$$A_s(p_x^{n_x} + \cdots + 1)p^n = B_s(p^n + \cdots + 1)p_x^{n_x}$$

I . When the value of $p$ does not change

$p$ becomes an odd prime since $p_x$ must be an odd prime satisfying $p_x \equiv 1 \pmod{4}$. Let $p = p_x$,

$$A_s(p^{n_x} + \cdots + 1)p^{n-n_x} = B_s(p^n + \cdots + 1)$$

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When \( n - n_x > 0 \), since \( B_s \) must be a power of \( p \), so if \( A_s = p^{n_x} + \cdots + 1 \), \( B_s = p^{n_s} \),
\[(p^{n_s} + \cdots + 1)(p^{n_x} + \cdots + 1)p^{n-n_x} = p^{n_s}(p^n + \cdots + 1)\]
When \( n - n_x = n_s \),
\[(p^{n_s} + \cdots + 1)(p^{n_x} + \cdots + 1) = p^n + \cdots + 1\]
It becomes unsuitable since the left side is obviously greater than the right side.
When \( n - n_x \neq n_s \), when \( p \) is transposed a contradiction arises since there is a prime factor \( p \) on either side.

When \( n - n_x \leq 0 \), if \( A_s = p_s^{n_s} + \cdots + 1 \), \( B_s = p_s^{n_x} \),
\[(p_s^{n_x} + \cdots + 1)/p_s^{n_s} \times (p^{n_x} + \cdots + 1)/p^{n_s} = (p^n + \cdots + 1)/p^n \quad \text{(E1)}\]
If \( f(n) = (p^n + \cdots + 1)/p^n = (p - p^{-n})/(p - 1) \),
\[\partial f(n)/\partial n = p^{-n}\log(p)/(p - 1)\]
Therefore, since \( \partial f(n)/\partial n > 0 \) holds when \( p > 1 \) in the domain of \( n \geq 1 \), \( f(n) \) is a monotonically increasing function of \( n \) in this domain. Since \( n_x \geq n \),
\[(p^{n_x} + \cdots + 1)/p^{n_x} \geq (p^n + \cdots + 1)/p^n \]
is established. In addition to this, since
\[(p_s^{n_x} + \cdots + 1)/p_s^{n_x} > 1\]
holds, the expression (E1) does not hold.

II. When the value of \( p \) change
When \( p = 1 \), \( A_s(p_x^{n_x} + \cdots + 1) = (n + 1)B_xp_x^{n_x} \)
Only when \( n = 1 \) this expression is correspond to the expression (A) where \( r = 1 \) so there are no solutions.

When \( p > 1 \), since \( p \neq p_x \) and \( B_s \) must be a power of \( p \), so if \( A_s = p^{n_x} + \cdots + 1 \), \( B_s = p^{n_s} \),
\[(p^{q_x} + \cdots + 1)(p_x^{n_x} + \cdots + 1)p^{n-q_x} = (p^n + \cdots + 1)p_x^{n_x} \quad \text{(E2)}\]

When \( n - q_x > 0 \), if \( p \) is a prime, this expression does not hold since there is no prime factor \( p \) on the right side. If \( p \) is a composite number, it is conceivable that \( p = p_x^{n'} \) where \( n' \) is an odd integer greater than 1. At this time,
\[(p^{q_x} + \cdots + 1)(p_x^{n_x} + \cdots + 1)p_x^{n'(n-q_x)} = (p^{n} + \cdots + 1)p_x^{n_x} \]
When \( n_x \neq n'(n - q_s) \), since there is a prime factor \( p_x \) on either side and a contradiction arises when \( p_x \) is transposed, \( n_x = n'(n - q_s) \) must be hold. Thereby,

\[
(p^{q_s} + \cdots + 1)(p_x^{n_x} + \cdots + 1) = p^n + \cdots + 1
\]

\[
(p^{q_s} + \cdots + 1)(p^{n-q_s} + \cdots + 1) = p^n + \cdots + 1
\]

is established. However, it becomes contradiction since the left side is plainly larger than the right side.

\[
(p^{q_s} + \cdots + 1)/p^{q_s} \times (p_x^{n_x} + \cdots + 1)/p_x^{n_x} = (p^n + \cdots + 1)/p^n
\]

When \( n - q_s < 0 \), similar to the proof of I, since \( f(n) \) is a monotonically increasing function of \( n \) in \( n \geq 1 \) and \( q_s > n \) when \( p > 1 \),

\[
(p^{q_s} + \cdots + 1)/p^{q_s} > (p^n + \cdots + 1)/p^n
\]

holds. In addition to this, when \( n_x \geq 1 \) since

\[
(p_x^{n_x} + \cdots + 1)/p_x^{n_x} > 1
\]

holds, the expression (E2) does not hold. According to I and II above, when \( \prod_{k=r+1}^{x}(A_k/B_k) \) is not an integer and \( s = r + 1 \) holds, the expression (D) does not hold.

When one point \((a,b,p,n)\) is \((1,1,1,1)\), since \( r = 0 \), that \( \prod_{k=r+1}^{x}(A_k/B_k) \) is not an integer is same that the condition (C) holds. If the condition (C) holds, when \( s > r + 1 \), \( G_s \neq G_r \) holds similarly. When \( s = r + 1 = 1 \), there are no solutions because the target nodes are in a layer of the distance 1 directly connected to the root of the graph. Therefore, if the condition (C) holds, except for \((a,b,p,n) = (1,1,1,1)\) there are no solutions satisfying the equation (A).

If the condition (C) does not hold, \( v = a/b \) when \( s = r \). Because the equation (A) must be satisfied at a point other than the point \((a,b,p,n) = (1,1,1,1)\), considering \( v \) becomes an integer,

\[
v = a/b = 2p^n/(p^n + \cdots + 1)
\]

\[
2p^n = v(p^n + \cdots + 1)
\]

Let \( z \) be an integer and if \( v = zp^n \) holds,

\[
2 = z(p^n + \cdots + 1)
\]

When \( p \equiv 1 \pmod{4}, p \geq 5 \) and \( n \equiv 1 \pmod{4}, n \geq 1 \),

\[
p^n + \cdots + 1 \geq 6
\]

At this time, it becomes inappropriate since \( z \) is not an integer. Therefore, if the condition (C) does not hold, except for \((a,b,p,n) = (1,1,1,1)\) there are no solutions satisfying the equation (A). From the above, there are no odd perfect numbers.
3. Complement

From the equation ⑤,

\[ 2b^p(p - 1) = a(p^{n+1} - 1) \]
\[ 2 = a(p^{n+1} - 1)/(bp^n(p - 1)) \]
\[ 2 = (p_1^{q_1+1} - 1)(p_2^{q_2+1} - 1) \ldots (p_r^{q_r+1} - 1)(p^{n+1} - 1) / (p_1^{q_1} p_2^{q_2} \ldots p_r^{q_r} p^n(p_1 - 1)(p_2 - 1) \ldots (p_r - 1)(p - 1)) \]
\[ 2(p_1^{q_1+1} - p_1^{q_1})(p_2^{q_2+1} - p_2^{q_2}) \ldots (p_r^{q_r+1} - p_r^{q_r})(p^{n+1} - p^n) \]
\[ = (p_1^{q_1+1} - 1)(p_2^{q_2+1} - 1) \ldots (p_r^{q_r+1} - 1)(p^{n+1} - 1) \]

We consider when \( r = 2 \).

\[ (p_1^{q_1+1} - 1)(p_2^{q_2+1} - 1)(p^{n+1} - 1) = 2(p_1^{q_1+1} - p_1^{q_1})(p_2^{q_2+1} - p_2^{q_2})(p^{n+1} - p^n) \]

Let \( s, t, u \) be integers,

\[ s = p_1^{q_1+1} - 1 \]
\[ t = p_2^{q_2+1} - 1 \]
\[ u = p^{n+1} - 1 \]

are.

\[ stu = 2(p_1^{q_1+1} - 1 - (p_1^{q_1} - 1))(p_2^{q_2+1} - 1 - (p_2^{q_2} - 1))(p^{n+1} - 1 - (p^n - 1)) \]
\[ stu = 2(s - (s + 1)/p_1 + 1)(t - (t + 1)/p_2 + 1)(u - (u + 1)/p + 1) \]
\[ pp_1p_2stu = 2((s + 1)p_1 - (s + 1))(t + 1)/p_2 + (t + 1))(u + 1)p + (u + 1)) \]
\[ pp_1p_2stu = 2(s + 1)(p_1 - 1)(p_2 - 1)(p - 1) \]
\[ stu/(s + 1)(t + 1)(u + 1) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1p_2p) \]

Since \( stu/(s + 1)(t + 1)(u + 1) \) is a monotonically increasing function for variables \( s, t \) and \( u \), if

\[ s \geq 3^{2+1} - 1 = 26, \ p_1 = 3, \ q_1 = 2 \]
\[ t \geq 7^{2+1} - 1 = 342, \ p_2 = 7, \ q_2 = 2 \]
\[ u \geq 5^2 - 1 = 24, \ p = 5, \ n = 1 \]

holds,

\[ stu/(s + 1)(t + 1)(u + 1) \geq 26 \times 342 \times 24/(27 \times 343 \times 25) = 7904/8575 \]
\[ 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1p_2p) = 2 \times 2 \times 6 \times 4/(3 \times 7 \times 5) = 32/35 \]

Since \( stu/(s + 1)(t + 1)(u + 1) \) is limited to 1 when \( s, t \) and \( u \) are infinite, \( stu/(s + 1)(t + 1)(u + 1) < 1 \)
If \( f(p_1, p_2, p) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p) \) holds, it is sufficient to consider a combination where \( f(p_1, p_2, p) < 1 \).

\[
f(3,7,5) = 2 \times 2 \times 6 \times 4/(3 \times 7 \times 5) = 32/35
\]

\[
f(3,11,5) = 2 \times 2 \times 10 \times 4/(3 \times 11 \times 5) = 32/33
\]

\[
f(3,13,5) = 2 \times 2 \times 12 \times 4/(3 \times 13 \times 5) = 64/65
\]

\[
f(3,17,5) = 2 \times 2 \times 16 \times 4/(3 \times 17 \times 5) = 256/255
\]

\[
f(3,7,13) = 2 \times 2 \times 6 \times 12/(3 \times 7 \times 13) = 96/91
\]

\[
f(3,5,17) = 2 \times 2 \times 4 \times 16/(3 \times 5 \times 17) = 256/255
\]

From the above, when \( r = 2 \), combinations \((p_1, p_2, p) = (3,7,5),(3,11,5),(3,13,5)\) can be considered.

Let \( q_k \) be 2 and \( n = 1 \), if \( g(p_1, p_2, p) = (p_1^3 - 1)(p_2^3 - 1)(p^2 - 1)/(p_1^3 p_2^3 p^2) \),

\[
g(3,7,5) = 26 \times 342 \times 24/(3^3 7^3 5^2) = 9704/8575 > 32/35
\]

\[
g(3,11,5) = 26 \times 1330 \times 24/(3^3 11^3 5^2) = 55328/59896
\]

\[
g(3,13,5) = 26 \times 2196 \times 24/(3^3 13^3 5^2) = 3904/4225
\]

Since the function \( g \) is the minimum in the case of \( q_k = 2 \) and \( n = 1 \), there is no solution \( q_k \) and \( n \) when \( g > f \), so the case of \((p_1, p_2, p) = (3,7,5)\) becomes unsuitable.

\[
stu/(s + 1)(t + 1)(u + 1)) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)
\]

\[
(p_1^{q_1+1} - 1) (p_2^{q_2+1} - 1)(p^{n+1} - 1)/(p_1^{q_1+1} p_2^{q_2+1} p^{n+1})
\]

\[
= 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)
\]

If \( F(p_1, p_2, p) = (p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p) \),

\[
F(p_1^{q_1+1} p_2^{q_2+1} p^{n+1}) = 2F(p_1, p_2, p)
\]
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5. References
   Hiroyuki Kojima "The world is made of prime numbers" Kadokawa Shoten, 2017
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