Proof that there are no odd perfect numbers

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Abstract
If $y$ is an odd perfect number, let $p$ be one of the prime factors of $y$, the exponent of $p$ be an integer $n (n \geq 1)$, the prime factors other than $p$ and different from each other be $p_1, p_2, \cdots, p_r, \cdots, p_s$ and the even exponent of $p_k$ be $q_k$.

$$y/p^n = (1 + p + p^2 + \cdots + p^n) \prod_{k=1}^{r}(1 + p_k + p_k^2 + \cdots + p_k^{q_k})/(2p^n) = \prod_{k=1}^{r} p_k^{q_k}$$

must be satisfied. Let $m$ be a non negative integer and $q$ be a positive integer,

$$n = 4m + 1$$
$$p = 4q + 1$$

Letting $a$ and $b$ be odd integers, satisfying following expressions,

$$a = \prod_{k=1}^{r}(1 + p_k + p_k^2 + \cdots + p_k^{q_k})$$
$$b = \prod_{k=1}^{r} p_k^{q_k}$$
$$a(p^n + \cdots + 1) - 2bp^n = 0$$

is established. This is a known content that has been proven by Euler. Let $s (s \geq r)$ be an integer, $v$ be a rational number,

$$v = \prod_{k=1}^{s}(1 + p_k + p_k^2 + \cdots + p_k^{q_k})/\prod_{k=1}^{s} p_k^{q_k}$$

holds. By the consideration of this research paper, we proved that when $v$ is not an integer, due to the uniqueness of $a(p^n + \cdots + 1)/(bp^n)$, etc there are no solutions that satisfies the equation $a(p^n + \cdots + 1) - 2bp^n = 0$ other than inappropriate solution $(a, b, p, n) = (1, 1, 1, 1)$. Then, because we proved that a contradiction arises when $v$ becomes an integer, we have obtained a conclusion that there are no odd perfect numbers.

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1. Introduction

The perfect number is one in which the sum of the divisors other than itself is the same value as itself and the smallest perfect number is

\[ 1 + 2 + 3 = 6 \]

It is 6. Whether odd perfect numbers exist or not is currently an unsolved problem in mathematics.

2. Proof

Let \( y \) be an odd perfect number, one of the prime factors of \( y \) be an odd prime \( p \) and an exponent of \( p \) be an integer \( n \) \((n \geq 1)\). Let the prime factors other than \( p \) and different from each other be \( p_1, p_2, \ldots, p_r \), \( q_k \) be the index of \( p_k \), and an integer \( a \) be the product of series of prime numbers other than prime \( p \).

\[
a = \prod_{k=1}^{r} (1 + p_k + p_k^2 + \cdots + p_k^{q_k}) \quad \text{①}
\]

The number of terms \( N \) of variable \( a \) is

\[
N = \prod_{k=1}^{r} (q_k + 1) \quad \text{②}
\]

When \( y \) is a perfect number,

\[ y = a(1 + p + p^2 + \cdots + p^n) - y \quad (n > 0) \]

is established.

\[
a \sum_{k=0}^{n} p^k / 2 = y
\]

\[
a \sum_{k=0}^{n} p^k / (2p^n) = y/p^n \quad \text{③}
\]

2.1. When \( q_k \) has at least one odd integer

Letting the number of terms where \( q_k \) is odd be a positive integer \( u \), because

\[ y/p^n = \prod_{k=1}^{r} p_k^{q_k} \]

is odd and the denominator on the left side of the expression ③ has a prime factor 2, from the expression ② variable \( a \) has prime factor 2 more than \( u \) and variable \( a \) is even. Therefore, \( n \) is even and \( u \) is 1 since \( \sum_{k=0}^{n} p^k \) must be an odd integer.
2.2. When all $q_k$ are even integers

$y/p^n$ is odd and the denominator on the left side of the expression $\textcircled{3}$ is even. Since $N$ is odd when $q_k$ are all even integers, variable $a$ is odd. Therefore, $\sum_{k=0}^{n} p^k \equiv 0 \pmod{2}$ is established and $n$ must be an odd integer since $\sum_{k=0}^{n} p^k$ is necessary to include one prime factor 2.

From 2.1, 2.2, in order for odd perfect numbers to exist, only one exponent of the prime factor of $y$ must be an odd integer. We consider the case of 2.2 below.

In order for $y$ to be an odd perfect number, the following expression must be established.

$$y/p^n = (1 + p + p^2 + \cdots + p^n) \prod_{k=1}^{r} (1 + p_k + p_k^2 + \cdots + p_k^{q_k})/(2p^n) = \prod_{k=1}^{r} p_k^{q_k}$$

However, $q_1,q_2,\cdots,q_r$ are all even integers.

Here, let $b$ be an odd integer.

$$b = \prod_{k=1}^{r} p_k^{q_k} \quad \textcircled{4}$$

A following expression is established.

$$y/p^n = a(1 + p + p^2 + \cdots + p^n)/(2p^n) = b$$

$$a(p^{n+1} - 1)/(2(p - 1)p^n) = b$$

$$(a - 2b)p^{n+1} + 2bp^n - a = 0 \quad \textcircled{5}$$

$$(ap - 2bp + 2b)p^n = a$$

Since $ap - 2bp + 2b$ is odd, $a/p^n$ is odd. Let $a/p^n$ be an odd integer $c$.

$$ap - 2bp + 2b = c \quad (c > 0) \quad \textcircled{6}$$

$$(2b - a)p = 2b - c$$

Since variable $a$ is odd, $2b - a$ is odd and $2b - a \neq 0$.

$$p = (2b - c)/(2b - a)$$
Since $n \geq 1$, 
\[ a - c = cp^n - c \geq cp - c > 0 \]
a > c
is.

From the equation (6),
\[ 2b(p - 1) - (ap - c) = 0 \]
\[ 2b - c(p^{n+1} - 1)/(p - 1) = 0 \]
Since $(p^n + \cdots + 1)/2$ is odd, $n = 4m + 1$ must be hold with $m$ as an integer.
\[ 2b(p - 1) = c(p^{n+1} - 1) \]
\[ 2b = c(p^n + \cdots + 1) \]
\[ 2b = c(p + 1)(p^{n-1} + p^{n-3} + \cdots + 1) \ldots (7) \]
Since $b$ is odd when $p + 1$ is not a multiple of 4, $p - 1$ must be a multiple of 4. A positive integer is taken as $q$.
\[ p = 4q + 1 \]
is established. The above conditions have been proved by Euler.

When $p > 1$,
\[ p^n - 1 < p^n \]
\[ (p^n - 1)/(p - 1) < p^n/(p - 1) \]
\[ p^{n-1} + \cdots + 1 < p^n/(p - 1) \]

Since $p$ is an odd prime number satisfying $p = 4q + 1$ and $p \geq 5$, 
\[ p^{n-1} + \cdots + 1 < p^n/4 \]
\[ 2b - a = c(p^n + \cdots + 1) - cp^n = c(p^{n-1} + \cdots + 1) \]
\[ 2b - a < cp^n = a/4 \]
\[ 2b < 5a/4 \]
a > $8b/5$ ... (8)
Let \( a_k \) and \( b_k \) be odd integers and if
\[
a_k = 1 + p_k + p_k^2 + \cdots + p_k^{q_k}, \quad b_k = p_k^{q_k}
\]
\[
a_k - b_k < b_k/(p_k - 1)
\]
\[
a_k < b_k p_k/(p_k - 1)
\]
\[
a = \prod_{k=1}^{r} a_k < \prod_{k=1}^{r} b_k p_k/(p_k - 1) = b \prod_{k=1}^{r} p_k/(p_k - 1)
\]
\[
a/b < \prod_{k=1}^{r} p_k/(p_k - 1)
\]

When \( r = 1 \), since \( a/b < 3/2 \) is established, it becomes inappropriate contrary to inequality \( \circ \).

From the expression \( \mathfrak{7} \),
\[
b = c(p + 1)/2 \times (p^{n-1} + p^{n-3} + \cdots + 1)
\]
holds. Since \((p + 1)/2\) is the product of only prime numbers of \( b \), let \( d_k \) be the index.
\[
(p + 1)/2 = \prod_{k=1}^{r} p_k^{d_k}
\]
\[
p = 2 \prod_{k=1}^{r} p_k^{d_k} - 1 \quad \mathfrak{9}
\]

From \( a = cp^n \) and the expression \( \mathfrak{7} \),
\[
2bp^n = a(p^n + \cdots + 1)
\]
\[
a(p^n + \cdots + 1)/(2bp^n) = 1 \quad (A)
\]

When \( r = 1 \),
\[
a = (p_1^{q_1+1} - 1)/(p_k - 1)
\]
\[
b = p_1^{q_1}
\]
The equation \( (A) \) does not hold since there are no odd perfect numbers when \( r = 1 \).
Let $R$ be a rational number.

$R = a\left(p^n + \cdots + 1\right)/(2bp^n)$

Define an operation [multiplication] and an operation [division] as follows.

An operation [multiplication] of $A_i/B_i$ is performed by multiplying $R$ by $A_i/B_i$ and changing $p$ and $n$ in the expression of $R$ to $p'$ and $n'$. If $R$ is changed to $R'$ by this operation,

$$R' = R \times A_i/B_i \times p^n/p'^n' \times (p'^n' + \cdots + 1)/(p^n + \cdots + 1)$$

$p' = 2 \prod_{k=1}^{r} p_k^{d_k} \times p_i^{d_i} - 1$

$0 \leq d_i \leq q_i$

Here, let $i$ be $i > r$.

An operation [division] of $A_j/B_j$ is performed by dividing $R$ by $A_j/B_j$ and changing $p$ and $n$ in the expression of $R$ to $p''$ and $n''$. If $R$ is changed to $R''$ by this operation,

$$R'' = R \times B_j/A_j \times p^n/p''n'' \times (p''n'' + \cdots + 1)/(p^n + \cdots + 1)$$

If $p_j$ is included in the expression (9), $p_j$ is deleted as $d_j = 0$.

$p'' = 2 \prod_{k=1}^{j-1} p_k^{d_k} \times \prod_{k=j+1}^{r} p_k^{d_k} - 1$

Here, $j$ is assumed to satisfy $1 \leq j \leq r$.

Let $A_k$ and $B_k$ to be odd integers.

$A_k = (p_k^{q_k+1} - 1)/(p_k - 1)$

$B_k = p_k^{q_k}$

When the operation [multiplication] of $A_{r+1}/B_{r+1}$ is performed on $R$, there are both cases that $p$ is increased or not changed. When this operation is performed, the rate of change of $R$ is $A_{r+1}p^n(p^n + \cdots + 1)/(B_{r+1}p^n(p^n + \cdots + 1))$, if $p$ after variation is $p'$. If the rate of change of $R$ is 1,

$A_{r+1}p^n(p^n + \cdots + 1)/(B_{r+1}p^n(p^n + \cdots + 1)) = 1$

$A_{r+1}p^n(p^n + \cdots + 1) = B_{r+1}p^n(p^n + \cdots + 1)$

This expression does not hold since the right side is not a multiple of $p$ when $p' > p$ and $A_{r+1} > B_{r+1}$ holds when $p' = p$. Due to this operation, $R$ becomes larger or smaller than the original value since the rate of change of $R$ does not become 1.
Assuming that \( R = 1 \) in some \( r \), letting \( t(t < r) \) be an integer and assuming that \( A_rA_{t+1} \ldots A_r \) is not a multiple of \( p \), \( R \) is performed the operations [division] of \( A_t/B_t, \ldots, A_{t+1}/B_{t+1}, A_t/B_t \). Furthermore, letting \( x(x > t) \) be an integer and \( R \) is performed the operations [multiplication] of \( A'_t/B'_t, A'_{t+1}/B'_{t+1}, \ldots, A_x/B_x \) and it is assumed that finally \( R = 1 \). At this time, assuming that \( n \) is changed when the operations [multiplication] or the operations [division] are performed.

\[
1 \times B_t p^n(p_{t-1}^{r-1} + \cdots + 1)/(A_t p_{t-1}^{r-1}(p^n + \cdots + 1)) \times \cdots B_{t+1} p_{t+2}^{r+2}(p_{t+1}^{r+1} + \cdots + 1)/(A_{t+1} p_{t+1}^{r+1}(p_{t+2}^{r+2} + \cdots + 1)) \times \cdots = 1
\]

\[
B_t B_{t+1} \ldots B_x A'_t A'_{t+1} \ldots A'_{x} p^n(p_x^{n_x} + \cdots + 1)
\]

When \( p_x = p \) and \( n_x < n \), it becomes a contradiction since the right side of above expression does not include the prime factor \( p \).

\[
A_1 \ldots A_r = cp^n
2B_1 \ldots B_r = c(p^n + \cdots + 1)
A_1 \ldots A_x = c'p_x^{n_x}
2B_1 \ldots B_x = c'(p_x^{n_x} + \cdots + 1)
\]

Assume that these equations hold. If these equations are substituted into the expression (B), the expression is established, so no contradiction arises by performing the operations [division].

Let \( s(s \geq r) \) be an integer and \( v \) be a rational number, if

\[
v = \prod_{k=1}^{s} (1 + p_k + p_k^2 + \cdots + p_k^q)/\prod_{k=1}^{s} p_k^q
\]

holds, we define a condition (C) as follows.

\( v \) is not an integer \( \cdots \)(C)

Consider a tree whose vertex is \( (a,b,p,n) = (1,1,1,1) \), and when the operations [multiplication] are performed, it becomes a child node. For example, consider a child node connected to a vertex as follows.

\( (a,b,p,n) = (13,9,5,5) \) as \( p_1 = 3, q_1 = 2 \) and \( d_1 = 1 \)

\( (a,b,p,n) = (13,9,17,9) \) as \( p_1 = 3, q_1 = 2 \) and \( d_1 = 2 \)

\( (a,b,p,n) = (57,49,97,13) \) as \( p_1 = 7, q_1 = 2 \) and \( d_1 = 2 \)
When an operation [multiplication] is performed from a point in a four-dimensional space \((a, b, p, n)\), the prime number of \(B_i\) can take all odd prime numbers, and its exponent \(q_i (q_i \mod 2 \equiv 0, q_i > 0)\), the variable \(d_i (0 \leq d_i \leq q_i)\) in the expression \(\mathcal{O}\), and the index \(n (n \mod 4 \equiv 1, n > 0)\) of \(p\) can assume all possible values. From a node, an operation [multiplication] of \(B_k\) that is not a prime number of \(B_i\) of the operations [multiplication] performed from the vertex to the node are performed. \(p\) does not necessarily have to be a prime number because this proof is not the one using only that \(p\) is a prime number. With these operations, since tree structure branches when an operation [multiplication] of a new prime number is performed, all nodes that can be considered as solutions can be connected from a vertex.

When the operation [multiplication] of which is the prime number of \(B_j\) of the last operation [multiplication] performed up to a certain node is performed, the operation [division] of the \(B_j\) must be performed and then the operation [multiplication] of \(B_j\) must be performed. That is, it returns from the child node to the parent node. By repeating the operations [division], it is possible to finally return to the vertex, so that it is possible to connect any two points in the four-dimensional space that can be considered as solutions by these operations.

Suppose that a tree structure is created by repeating the operations [multiplication] as described above. Here, a case is considered in which a solution exists at a certain point in this four-dimensional space \((a, b, p, n)\) and a solution exists at other points connected by only the operations [multiplication]. If \(R = 1\) holds again when the operations [multiplication] are performed from one point where \(R = 1\),

\[
1 \times A_{r+1}p^n(\text{pr}_1^nr+1\cdots+1)/(B_{r+1}\text{pr}_1^nr+1(p^n\cdots+1)) \times A_{r+2}\text{pr}_1^nr+2\cdots+1)/(B_{r+2}\text{pr}_1^nr+2(p^n\cdots+1)) \times \cdots \times A_{x}\text{px}_1^nx+1\cdots+1) = 1
\]

\[
A_{r+1}A_{r+2}...A_x/(B_{r+1}B_{r+2}...B_x) = p^n\text{px}(p^n\cdots+1)/(p^n\text{px}(p^n\cdots+1))
\]

\[
A_1A_2...A_x(p^n\cdots+1)/(B_xB_2...B_x^n) = A_1A_2...A_x(p^n\cdots+1)/(B_xB_2...B_x^n) \quad (D)
\]
Assume that \( G_r = A_1 A_2 ... A_r (p^n + \cdots + 1)/(B_1 B_2 ... B_r p^n) \) holds. Assume that \( q_k \) becomes \( q_k - h_k \) and \( n \) becomes \( n - h \) \( (n - h \geq 0) \) by changing \( q_k \) and \( n \) with respect to \( G_r \). Here, \( h_k \) is an even integer and \( h \) is an integer to be a multiple of 4 or be \( n \). Then, \( G_s \) is a function obtained by performing the operations [multiplication] \( s - r \) \( (s > r) \) times. If \( h \) equals \( n \), since it means that \( p \) has been deleted, the operations [multiplication] are performed so that \( p \) becomes a new value. \( G_s \) represents all points \((a, b, p, n)\) that can be solutions. At this time, if there is no change at \( G_s = G_r \),
\[
G_s/G_r = p_{r+1}^{q_{r+1}} \times ... \times p_s^{q_s}/((p_{r+1}^{q_{r+1}} + \cdots + 1) \times ... \times (p_s^{q_s} + \cdots + 1)) \times p_1^{q_1} \times p_2^{q_2} \times ... \times p_r^{q_r} p^n (p_1^{q_1-h_1} + \cdots + 1) ... (p_r^{q_r-h_r} + \cdots + 1)(p^{n-h} + \cdots + 1)/(p_1^{q_1-h_1} \times ... \times p_r^{q_r-h_r}p^{n-h}(p_1^{q_1} + ... + 1) ... (p_r^{q_r} + ... + 1)(p^n + ... + 1)) = 1
\]
\[
p_{r+1}^{q_{r+1}} \times ... \times p_s^{q_s} p_1^{q_1} \times ... \times p_r^{h_r}(p_1^{q_1-h_1} + \cdots + 1) ... (p_r^{q_r-h_r} + \cdots + 1)(p^n + ... + 1) = (p_1^{q_1} + ... + 1) ... (p_r^{q_r} + ... + 1)(p^n + ... + 1)(p_{r+1}^{q_{r+1}} + ... + 1) ... (p_s^{q_s} + ... + 1)
\]
Since \( \prod_{k=1}^r A_k = c p^n \) holds,
\[
p_{r+1}^{q_{r+1}} \times ... \times p_s^{q_s} p_1^{q_1} \times ... \times p_r^{h_r}(p_1^{q_1-h_1} + \cdots + 1) ... (p_r^{q_r-h_r} + \cdots + 1)(p^n + ... + 1)
\]
\[
= c p^{n-h}(p^n + ... + 1)(p_{r+1}^{q_{r+1}} + ... + 1) ... (p_s^{q_s} + ... + 1)
\]
When \( h_k < 0 \) \( (1 \leq k \leq r) \), being multiplied by \( p_k^{-h_k} \) so that both sides become integers. When \( \prod_{k=r+1}^r (A_k/B_k) \) is not an integer, if both sides are divided by the prime numbers from \( p_{r+1} \) to \( p_s \), at least one prime number among the prime numbers from \( p_{r+1} \) to \( p_s \) is left on the left side. Since \( c \) and \( p^n + ... + 1 \) are products of prime numbers from \( p_1 \) to \( p_r \) and in the case of \( s > r + 1 \) the left side has some prime numbers that are not on the right side as factors, this expression does not hold.

In the case of \( s = r + 1 \), letting \( w \) \( (1 \leq w \leq s) \) be an integer, when \( p \neq p_w \), this expression does not hold similarly. From the above, if \( s > r + 1 \) or \( p \neq p_w \), the expression (C) does not hold because \( G_r \) must be represented uniquely. When \( s = r + 1 \) and the value of \( p \) is changed from the value of one point and \( p = p_w \), in the expression (D) as \( x = s \) and \( p = p_x \),
\[
A_s (p^n x + ... + 1) p^{n-n_x} = B_s (p^n + ... + 1)
\]
Ⅰ. When $n - n_x > 0$

When $B_s$ is not a power of $p$, this expression does not hold because the right side is not a multiple of $p$.

When $B_s$ is a power of $p$, if $A_s = p^{q_s} + \cdots + 1$ and $B_s = p^{q_s}$,

$$(p^{q_s} + \cdots + 1)(p^{n_x} + \cdots + 1)p^{n-q} = (p^n + \cdots + 1)p^{n_x} \ldots \text{(E1)}$$

When $n - n_x = q_s$,

$$(p^{n-n_x} + \cdots + 1)(p^{n_x} + \cdots + 1) = p^n + \cdots + 1$$

is established. This expression does not hold since the left side is larger than the right side obviously.

When $n - n_x \neq q_s$, the expression (E1) is inconsistent because the prime factor $p$ exists only on the left or right side.

From the above, the expression (E1) does not hold when $n - n_x > 0$.

Ⅱ. When $n - n_x \leq 0$

If $A_s = p^{q_s} + \cdots + 1$ and $B_s = p^{q_s}$,

$$(p^{q_s} + \cdots + 1)/p^{q_s} \times (p^{n_x} + \cdots + 1)/p^{n_x} = (p^n + \cdots + 1)/p^n \ldots \text{(E2)}$$

When $n - n_x \leq 0$ and $p > 1$, if

$$f(n) = (p^n + \cdots + 1)/p^n = (p - p^{-n})/(p - 1)$$

$$\partial f(n)/\partial n = p^{-n\log(p)}/(p - 1)$$

Therefore, since $\partial f(n)/\partial n > 0$ holds in the domain of $n \geq 1$, $f(n)$ is a monotonically increasing function of $n$ in this domain. Since $n_x \geq n$,

$$(p^{n_x} + \cdots + 1)/p^{n_x} \geq (p^n + \cdots + 1)/p^n$$

is established. In addition to this, when $q_s \geq 2$ since

$$(p^{q_s} + \cdots + 1)/p^{q_s} > 1$$

holds, the expression (E2) does not hold.

From the above I and II, when $s = r + 1$ and $\prod_{k=r+1}^{s} (A_k/B_k)$ is not an integer, the expression (D) does not hold.

When one point $(a, b, p, n)$ is $(1,1,1,1)$, since $r = 0$, that $\prod_{k=r+1}^{s} (A_k/B_k)$ is not an integer is same that the condition (C) holds. If the condition (C) holds, when $s > r + 1$, $G_s \neq G_r$ holds similarly. When $s = r + 1 = 1$, there are no solutions because the target nodes are the second rank connected to the vertex of the tree. Therefore, if the condition (C) holds, except for $(a, b, p, n) = (1,1,1,1)$ there are no solutions satisfying the equation (A).
If the condition (C) does not hold, \( v = \frac{a}{b} \) when \( s = r \). Because the equation (A) must be satisfied at a point other than the point \((a, b, p, n) = (1, 1, 1, 1)\), considering \( v \) becomes an integer,
\[
v = \frac{a}{b} = \frac{2p^n}{p^n + \cdots + 1}
\]
\[
2p^n = v(p^n + \cdots + 1)
\]
Let \( z \) be an integer and if \( v = zp^n \) holds,
\[
2 = z(p^n + \cdots + 1)
\]
When \( p \equiv 1 \pmod{4}, \ p \geq 5 \) and \( n \equiv 1 \pmod{4}, \ n \geq 1, \)
\[
p^n + \cdots + 1 \geq 6
\]
At this time, it becomes inappropriate since \( z \) is not an integer. Therefore, if the condition (C) does not hold, except for \((a, b, p, n) = (1, 1, 1, 1)\) there are no solutions satisfying the equation (A). From the above, there are no odd perfect numbers.
3. Complement

From the equation (5),

\[ 2bp^n(p - 1) = a(p^{n+1} - 1) \]
\[ 2 = a(p^{n+1} - 1)/(bp^n(p - 1)) \]
\[ 2 = (p_1^{q_1+1} - 1)(p_2^{q_2+1} - 1) \ldots (p_r^{q_r+1} - 1)(p^{n+1} - 1) \]
\[ / (p_1^{q_1}p_2^{q_2} \ldots p_r^{q_r}p^n(p_1 - 1)(p_2 - 1) \ldots (p_r - 1)(p - 1)) \]
\[ 2(p_1^{q_1+1} - p_1^{q_1})(p_2^{q_2+1} - p_2^{q_2}) \ldots (p_r^{q_r+1} - p_r^{q_r})(p^{n+1} - p^n) \]
\[ = (p_1^{q_1+1} - 1)(p_2^{q_2+1} - 1) \ldots (p_r^{q_r+1} - 1)(p^{n+1} - 1) \]

We consider when \( r = 2 \).

\( (p_1^{q_1+1} - 1)(p_2^{q_2+1} - 1)(p^{n+1} - 1) = 2(p_1^{q_1+1} - p_1^{q_1})(p_2^{q_2+1} - p_2^{q_2})(p^{n+1} - p^n) \)

Let \( s, t, u \) be integers,

\[ s = p_1^{q_1+1} - 1 \]
\[ t = p_2^{q_2+1} - 1 \]
\[ u = p^{n+1} - 1 \]

are.

\[ stu = 2(p_1^{q_1+1} - 1 - (p_1^{q_1} - 1))(p_2^{q_2+1} - 1 - (p_2^{q_2} - 1))(p^{n+1} - 1 - (p^n - 1)) \]
\[ stu = 2(s - (s + 1)/p_1 + 1)(t - (t + 1)/p_2 + 1)(u - (u + 1)/p + 1) \]
\[ pp_1p_2stu = 2((s + 1)p_1 - (s + 1))(t + 1)p_2 + (t + 1)((u + 1)p + (u + 1)) \]
\[ pp_1p_2stu = 2(s + 1)(p_1 - 1)(t + 1)(p_2 - 1)(u + 1)(p - 1) \]
\[ stu/((s + 1)(t + 1)(u + 1)) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1p_2p) \]

Since \( stu/((s + 1)(t + 1)(u + 1)) \) is a monotonically increasing function for variables \( s, t \) and \( u \), if

\[ s \geq 3^{2+1} - 1 = 26, \ p_1 = 3, \ q_1 = 2 \]
\[ t \geq 7^{2+1} - 1 = 342, \ p_2 = 7, \ q_2 = 2 \]
\[ u \geq 5^2 - 1 = 24, \ p = 5, \ n = 1 \]

holds,

\[ stu/((s + 1)(t + 1)(u + 1)) \geq 26 \times 342 \times 24/(27 \times 343 \times 25) = 7904/8575 \]
\[ 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1p_2p) = 2 \times 2 \times 6 \times 4/(3 \times 7 \times 5) = 32/35 \]

Since \( stu/((s + 1)(t + 1)(u + 1)) \) is limited to 1 when \( s, t \) and \( u \) are infinite,

\[ stu/((s + 1)(t + 1)(u + 1)) < 1 \]
If \( f(p_1, p_2, p) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p) \) holds, it is sufficient to consider a combination where \( f(p_1, p_2, p) < 1 \).

\[
\begin{align*}
  f(3,7,5) &= 2 \times 2 \times 6 \times 4/(3 \times 7 \times 5) = 32/35 \\
  f(3,11,5) &= 2 \times 2 \times 10 \times 4/(3 \times 11 \times 5) = 32/33 \\
  f(3,13,5) &= 2 \times 2 \times 12 \times 4/(3 \times 13 \times 5) = 64/65 \\
  f(3,17,5) &= 2 \times 2 \times 16 \times 4/(3 \times 17 \times 5) = 256/255 \\
  f(3,7,13) &= 2 \times 2 \times 6 \times 12/(3 \times 7 \times 13) = 96/91 \\
  f(3,5,17) &= 2 \times 2 \times 4 \times 16/(3 \times 5 \times 17) = 256/255 \\
\end{align*}
\]

From the above, when \( r = 2 \), combinations \((p_1, p_2, p) = (3,7,5), (3,11,5), (3,13,5)\) can be considered.

Let \( q_k \) be 2 and \( n = 1 \), if \( g(p_1, p_2, p) = (p_1^3 - 1)(p_2^3 - 1)(p^2 - 1)/(p_1^3 p_2^3 p^2) \),

\[
\begin{align*}
  g(3,7,5) &= 26 \times 342 \times 24/(3^3 7^3 5^2) = 7904/8575 > 32/35 \\
  g(3,11,5) &= 26 \times 1330 \times 24/(3^3 11^3 5^2) = 55328/59895 \\
  g(3,13,5) &= 26 \times 2196 \times 24/(3^3 13^3 5^2) = 3904/4225 \\
\end{align*}
\]

Since the function \( g \) is the minimum in the case of \( q_k = 2 \) and \( n = 1 \), there is no solution \( q_k \) and \( n \) when \( g > f \), so the case of \((p_1, p_2, p) = (3,7,5)\) becomes unsuitable.

\[
\begin{align*}
  s t u ((s + 1) (t + 1) (u + 1)) &= 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p) \\
  (p_1^{q_1+1} - 1)(p_2^{q_2+1} - 1)(p^{n+1} - 1)/(p_1^{q_1+1} p_2^{q_2+1} p^{n+1}) &= 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p) \\
\end{align*}
\]

If \( F(p_1, p_2, p) = (p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p) \),

\[
F(p_1^{q_1+1}, p_2^{q_2+1}, p^{n+1}) = 2F(p_1, p_2, p)
\]
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5. References

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