Refutation of lifting countable to uncountable arithmetic

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Abstract: In reverse mathematics (RM) we evaluate an axiom which is not tautologous. Because RM derives theorems from axioms, if an axiom is refuted, then its derived theorem is also refuted, hence refuting RM itself and derived conjectures such as “uplifting” countable mathematics to uncountable mathematics. These results form a non tautologous fragment of the universal logic $VŁ4$.

We assume the method and apparatus of Meth8/$VŁ4$ with $\top$ as tautology as the designated proof value, $\bot$ as contradiction, $\bot$ as truthity (non-contingency), and $\bot$ as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET $\neg$ Not, $\sim$; $\lor$ Or, $\lor$, $\lor$; $\neg$ Not Or; $\land$ And, $\land$, $\land$, $\land$; $\neg$ Not And;
$\rightarrow$ Imply, greater than, $\rightarrow$, $\rightarrow$, $\rightarrow$, $\rightarrow$; $\not\rightarrow$ Not Imply, less than, $\in$, $\in$, $\in$, $\in$; $\not\rightarrow$ Not Imply, $\not\rightarrow$, $\not\rightarrow$, $\not\rightarrow$;
$\equiv$ Equivalent, $\equiv$, $\equiv$, $\equiv$, $\equiv$; $\not\equiv$ Not Equivalent, $\not\equiv$, $\not\equiv$;
$\%$ possibility, for one or some, $\exists$, $\exists$, $\exists$, $\exists$; $\#$ necessity, for every or all, $\forall$, $\forall$, $\forall$, $\forall$;
$(z=z)$ $\top$ as tautology, $\top$, ordinal 3; $(z\not=z)$ $\bot$ as contradiction, $\bot$, Null, $\bot$, $\bot$;
$(%z>#z)$ $\bot$ as non-contingency, $\Delta$, ordinal 1; $(%z<#z)$ $\bot$ as contingency, $\nabla$, ordinal 2;
$\not(y<x)$ $(x\leq y)$, $(x\leq y)$, $(x\leq y)$, $(A=B)$ $(A\not=B)$.
Note for clarity, we usually distribute quantifiers onto each designated variable.


Abstract. Ordinary, i.e. non-set theoretic, mathematics is generally formalised in second-order arithmetic, a direct descendant of Hilbert-Bernays’ Grundlagen der Mathematik with a language restricted to the countable: only natural numbers and sets thereof are directly given, while uncountable objects are available indirectly via countable representations. For instance, Turing’s computational framework, nowadays called computability theory, and the associated program Reverse Mathematics (essentially) take place in second-order arithmetic. The aim of this paper is to lift some of these ‘countable’ results to uncountable mathematics. In particular, we show that with little modification recursive counterexamples from computability theory and reversals from Reverse Mathematics, second-order/countable as they may be, yield interesting results in higher-order/uncountable mathematics. We shall treat the following topics/theorems: the montone convergence theorem/Specker sequences, compact and closed sets in metric spaces, the Rado selection lemma, the ordering and algebraic closures of fields, and ideals of rings.

1. Introduction Computability theory has its roots in the seminal work of Turing, providing an intuitive notion of computation based on what we nowadays call Turing machines ... Now, classical (resp. higher-order) computability theory deals with the computability of sets of natural numbers (resp. higher-order objects). In classical computability theory, a recursive counterexample to a theorem (formulated in an appropriate language) shows that the latter does not hold when restricted to computable sets. An historical overview may be found in the introduction of ...

Recursive counterexamples turn out to be highly useful in the Reverse Mathematics program (RM hereafter ...). Indeed, the aim of RM is to determine the minimal axioms needed to prove a given theorem of ordinary mathematics, often resulting in an equivalence between these axioms and the theorem; recursive counterexamples often (help) establish the ‘reverse’ implication (or: reversal) from the theorem at hand to the minimal axioms. ...

As is well-known, both (classical) RM and recursion theory are essentially restricted to the language of second-order arithmetic, i.e. natural numbers and sets thereof; uncountable objects are
(only) available via countable representations. Historically, second-order arithmetic stems from the logical system \( H \) in the Grundlagen der Mathematik, in which Hilbert and Bernays formalise large swathes of mathematics. It is then a natural, if somewhat outlandish, question whether second-order/countable results can yield any interesting results in higher-order/uncountable RM and computability theory. In this paper, we show that recursive counterexamples and reversals are readily modified to provide interesting implications in higher-order arithmetic. As an example, we show that a well-known recursive counterexample by Ershov about countable fields yields a similar result about uncountable fields. We treat the following theorems/topics: (a) montone convergence theorem/Specker sequences …, (b) compactness of metric space…, (c) closed sets in metric spaces …, (d) Rado selection lemma …, (e) ordering of field …, (f) algebraic closures of fields …, (g) ideals of rings … We emphasise that our ‘lifted’ results are not always optimal; we even provide a counterexample … We stress that our aim is to show that with little modification recursive counterexamples and reversals, second-order/countable as they may be, also establish results in higher-order/uncountable mathematics. One readily re-obtains the original results by applying the canonical embedding from higher- to second-order arithmetic, called ECF … As a bonus, our results pertaining to metric spaces suggest that the latter can only be reasonably studied in weak systems via countable representations (aka codes) in the language of second-order arithmetic.

As will become clear below, there is no ‘size restriction’ on our results: for instance, we first lift Specker sequences as in item (a) to Specker nets indexed by Baire space …; the same proof then immediately provides a lifting to Specker nets indexed by sets of any cardinality (expressible in the language at hand). Intuitively, our results constitute some kind of empirical version of the Löwenheim-Skolem theorem, which states that models of any cardinality exist.

As to naming, Reverse Mathematics derives its name from the ‘reverse’ way of doing mathematics it embodies, namely to derive theorems from axioms rather than the other way around. … Perhaps the results in this paper should therefore be baptised upwards mathematics, as it pushes theorems upwards in the (finite) type hierarchy.

Finally, it is now an interesting question which other techniques of classical computability theory lift with similar ease into the higher-order world. Similarly, it is a natural question whether one can obtain a meta-theorem that encompasses all the below liftings, i.e. ‘invert the ECF-translation’ as it were. We do not have an answer to these questions.

### 2.2. Some axioms of higher-order RM.

We introduce some functionals which constitute the counterparts of some of the Big Five systems, in higher-order RM. … First of all, ACA\(_0\) is readily derived from:

\[
(\exists \mu^2)(\forall f^1)(\exists n)(f(n)=0) \rightarrow [(f(\mu(f))=0) \land (\forall i<\mu(f)f(i)\neq 0) \land [(\forall n)(f(n)\neq 0) \rightarrow \mu(f)=0]] (\mu^2.1)
\]

\[
\text{LET } p, \ q, \ r, \ s, \ t, \ u \\
\text{f, f}^1, \ i, \ n, \ \mu, \ \mu^2
\]

\[
(\%u\&\%q\&((p\&\%s)=(s@s))>((((p&@t\&p))=(s@s)) \\
\&((\#r<(t\&p))&((p\&\#r)@(s@s)))) \\
&(((\#s&(p&s))@(s@s))>((t&p)=(s@s))) ;
\]

\[
\begin{array}{cccccccc}
TTTT & TTTT & TTTT & TTTT \\
TTTT & TTTT & TTTT & TTTT \\
TTCC & TTCT & TTCT & TTCT \\
TTCC & TTCT & TTCT & TTCT
\end{array} (\mu^2.2)
\]

**Remark \(\mu^2.2\):** Eq. \(\mu^2.2\) as rendered is not tautologous, refuting it as an axiom.

Because RM derives theorems from axioms, if an axiom is refuted, then that theorem in RM is refuted, hence refuting RM and derived conjectures such as “uplifting” countable mathematics to uncountable mathematics.