Refutation of the law of excluded middle as a no-go theorem

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Abstract: We evaluate the seminal theorem to reverse mathematics (RM). It is not tautologous, hence refuting RM. By extension the following are also not confirmed: the Tietze extension theorem; Ekeland’s variational principle; and the law of excluded middle as a no-go theorem. These results form a non tautologous fragment of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, F as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ¬ Not, ¬; + Or, ∨, ⊃, ⊔; - Not Or; & And, ∧, ∩, ⊓, ⊠; \ Not And;
> Imply, greater than, →, ⇒, ⊃, ⊖, ⊑; < Not Imply, less than, ∈, ⊂, ⊆, ⊆, ≤;
= Equivalent, ≡, ⊖, ⊖, ⊖, ≈; @ Not Equivalent, ⊖, ⊖;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) T as tautology, T, ordinal 3; (z@z) F as contradiction, Ø, Null, ⊥, zero;
(%z>#z) N as non-contingency, Δ, ordinal 1; (%z<#z) C as contingency, ∇, ordinal 2;
~( y < x) ( x ≤ y), ( x ⊆ y), ( x ∈ y); (A=B) (A~B).
Note for clarity, we usually distribute quantifiers onto each designated variable.


Abstract. The aim of Reverse Mathematics is to find the minimal axioms needed to prove theorems of ordinary, i.e. non-set theoretic, mathematics. This development generally takes place in the language of second-order arithmetic in which uncountable objects are represented via countable ‘codes’. Hence, it is paramount that this ‘coding practise’ of RM does not change the logical strength of a given theorem, lest the wrong minimal axioms be identified.

We show that the coding of (continuous) functions fundamentally distorts the logical strength of Ekeland’s variational principle and Tietze’s extension theorem. The latter are quite elementary compared to e.g. theorems from measure theory or topology where the coding practise may be claimed (or even expected) to be problematic. Another novelty is that ... all previously known problems with coding arise from the class of coded objects being ‘smaller’ than the class of actual objects in weak systems of higher-order arithmetic; by contrast, the problems with coding identified in this paper seem to emerge from the class of coded objects being ‘larger’ than the class of actual objects. Finally, we obtain our results via a so-called no-go theorem that seems to essentially depend on the law of excluded middle.

1. Introduction … In this paper, we establish a ... hitherto unknown problem with regard to coding: we identify two theorems, namely Ekeland’s variational principle and the Tietze extension theorem, for which the coding of (continuous) functions greatly distorts the logical strength of the theorems, and hence the RM classification. It would perhaps not come as a surprise that the coding of topologies or measurable sets has its problems ..., but the aforementioned theorems are quite elementary in comparison.

Next, as to the title of this paper, our proofs seem to be based on the law of excluded middle in an essential way, as is clear from Theorem 1.1 below. Moreover, a ‘no-go theorem’ is a mathematical result that is claimed to imply that a certain state of the (physical) world is impossible. Theorem 1.1
is a ‘mathematical’ no-go theorem, as it implies that dropping a continuity condition cannot increase the logical strength of certain theorems. The proof of this theorem hinges on the law of excluded middle, as is readily apparent.

**Theorem 1.1.** If RCA\(^0\) proves \((\forall f \in C(R))A(f)\) and ACA\(^0\) proves \((\forall f : R \to R)A(f)\), then RCA\(^0\) proves \((\forall f : R \to R)A(f)\).

[The same holds for RCA\(^0\) replaced by RCA\(^0\) + WKL or ‘R \to R’ replaced by ‘[0, 1] \to R’ or ‘\(\mathbb{N}\) \to \(\mathbb{N}\)’.]

\[
\text{LET } p, q, r, s, t, u: \text{ A, C, R, f, ACA\(^0\), RCA\(^0\).}
\]

\[
((u>(#s<(q\&r)\&(p\&s))))&(t>(#s>(r>r))&(p\&s)))>(u>(#s>(r>r))&(p\&s)) ;
\]

\[
\text{ Remark 1.1.2: Eq. 1.1.2 as rendered is not tautologous. This refutes the Th. 1.1.}
\]

It is essential that we explain why Theorem 1.1, trivial as it seems, is interesting at all. As it happens, Theorem 1.1 applies to (various versions of) the Tietze extension theorem and Ekeland’s variational principle, which have been studied in second-order RM . . . As a result, the RM of these theorems will be seen to be very different when working in higher-order rather than second-order arithmetic. . . In some cases, our higher-order versions are provable in a conservative extension of WKL\(^0\). . .[T]his contributes to a ‘very important direction of research’, namely a partial realisation of Hilbert’s program for the foundations of mathematics.

Because Th.1.1, the first theorem of RM, is refuted, the Tietze extension theorem, Ekeland’s variational principle, and the law of excluded middle as a no-go theorem are also refuted.