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Preprocessing quaternion data in quaternion spaces using the quaternion domain Fourier transform

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***Abstract*—Recently a new type of hypercomplex Fourier transform has been suggested. It consequently transforms quaternion valued signals (for example electromagnetic scalar-vector potentials, color data, space-time data, etc.) defined over a quaternion domain (space-time or other 4D domains) from a quaternion "position" space to a quaternion "frequency" space. Therefore the quaternion domain Fourier transform (QDFT) uses the full potential provided by hypercomplex algebra in higher dimensions, such as 3D and 4D transformation covariance. The QDFT is explained together with its main properties relevant for applications such as quaternionic data preprocessing.**

I. INTRODUCTION

The electromagnetic field equations were originally formulated by J. C. Maxwell [22] in the language of Hamilton's quaternions [12]. Later, among many other applications, quaternions began to play an important role in aerospace engineering [21], color signal processing [10], and in material science for texture analysis [1], [23].

Quaternion Fourier transforms (QFT) are since over 20 years a mathematically well researched and frequently applied subject [5]. Yet interesting enough most publications on QFTs concentrate on transformations for signals with domain $\mathbb{R}^2$. Motivated by private communication with T.L. Saaty related to *quaternion valued functions over the domain of quaternions*, we establish here a genuine Fourier transform with a quaternionic kernel operating on such functions.

This paper follows the treatment in [18] (which should be consulted for further details) and begins by introducing quaternions and their relevant properties, including quaternion domain functions in Section II. The quaternion domain Fourier transform (QDFT) is defined in Section III. Many application relevant properties of the QDFT are investigated in Section IV.

II. DEFINITION AND PROPERTIES OF QUATERNIONS $\mathbb{H}$

A. Basic facts about quaternions

Gauss, Rodrigues and Hamilton's four-dimensional (4D) *quaternion algebra* $\mathbb{H}$ is defined over $\mathbb{R}$ with three imaginary units $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$:

$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k},\ \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i},$$
$$\boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j},\ \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1. \tag{1}$$

Every quaternion can be written explicitly as a linear combination

$$q = q_r + q_i\boldsymbol{i} + q_j\boldsymbol{j} + q_k\boldsymbol{k} \in \mathbb{H}, \quad q_r, q_i, q_j, q_k \in \mathbb{R}, \tag{2}$$

and has a *quaternion conjugate*

$$\tilde{q} = q_r - q_i\boldsymbol{i} - q_j\boldsymbol{j} - q_k\boldsymbol{k}, \qquad \widetilde{pq} = \tilde{q}\tilde{p}. \tag{3}$$

This leads to the *norm* of $q \in \mathbb{H}$

$$|q| = \sqrt{q\tilde{q}} = \sqrt{q_r^2 + q_i^2 + q_j^2 + q_k^2}, \qquad |pq| = |p||q|. \tag{4}$$

The *inverse* of a non-zero quaternion $q \neq \mathbb{H}$ is

$$q^{-1} = \frac{\tilde{q}}{|q|^2}. \tag{5}$$

The *(symmetric) scalar part* of a quaternion is defined as

$$\langle q \rangle_0 = Sc(q) = q_r = \frac{1}{2}(q + \tilde{q}),$$
$$Sc(pq) = Sc(qp) = Sc(\tilde{p}\tilde{q}),$$
$$Sc(pqr) = Sc(qrp) = Sc(rpq). \tag{6}$$

Every quaternion $a \in \mathbb{H}$, $a \neq 0$, can be written as scalar part plus *(pure) vector part*

$$a = a_r + a_i\boldsymbol{i} + a_j\boldsymbol{j} + a_k\boldsymbol{k} = a_r + \mathbf{a}$$
$$= |a|(\cos\alpha + \frac{\mathbf{a}}{|a|}\sin\alpha) = |a|e^{\hat{\mathbf{a}}\alpha}, \tag{7}$$

with $\hat{\mathbf{a}} = \mathbf{a}/|a|$, $\cos\alpha = a_r/|a|$, $\alpha \in [0, \pi)$. A *scalar product* of quaternions can be defined for $x, y \in \mathbb{H}$ as

$$x \cdot y = Sc(\tilde{x}y) = x_r y_r + x_i y_i + x_j y_j + x_k y_k. \tag{8}$$

Two quaternions interpreted as elements of $\mathbb{R}^4$ are defined to be *orthogonal*, if and only if their scalar product is zero

$$x \perp y \quad \Leftrightarrow \quad x \cdot y = 0. \tag{9}$$

Pure quaternions have zero scalar part. A *(normed) unit pure quaternion* $\mathbf{q}$ squares to -1

$$\mathbf{q}^2 = -(q_i^2 + q_j^2 + q_k^2) = -1. \tag{10}$$

The set of unit pure quaternions is isomorphic to the unit sphere $S^2 \subset \mathbb{R}^3$.

If we interpret the four real coefficients of $x \in \mathbb{H}$, $x_r, x_i, x_j, x_k \in \mathbb{R}$ as *coordinates* in $\mathbb{R}^4$, with *infinitesimal* *volume element* $d^4x = dx_r dx_i dx_j dx_k$, then the substitution $z = ax$, $a \in \mathbb{H}$, yields [18]

$$z = ax \quad \Rightarrow \quad d^4z = |a|^4 d^4x, \qquad d^4x = |a|^{-4} d^4z, \quad (11)$$

assuming $a \neq 0$ for the last identity.

For the transformation $z = axb$, $a, b, x \in \mathbb{H}$, we obtain [18]

$$d^4z = |a|^4|b|^4 d^4x = |ab|^4 d^4x. \quad (12)$$

As expected the *rotation* (19) does not change the infinitesimal volume element

$$z = axa^{-1} \quad \Rightarrow \quad d^4x = |aa^{-1}|^4 d^4x = d^4x. \quad (13)$$

We follow [19] in defining the following *derivative operators*

$$\tilde{\partial} = \partial_{x_r} + \partial_{x_i}\boldsymbol{i} + \partial_{x_j}\boldsymbol{j} + \partial_{x_k}\boldsymbol{k}, \quad (14)$$
$$\partial = \partial_{x_r} - \partial_{x_i}\boldsymbol{i} - \partial_{x_j}\boldsymbol{j} - \partial_{x_k}\boldsymbol{k}, \quad (15)$$

where $\partial_{x_r} = \partial/\partial x_r$, etc. are scalar partial derivatives. We further define the three-dimensional *Dirac operator*

$$D = \tilde{\partial} - \partial_{x_r} = \partial_{x_i}\boldsymbol{i} + \partial_{x_j}\boldsymbol{j} + \partial_{x_k}\boldsymbol{k}, \qquad \tilde{\partial} = \partial_{x_r} + D. \quad (16)$$

The *orthogonal planes split* of $q \in \mathbb{H}$ with pure unit quaternion $f = g = I$, $I \in \mathbb{H}$, $I^2 = -1$, [14], [16], [17] is defined as

$$q_\pm = \frac{1}{2}(q \pm IqI), \qquad q_- = q_r + q_I I,$$
$$q_+ = q_J J + q_K K = (q_J + q_K I)J, \quad (17)$$

with *rotation operator* $R = (\boldsymbol{i} + I)\boldsymbol{i}$, $J = R\boldsymbol{j}R^{-1}$ and $K = R\boldsymbol{k}R^{-1}$, $J^2 = K^2 = -1$, $q_r, q_I, q_J, q_K \in \mathbb{R}$, similar to [17]. Note, that there is a *gauge freedom* in this split by changing $R \to R\exp(I\varphi/2)$, $\varphi \in [0, 2\pi)$, i.e. a rotation freedom in the q_+-plane. The units $\{1, I, J, K\}$ form another *equivalent representation* of quaternions $\mathbb{H}$. Note further, that the q_- part *commutes* with I, whereas the q_+ part *anticommutes*

$$q_- I = I q_-, \qquad q_+ I = -I q_+. \quad (18)$$

Note finally, that the split (17) with $I = (\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})/\sqrt{3}$ as grey line direction, yields the conventional split of a color image into luminance and chrominance components [10].

B. Quaternions and reflections and rotations in three and four dimensions

The *geometry of reflections and rotations* in three and four dimensions, expressed in the language of quaternions is discussed in [7], [17], [23]. We give an overview of how important *orthogonal transformations* in three-dimensional and four-dimensional Euclidean space can be expressed by means of quaternions.

A three-dimensional *rotation* of the vector part $\mathbf{x}$ of the quaternion $x \in \mathbb{H}$ by the angle 2α around the axis $\hat{\mathbf{a}}$ (compare eq. (7)), leaving the scalar part x_r invariant, is given by [18]

$$x' = axa^{-1}. \quad (19)$$

We further note, that the transformation

$$x' = axb, \qquad a = e^{\alpha\hat{\mathbf{a}}}, \qquad b = e^{\beta\hat{\mathbf{a}}}, \quad (20)$$

rotates the x_--part by the angle $\alpha + \beta$ in the q_--plane (determined by (17), setting $I = \hat{\mathbf{a}}$), and rotates the x_+-part by $\alpha - \beta$ in the q_+-plane.

The *4D reflection* at the *real line* is given by quaternion conjugation $x \to \tilde{x}$, leaving the real line pointwise invariant.

The *4D reflection* at the *3D hyperplane* of pure quaternions is therefore given by $x \to -\tilde{x}$, leaving the 3D hyperplane of pure quaternions pointwise invariant.

A *reflection* at a (pointwise invariant) *general line* in $\mathbb{R}^4$ in the direction of the unit quaternion $a \in \mathbb{H}$, $|a| = 1$, is given by $x \to a\tilde{x}a$.

A *reflection* at the (pointwise invariant) *three-dimensional hyperplane* orthogonal to the direction in four dimensions specified by the unit quaternion a, $|a| = 1$, is given by $x \to -a\tilde{x}a$.

A *general rotation* in $\mathbb{R}^4$ is given by

$$x \to axb, \qquad a, b \in \mathbb{H}, \qquad |a| = |b| = 1. \quad (21)$$

To understand the *geometry of this rotation* [17], we rewrite the unit quaternions a, b as

$$a = e^{\alpha\hat{\mathbf{a}}}, \qquad b = e^{\beta\hat{\mathbf{b}}}. \quad (22)$$

The pure unit quaternions $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ define two orthogonal two-dimensional rotation planes in $\mathbb{R}^4$, where without restriction of generality we assume $\hat{\mathbf{a}} \neq \hat{\mathbf{b}}$, because the case $\hat{\mathbf{a}} = \hat{\mathbf{b}}$ has already been discussed in (20). The $q_+^{a,b}$ plane with orthogonal basis and projection

$$q_+^{a,b}\text{basis}: \{\hat{\mathbf{a}} - \hat{\mathbf{b}}, 1 + \hat{\mathbf{a}}\hat{\mathbf{b}}\}, \qquad q_+^{a,b} = \frac{1}{2}(q + \hat{\mathbf{a}}q\hat{\mathbf{b}}), \quad (23)$$

and the orthogonal $q_-^{a,b}$ plane orthogonal basis and projection

$$q_-^{a,b}\text{basis}: \{\hat{\mathbf{a}} + \hat{\mathbf{b}}, 1 - \hat{\mathbf{a}}\hat{\mathbf{b}}\}, \qquad q_-^{a,b} = \frac{1}{2}(q - \hat{\mathbf{a}}q\hat{\mathbf{b}}), \quad (24)$$

such that $q = q_+^{a,b} + q_-^{a,b}$, for all $q \in \mathbb{H}$. The transformation $x \to axb$ of (21) then means geometrically a rotation by the angle $\alpha - \beta$ in the $q_+^{a,b}$ plane (around the $q_-^{a,b}$ plane as axis) and a rotation by the angle $\alpha + \beta$ in the $q_-^{a,b}$ plane (around the $q_+^{a,b}$ plane as axis). This also tells us, that for $\alpha = \beta$ the rotation degenerates to a single two-dimensional rotation by 2α in the $q_-^{a,b}$ plane, and for $\alpha = -\beta$ it degenerates to a single two-dimensional rotation by 2α in the $q_+^{a,b}$ plane.

A *general rotary reflection* (rotation reflection) in $\mathbb{R}^4$ is given by

$$x \to a\tilde{x}b, \qquad a, b \in \mathbb{H}, \qquad |a| = |b| = 1. \quad (25)$$

This rotary reflection has the pointwise invariant line through $a + b$. In the remaining three-dimensional hyperplane, orthogonal to the $a + b$ line, the axis of the rotary reflection is the line in the direction $a - b$, because $a\widetilde{(a - b)}b = -(a - b)$. The rotation plane of the rotary reflection is spanned by the two orthogonal quaternions $v_{1,2} = [a, b](1 \pm \tilde{a}b)$, $[a, b] = ab - ba$, and the angle of rotation is $\Gamma = \pi - \text{acrccos}(Sc(\tilde{a}b))$, [17].

C. Quaternion domain functions

Every real valued quaternion domain function f maps $\mathbb{H} \to \mathbb{R}$:

$$f : x \mapsto f(x) \in \mathbb{R}, \quad \forall x \in \mathbb{H}. \tag{26}$$

Every *quaternion valued quaternion domain function* f maps $\mathbb{H} \to \mathbb{H}$, its four coefficient functions f_r, f_i, f_j, f_k, are in turn real valued quaternion domain functions:

$$f : x \mapsto f(x) = f_r(x) + f_i(x)\boldsymbol{i} + f_j(x)\boldsymbol{j} + f_k(x)\boldsymbol{k} \in \mathbb{H}. \tag{27}$$

We define for two functions $f, g : \mathbb{H} \to \mathbb{H}$ the following *quaternion valued inner product*

$$(f, g) = \int_{\mathbb{H}} f(x)\tilde{g}(x)d^4x \tag{28}$$

with $d^4x = dx_r dx_i dx_j dx_k \in \mathbb{R}$. Note that quaternion conjugation yields

$$\widetilde{(f, g)} = (g, f). \tag{29}$$

This means that the real scalar part of the *inner product* (f, g) is *symmetric*

$$\langle f, g\rangle = \frac{1}{2}[(f, g) + (g, f)] = \int_{\mathbb{H}} \langle f(x)\tilde{g}(x)\rangle_0 d^4x \in \mathbb{R},$$
$$\langle f, g\rangle = \langle g, f\rangle. \tag{30}$$

We further define the $L^2(\mathbb{H};\mathbb{H})$-*norm*[1] as

$$||f|| = \sqrt{(f, f)} = \sqrt{\langle f, f\rangle} = \sqrt{\int_{\mathbb{H}} |f(x)|^2 d^4x} \geq 0. \tag{31}$$

The *quaternion domain module* $L^2(\mathbb{H};\mathbb{H})$ is the set of all finite $L^2(\mathbb{H};\mathbb{H})$-norm functions

$$L^2(\mathbb{H};\mathbb{H}) = \{f | f : \mathbb{H} \to \mathbb{H}, ||f|| \leq \infty\}. \tag{32}$$

The *convolution* of two functions $f, g \in L^2(\mathbb{H};\mathbb{H})$ is defined as

$$(f * g)(x) = \int_{\mathbb{H}} f(y)g(x - y)d^4y. \tag{33}$$

For unit norm signals $f \in L^2(\mathbb{H};\mathbb{H})$, $||f|| = 1$, we define the *effective spatial width* or spatial uncertainty (or signal width) of f *in the direction* of the unit quaternion $a \in \mathbb{H}$, $|a| = 1$, as the square root of the variance of the energy distribution of f along the a-axis

$$\Delta x_a = ||(x \cdot a)f|| = \sqrt{\int_{\mathbb{H}} (x \cdot a)^2 |f(x)|^2 d^4x}. \tag{34}$$

Also for unit norm signals f, we define the *effective spatial width* (spatial uncertainty) as the square root of the variance of the energy distribution of f

$$\Delta x = ||xf|| = \sqrt{\int_{\mathbb{H}} |x|^2 |f(x)|^2 d^4x}. \tag{35}$$

[1] Note that in equation (13) of [14] the square root is missing over the integral in the definition of the $L^2(\mathbb{R}^2;\mathbb{H})$-norm.

III. The quaternion domain Fourier transform

Since the traditional quaternion Fourier transform (QFT) [9], [11], [14] is only defined for real or quaternion valued signals over the domain $\mathbb{R}^2$, we newly define the *quaternion domain Fourier transform* (QDFT) for $h \in L^1(\mathbb{H};\mathbb{H})$ as

$$\mathcal{F}\{h\}(\omega) = \hat{h}(\omega) = \frac{1}{(2\pi)^2} \int_{\mathbb{H}} h(x)e^{-Ix\cdot\omega} d^4x, \tag{36}$$

with $x, \omega \in \mathbb{H}$, $d^4x = dx_r dx_i dx_j dx_k \in \mathbb{R}$, and some constant $I \in \mathbb{H}$, $I^2 = -1$. The constant unit pure quaternion I can be chosen specific for each problem.

Note that the QDFT of (36) is *steerable* due to the free choice of the unit pure quaternion unit $I \in S^2$.

This QDFT definition is *left linear*

$$\mathcal{F}\{\alpha h + \beta g\}(\omega) = \alpha\hat{h}(\omega) + \beta\hat{g}(\omega), \tag{37}$$

for $g, h \in L^1(\mathbb{H};\mathbb{H})$ and constants $\alpha, \beta \in \mathbb{H}$.

Applying the orthogonal planes split (17) to the signal function $h = h_+ + h_-$ and to the QDFT $\hat{h}$ we find

$$\hat{h}(\omega) = \hat{h}_+(\omega) + \hat{h}_-(\omega), \tag{38}$$

$$\hat{h}_+(\omega) = \frac{1}{(2\pi)^2} \int_{\mathbb{H}} h_+(x)e^{-Ix\cdot\omega} d^4x$$
$$= \frac{1}{(2\pi)^2} \int_{\mathbb{H}} e^{+Ix\cdot\omega} h_+(x) d^4x, \tag{39}$$

$$\hat{h}_-(\omega) = \frac{1}{(2\pi)^2} \int_{\mathbb{H}} h_-(x)e^{-Ix\cdot\omega} d^4x$$
$$= \frac{1}{(2\pi)^2} \int_{\mathbb{H}} e^{-Ix\cdot\omega} h_-(x) d^4x. \tag{40}$$

Example. Following the suggestion of T. L. Saaty, we QDFT transform the *functional quaternion equation*

$$h(ax) = bh(x), \quad h : \mathbb{H} \to \mathbb{H}, \tag{41}$$

with quaternion constants $a, b \in \mathbb{H}$. We define the auxiliary function $h_a(x) = h(ax)$ and compute [18]

$$\hat{h}_a(\omega) = |a|^{-4}\hat{h}(\tilde{a}^{-1}\omega). \tag{42}$$

Using relationship (42) and left linearity we arrive at the QDFT of (41)

$$|a|^{-4}\hat{h}(\tilde{a}^{-1}\omega) = b\hat{h}(\omega), \tag{43}$$

or equivalently

$$\hat{h}(\tilde{a}^{-1}\omega) = |a|^4 b\hat{h}(\omega), \tag{44}$$

which seems neither less nor more complicated to solve than the original equation (41).

An application of (42) is the four-dimensional inversion at the origin $x \to -x$ which results in

$$\hat{h}_{-1}(\omega) = \hat{h}(-\omega). \tag{45}$$

The QDFT can *separate* the two components of a "complex" signal $f : \mathbb{H} \to \mathbb{R} + \boldsymbol{i}\mathbb{R}$, $f(x) = f_r(x) + \boldsymbol{i}f_i(x)$, into *even* and *odd components* with respect to the inversion $x \to -x$. Let

$$f(x) = f_r(x) + \boldsymbol{i}f_i(x)$$
$$= f_r^e(x) + f_r^o(x) + \boldsymbol{i}(f_i^e(x) + \boldsymbol{i}f_i^o(x)), \tag{46}$$

with

$$f_r^e(-x) = f_r^e(x) = \frac{1}{2}(f_r(x) + f_r(-x)),$$
$$f_r^o(-x) = -f_r^o(x) = \frac{1}{2}(f_r(x) - f_r(-x)),$$
$$f_i^e(x) = f_i^e(-x), \qquad f_i^o(x) = -f_i^o(x). \tag{47}$$

Then for $I = \boldsymbol{j}$ (we could also set $I = \boldsymbol{k}$ or any other pure quaternion $\perp \boldsymbol{i}$) we have by linearity

$$\hat{f}(\omega) = \hat{f}_r^e(\omega) + \hat{f}_r^o(\omega) + \boldsymbol{i}(\hat{f}_i^e(\omega) + \boldsymbol{i}\hat{f}_i^o(\omega))$$
$$= \int_{\mathbb{H}} f_r^e(x)\cos(x\cdot\omega)d^4\omega + \boldsymbol{j}\int_{\mathbb{H}} f_r^o(x)\sin(x\cdot\omega)d^4\omega$$
$$+\boldsymbol{i}\int_{\mathbb{H}} f_i^e(x)\cos(x\cdot\omega)d^4\omega$$
$$+\boldsymbol{k}\int_{\mathbb{H}} f_i^o(x)\sin(x\cdot\omega)d^4\omega. \tag{48}$$

Compare [20] for a similar approach to the symmetry analysis of signals $f : \mathbb{R} \to \mathbb{C}$.

IV. Properties of the QDFT

Properties of the QDFT that can easily be established are transform *inversion* (Plancherel theorem)

$$h(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{H}} \hat{h}(x)e^{+Ix\cdot\omega}d^4\omega, \tag{49}$$

a *shift theorem* for $g(x) = h(x-a)$, constant $a \in \mathbb{H}$,

$$\hat{g}(\omega) = \hat{h}(\omega)e^{-Ia\cdot\omega}, \tag{50}$$

and a *modulation theorem* for $m(x) = h(x)e^{Ix\cdot\omega_0}$, constant $\omega_0 \in \mathbb{H}$,

$$\hat{m}(\omega) = \hat{h}(\omega - \omega_0). \tag{51}$$

Linear combinations with constant quaternion coefficients $\alpha, \beta \in \mathbb{H}$ from the *right* lead due to (18) to

$$\mathcal{F}\{h\alpha + g\beta\}$$
$$= \hat{h}(\omega)\alpha_+ + \hat{h}(-\omega)\alpha_- + \hat{g}(\omega)\beta_+ + \hat{g}(-\omega)\beta_-. \tag{52}$$

We define $g_l(x) = \partial_{x_l}h(x)$, $l \in \{r, i, j, k\}$ for the *partial derivative* of the signal function h and obtain its QDFT as

$$\widehat{g_l}(\omega) = \hat{h}(\omega)I\omega_l. \tag{53}$$

For example for $l = r$ we obtain

$$\widehat{\partial_{x_r}h}(\omega) = \omega_r\hat{h}(\omega)I. \tag{54}$$

This leads to the QDFT of the *derivative operators*

$$\widehat{\tilde{\partial}^m h}(\omega) = \omega^m\hat{h}(\omega)I^m, \qquad \widehat{\partial^m h}(\omega) = \widetilde{\omega}^m\hat{h}(\omega)I^m,$$
$$m \in \mathbb{N}. \tag{55}$$

Applying the *derivative operators from the right* to the signal function h we further obtain

$$\widehat{h\tilde{\partial}^m}(\omega) = \widehat{h\omega^m}(\omega)I^m, \qquad \widehat{h\partial^m}(\omega) = \widehat{h\widetilde{\omega}^m}(\omega)I^m,$$
$$m \in \mathbb{N}. \tag{56}$$

QDFT transformations of the *Dirac operator* D applied from the left and right, respectively, give

$$\widehat{D^m h}(\omega) = \boldsymbol{\omega}^m\hat{h}(\omega)I^m, \qquad \widehat{hD^m}(\omega) = \widehat{h\boldsymbol{\omega}^m}(\omega)I^m,$$
$$m \in \mathbb{N}. \tag{57}$$

where the pure quaternion part of the quaternion frequency ω is $\boldsymbol{\omega} = \omega - \omega_r$.

The QDFT of m-fold *powers of coordinates* x_l, $l \in \{r, i, j, k\}$, $m \in \mathbb{N}$, times the signal function h leads to (dual to (53))

$$\widehat{x_l^m h}(\omega) = \partial_{\omega_l}^m\hat{h}(\omega)I^m. \tag{58}$$

For example for $l = r$ we obtain

$$\widehat{x_r h}(\omega) = \partial_{\omega_r}\hat{h}(\omega)I. \tag{59}$$

If $P(x_r, x_i, x_j, x_k) = \sum_{m_r,m_i,m_j,m_k}\alpha_{m_r,m_i,m_j,m_k} x_r^{m_r}x_i^{m_i}x_j^{m_j}x_k^{m_k}$, with quaternion coefficients $\alpha_{m_r,m_i,m_j,m_k} \in \mathbb{H}$, is a *polynomial* of the four *coordinates* $\{x_r, x_i, x_j, x_k\}$, then the QDFT yields

$$\mathcal{F}\{P(x_r, x_i, x_j, x_k)h\}(\omega) = \sum_{m_r,m_i,m_j,m_k}$$
$$\alpha_{m_r,m_i,m_j,m_k}\partial_{\omega_r}^{m_r}\partial_{\omega_i}^{m_i}\partial_{\omega_j}^{m_j}\partial_{\omega_k}^{m_k}\hat{h}(\omega)I^{m_r+m_i+m_j+m_k}. \tag{60}$$

For example for $P(x) = a\cdot x = a_r x_r + a_i x_i + a_j x_j + a_k x_k$ we obtain

$$\mathcal{F}\{(a\cdot x)h\}(\omega) = (a\cdot\tilde{\partial}_\omega)\hat{h}(\omega)I, \tag{61}$$

with $\tilde{\partial}_\omega = \partial_{\omega_r} + \partial_{\omega_i}\boldsymbol{i} + \partial_{\omega_j}\boldsymbol{j} + \partial_{\omega_k}\boldsymbol{k}$ and $a\cdot\tilde{\partial}_\omega = a_r\partial_{\omega_r} + a_i\partial_{\omega_i} + a_j\partial_{\omega_j} + a_k\partial_{\omega_k}$. We have the *dual* to (60) result that

$$\mathcal{F}\{P(\partial_{x_r}, \partial_{x_i}, \partial_{x_j}, \partial_{x_k})h\}(\omega) = \sum_{m_r,m_i,m_j,m_k}\alpha_{m_r,m_i,m_j,m_k}$$
$${\omega_r}^{m_r}{\omega_i}^{m_i}{\omega_j}^{m_j}{\omega_k}^{m_k}\hat{h}(\omega)I^{m_r+m_i+m_j+m_k}, \tag{62}$$

with the special case (dual to (61))

$$\mathcal{F}\{(a\cdot\tilde{\partial})h\}(\omega) = (a\cdot\omega)\hat{h}(\omega)I. \tag{63}$$

Note that (62) shows how the QDFT (with $t = x_0$, $x_1 = x_i$, $x_2 = x_j$, $x_3 = x_k$) can be used to *treat* important *partial differential equations in physics*, e.g. the heat equation, wave equation, Klein-Gordon equation, the Maxwell equations in vacuum, free particle Schrödinger and Dirac equations [24]–[27].

Equation (60) leads further (dual to left side of (55)) to,

$$\widehat{xh}(\omega) = \tilde{\partial}\hat{h}(\omega)I, \qquad \widehat{x^m h}(\omega) = \tilde{\partial}^m\hat{h}(\omega)I^m,$$
$$m \in \mathbb{N}. \tag{64}$$

Multiplying instead with the *quaternion conjugate* $\tilde{x}$ we obtain (*dual* to right side of (55))

$$\widehat{\tilde{x}h}(\omega) = \partial\hat{h}(\omega)I, \quad \widehat{\tilde{x}^m h}(\omega) = \partial^m\hat{h}(\omega)I^m,$$
$$m \in \mathbb{N}. \tag{65}$$

Taking only the *pure vector part* of x, $\mathbf{x} = x - x_r$ we obtain (*dual* to (57))

$$\widehat{\mathbf{x}h}(\omega) = D_\omega \hat{h}(\omega) I, \quad \widehat{\mathbf{x}^m h}(\omega) = D_\omega^m \hat{h}(\omega) I^m, \quad m \in \mathbb{N}, \tag{66}$$

where $D_\omega = \partial_{\omega_i}\boldsymbol{i} + \partial_{\omega_j}\boldsymbol{j} + \partial_{\omega_k}\boldsymbol{k}$.

We further obtain the following *QDFT Plancherel identity*, which expresses, that the quaternion valued *inner product* (28) of two quaternion domain module functions $f, g \in L^2(\mathbb{H};\mathbb{H})$ is given by the quaternion valued inner product of the corresponding QDFTs $\hat{f}$ and $\hat{g}$

$$(f, g) = (\hat{f}, \hat{g}). \tag{67}$$

As corollaries we get the corresponding *QDFT Plancherel identity* for the *scalar inner product* of equation (30)

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle, \tag{68}$$

as well as the *QDFT Parseval identity*

$$||f|| = ||\hat{f}||. \tag{69}$$

The QDFT Parseval identity means, that the QDFT preserves the signal energy when applied in signal processing.

We now define analogous to (34) for unit norm signals $f \in L^2(\mathbb{H};\mathbb{H})$, $||f|| = 1$, the *effective spectral width* (or band width) of f *in the direction* of the unit quaternion $a \in \mathbb{H}$, $|a| = 1$, as the square root of the variance of the frequency spectrum of f along the a-axis

$$\Delta\omega_a = ||(\omega \cdot a)\hat{f}|| = \sqrt{\int_{\mathbb{H}} (\omega \cdot a)^2 |\hat{f}(\omega)|^2 d^4\omega}. \tag{70}$$

We further define the he *effective spectral width* (frequency uncertainty) as the square root of the variance of the energy distribution of $\hat{f}$

$$\Delta\omega = ||\omega\hat{f}|| = \sqrt{\int_{\mathbb{H}} |\omega|^2 |\hat{f}(\omega)|^2 d^4\omega}. \tag{71}$$

We can now state the *directional uncertainty principle* for the QDFT of unit norm signals $f \in L^2(\mathbb{H};\mathbb{H})$, $||f|| = 1$ as

$$\Delta x_a \Delta\omega_b \geq \frac{|a \cdot b|}{2}. \tag{72}$$

The *uncertainty principle* takes the form

$$\Delta x \Delta\omega \geq 1. \tag{73}$$

Equality holds in (72) and (73) for *Gaussian signals* [18].

The QDFT of the *convolution* (33) of two functions $f, g \in L^2(\mathbb{H};\mathbb{H})$ results in

$$\widehat{(f * g)}(\omega) = (2\pi)^2[\hat{f}(\omega)\hat{g}_-(\omega) + \hat{f}(-\omega)\hat{g}_+(\omega)]. \tag{74}$$

Note that for $\hat{g}_+(\omega) = 0$ or if $\hat{f}(\omega) = \hat{f}(-\omega)$ we obtain

$$\widehat{(f * g)}(\omega) = (2\pi)^2 \hat{f}(\omega)\hat{g}(\omega). \tag{75}$$

An application of the QDFT convolution (74) is, e.g., the *fast convolution* (via simple multiplication of the QDFTs in the Fourier domain) of a *quaternion domain signal* $f : \mathbb{H} \to \mathbb{H}$ with a *pair of complex filters* $g_1(x) = g_{1,r}(x) + g_{1,i}(x)\boldsymbol{i} = g_-(x)$, $g_2(x) = g_{2,r}(x) + g_{2,i}(x)\boldsymbol{i} = g_+(x)(-\boldsymbol{j})$, choosing $I = \boldsymbol{i}$ in (36).

Next, we study the *covariance properties* of the QDFT under *orthogonal transformations*. We find that a *three-dimensional rotation* (19) of the argument $g(x) = h(a^{-1}xa)$ leads to

$$\hat{g}(\omega) = \hat{h}(a^{-1}\omega a). \tag{76}$$

The *reflection at* the pointwise invariant *real scalar line* $x \to \tilde{x}$, $g(x) = h(\tilde{x})$ gives

$$\hat{g}(\omega) = -\hat{h}(\widetilde{\omega}). \tag{77}$$

The *reflection at* the three-dimensional *hyperplane of pure quaternions* $x \to -\tilde{x}$, $g(x) = h(-\tilde{x})$ results in

$$\hat{g}(\omega) = -\hat{h}(-\widetilde{\omega}). \tag{78}$$

The *reflection at* the pointwise invariant *line through* $a \in \mathbb{H}$, $|a| = 1$, $x \to a\tilde{x}a$, $g(x) = h(a\tilde{x}a)$ gives

$$\hat{g}(\omega) = -\hat{h}(\tilde{a}^{-1}\widetilde{\omega}\tilde{a}^{-1}) = -\hat{h}(a\widetilde{\omega}a), \tag{79}$$

because $\tilde{a}^{-1} = a$ for $|a| = 1$. The *reflection at* the three-dimensional *hyperplane orthogonal to* the line through $a \in \mathbb{H}$, $|a| = 1$ $x \to -a\tilde{x}a$, $g(x) = h(-a\tilde{x}a)$ results in

$$\hat{g}(\omega) = -\hat{h}(-a\widetilde{\omega}a). \tag{80}$$

A general *four-dimensional rotation* in $\mathbb{R}^4$, $x \to axb$, $a, b \in \mathbb{H}$, $|a| = |b| = 1$, $g(x) = h(axb)$ leads to

$$\hat{g}(\omega) = \hat{h}(a\omega b). \tag{81}$$

We have thus studied the behavior of the QDFT under *all point group transformations* in three and four dimensions (reflections, rotations, rotary reflections, inversions), which are of importance in crystallography. We note, that quaternions have already been employed for the description of crystallographic symmetry in [1] and for the description of root systems of finite groups in three and four dimensions in [8].

V. Conclusion

We first *reviewed quaternion algebra*, orthogonal transformations expressed in quaternion algebra, and quaternion calculus.

We established the *steerable quaternion domain Fourier transform* (QDFT) with a free choice a single constant pure unit quaternion in the kernel. We examined the *properties* of left and right linearity, orthogonal plane split property, and gave an example of the QDFT of a functional equation. Further properties studied are the inverse QDFT, shift and modulation theorems, the QDFT of quaternion coordinate polynomials multiplied with quaternion domain signals[2], as well of products with powers of the signal argument x,

[2]Note that real and complex polynomial generated moment invariants have recently been successfully used for translation, rotation and scale invariant normalized moment description of vector field features, including flow fields [2]–[4].

and the corresponding dual properties (polynomials of partial differential operators, quaternion derivatives and Dirac derivatives). We found that the QDFT can separate the symmetry components of complex signals, and can be applied to many partial differential euqations in physics. Quaternion *non-commutativity* means, that multiplication from the right and left need to be distinguished carefully. Next we established Parseval and Plancherel identities, uncertainty principles and convolution properties for the QDFT. The convolution allows e.g. fast filtering with pairs of complex filters. Finally we studied the covariance properties of the QDFT under orthogonal transformations of the signal arguments, which may a.o. be of importance for applications in crystallography.

We expect that this new quaternionic Fourier transformation may find rich *applications* in mathematics (e.g. higher dimensional holomorphic functions [19]) and physics, including relativity and spacetime physics, in three-dimensional color field processing, neural signal processing, space color video processing, crystallography, quaternion analysis, and for the solution of many types of quaternionic differential equations. We further expect that the QDFT can be successfully *extended* to *localized transforms*, e.g., quaternion domain window Fourier transforms, and continuous quaternionic wavelets and quaternionic ridgelets [6][3]. Further research should be done into *operator versions* of the QDFT, and its related *linear canonical transforms*, which may open up many further areas of interesting applications, including quantum physics and quantum information processing. Especially for applications, *discretization* and *fast implementation* with pairs of complex fast Fourier transforms will be of great interest.

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For in the gospel the righteousness of God is revealed – a righteousness that is by faith from first to last, just as it is written: The righteous will live by faith. [28].

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