A New Exact Solution to the Bratu Boundary Value Problem

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Abstract

In this paper the problem of finding other types of excat solution to the Bratu boundary value problem is solved. In this context a new solution in terms of trigonometric functions has for the first time been computed explicitly and directly for this problem, using a simple change of variable. The exact and explicit general solution obtained for the Bratu equation may be used for a large variety of initial and boundary conditions.

Keywords: Bratu equation, boundary value problem, exact and explicit solution.

Introduction

The problem of Bratu, a famous nonlinear two-point boundary value problem, arises originally from the theory of combustion and later used in many physical applications and expansion theory of the universe of Chandrasekhar [1-5]. Although the Bratu nonlinear two-point boundary value problem

$$u''(x) + \lambda e^{u(x)} = 0 \tag{1}$$

with the conditions

$$u(0) = u(1) = 0 \tag{2}$$

was established more than a century ago, it continues to be the subject of intensive study [1-5]. The literature shows a rich and various analysis of this problem. The Bratu problem has been used well to test the validity of a large number of approximation methods with more or less success. Although the exact solution is also known for more than a century, the general solution of the differential equation has been established very recently by our research group [1] thus allowing an application of a wide variety of initial and boundary conditions. Despite this advancement one may ask whether this problem may exhibit other types of solution. The present work is proposed to show that the problem of Bratu may exhibit another type of solution in explicit and direct manner. In this respect, a simple change of variable is used to transform the original Bratu differential equation in an integrable quadratic Liénard type differential equation (section 2) such that the exact and explicit general solution to the Bratu equation may be computed (section 3). The determination of the two constants of integration by application of the two boundary conditions leads to get the new exact and explicit solution to the Bratu boundary value problem (section 4) and a conclusion is finally carried out for the work.

2. Equivalence of Bratu equation with quadratic Liénard type equation

The objective in this section is to map the elliptic equation of Bratu into integrable quadratic Liénard type equation. For a general formulation we consider the generalized Bratu elliptic equation [1]

$$u''(x) + \lambda e^{\beta u(x)} = 0 \tag{3}$$

where λ and β are arbitrary parameters. For $\lambda > 0$, and $\beta = 1$, one may recover [1-5] the Bratu differentiel equation (1). In this way the following theorem may be proved.

Theorem 1. *Let*

$$y^q(x) = e^{u(x)} \tag{4}$$

be a simple change of variable, where q is an arbitrary parameter. Then (3) is mapped into

$$y''(x) - \frac{y'^2(x)}{y(x)} + \frac{\lambda}{q} y^{\beta q+1} = 0 \tag{5}$$

Proof. From (4) one may compute the first derivative

$$u'(x) = q\, y^{-1}(x) y'(x) \tag{6}$$

Hence the second derivative of $u(x)$ with respect to x takes the form

$$u''(x) = q\frac{y''(x)}{y(x)} - q\frac{y'^2(x)}{y^2(x)} \tag{7}$$

Substituting (7) into (3) leads to obtain immediatly (5). The equation (5) is a generalized quadratic Liénard type equation which may be solved for some values of q. Once the solution $y(x)$ is known, one may compute easily the exact and general solution of (3) from which the general solution of the Bratu differential equation (1) may be deduced.

3. Exact and general solutions of the Bratu equation

3.1 Solution in terms of trigonometric function

For $\beta q = -1$, the equation (5) reduces to

$$y''(x) - \frac{y'^2(x)}{y(x)} - \beta\lambda = 0 \tag{8}$$

The solution of (8) may be written in the form [6]

$$y(x) = -\frac{2\beta\lambda}{K_1^2}\cos^2\left(\frac{K_1(x+K_2)}{2}\right) \tag{9}$$

where K_1 and K_2 are arbitrary parameters. From (9) and using (4) one may get the solution of the generalized equation (3) as

$$u(x) = \ln\left|\left[-\frac{2\beta\lambda}{K_1^2}\cos^2\left(\frac{K_1(x+K_2)}{2}\right)\right]^{-\frac{1}{\beta}}\right|$$

which may take the form

$$u(x) = -\frac{1}{\beta}\ln\left[\frac{2|-\beta\lambda|}{K_1^2}\cos^2\left(\frac{K_1(x+K_2)}{2}\right)\right] \qquad (10)$$

as $y^q(x) \succ 0$. Then the following theorem is proved.

Theorom 2. *Consider the equation (9). Then the exact and general solution to equation (3) becomes equation (10).*

Knowing that $\beta = 1$, for the Bratu equation, one may get the exact and general solution of (1) as

$$u(x) = -\ln\left[\frac{2\lambda}{K_1^2}\cos^2\left(\frac{K_1(x+K_2)}{2}\right)\right] \qquad (11)$$

In this respect the following result is obtained.

Theorem 3. *Consider equation (10). If $\beta = 1$, then the exact and general solution to the Bratu equation (1) takes the form (11).*

3.2. Solution in terms of hyperbolic function

In this case the solution to (8) becomes [6]

$$y(x) = -\frac{\beta\lambda}{2K_1^2}\left[\cosh\left(\frac{1}{2K_1(x+K_2)}\right) - 1\right] \qquad (12)$$

from which the general solution of the generalized Bratu equation (3) becomes

$$u(x) = -\frac{1}{\beta}\ln\left|-\frac{\beta\lambda}{2K_1^2}\left[\cosh\left(\frac{1}{2K_1(x+K_2)}\right) - 1\right]\right| \qquad (13)$$

Using (13) one may obtain the exact and general solution to the Bratu equation (1) as

$$u(x) = -\ln\left|-\frac{\lambda}{2K_1^2}\left[\cosh\left(\frac{1}{2K_1(x+K_2)}\right) - 1\right]\right| \qquad (14)$$

Therefore the following result is proved.

Theorem 4. *If (12) is the exact and general solution to (8) then the exact and general solution to the generalized Bratu equation (3) is equation (13).*

The following theorem has been also shown in this way.

Theorem 5. *If (13) holds, then the exact and general solution to the Bratu equation (1) may be written as (14).*

The solution (14) can not be used to solve the Bratu boundary value problem. Therefore, only the solution (11) may be used to solve the Bratu nonlinear two-point boundary value problem.

4. Solution to Bratu boundary value problem in terms of trigonometric function

In this section the Bratu boundary value problem is solved. Using the first boundary condition $u(0) = 0$, one may get from the general solution (10) the relation

$$\frac{2|-\beta\lambda|}{K_1^2}\cos^2\left(\frac{K_1K_2}{2}\right)=1$$

from which one may secure

$$\cos^2\left(\frac{K_1K_2}{2}\right)=\frac{K_1^2}{2|-\beta\lambda|} \tag{15}$$

Applying $u(1)=0$, leads to

$$\frac{2|-\beta\lambda|}{K_1^2}\cos^2\left(\frac{K_1(1+K_2)}{2}\right)=1$$

from which one may obtain

$$\cos^2\left(\frac{K_1(1+K_2)}{2}\right)=\frac{K_1^2}{2|-\beta\lambda|} \tag{16}$$

Equating equation (15) and equation (16) leads to the relation

$$\cos^2\left(\frac{K_1K_2}{2}\right)=\cos^2\left(\frac{K_1(1+K_2)}{2}\right) \tag{17}$$

which is only valid for

$$\frac{K_1}{2}=2n\pi \tag{18}$$

where n is an integer different from zero. From (18) one may ensure

$$K_1=4n\pi \tag{19}$$

so that K_2 may take the value

$$K_2=\frac{1}{2n\pi}\arccos\left(\pm 2n\pi\sqrt{\frac{2}{|-\beta\lambda|}}\right) \tag{20}$$

Therefore the exact solution of the Bratu boundary value problem for the generalized equation (3) becomes

$$u(x)=u_n(x)=-\frac{1}{\beta}\ln\left[\frac{|-\beta\lambda|}{8n^2\pi^2}\cos^2\left(2n\pi x+\arccos\left(\pm 2n\pi\sqrt{\frac{2}{|-\beta\lambda|}}\right)\right)\right] \tag{21}$$

The above shows the following theorem.

Theorem 6. *Consider equation (10). If equation (2) holds, then the exact solution to the boundary value problem of Bratu-type associated to (3) becomes (21).*

From (21), knowing that $\beta=1$, the exact solution to the original Bratu boundary value problem (1) may take the definitive form

$$u(x)=u_n(x)=-\ln\left[\frac{\lambda}{8n^2\pi^2}\cos^2\left(2n\pi x+\arccos\left(\pm 2n\pi\sqrt{\frac{2}{\lambda}}\right)\right)\right] \tag{22}$$

The above shows also the following result.

Theorem 7. *Consider equation (21). If $\beta=1$, then the exact solution to the original Bratu nonlinear two-point boundary value problem takes the form (22).*

As can be seen, the solution (22) is not reported previously in literature. Therefore such a solution is obtained for the first time for the Bratu nonlinear two-point boundary value problem. Figure 1 shows the behavior of solution (22). As one can also notice, the developed theory can be used with accuracy and efficiency to secure new exact and explicit solutions to the Wazwaz boundary value problems of Bratu type [3] in a direct and straightforward manner. It suffices for the first Bratu type Bratu [3] to substitute $\lambda = -\pi^2$, and $\beta = 1$, into equation (21) to obtain the new solution

$$u(x) = u_n(x) = -\ 2\ln\left[\frac{1}{2n\sqrt{2}}\cos\left(2n\pi x + \arccos\left(\pm 2n\sqrt{2}\right)\right)\right] \qquad (23)$$

Figure 2 shows the behavior of solution (23). For the second Bratu type problem, one may need to put $\lambda = \pi^2$, and $\beta = -1$, into (21) to get the new solution

$$u(x) = u_n(x) = 2\ln\left[\frac{1}{2n\sqrt{2}}\cos\left(2n\pi x + \arccos\left(\pm 2n\sqrt{2}\right)\right)\right] \qquad (24)$$

Figure 3 shows the behavior of solution (24). The current theory may also in principle solve the Wazwaz initial value problem of the Bratu- type [3] using equation (10) and also (14). It is worth to note that the present method does not need to solve a transcendental equation as it is the case in the usual Bratu solution. From the above we may validly conclude this work.

Conclusion

The Bratu nonlinear two-point boundary value problem is a well-known problem with exact solution widely investigated in the literature. In this work we have proved for the first time by a simple change of variable, the existence of a new exact and explicit solution to this problem in terms of trigonometric functions. The general solution obtained for the Bratu equation may then be used for various initial and boundary conditions to deepen the analytical properties of this problem.

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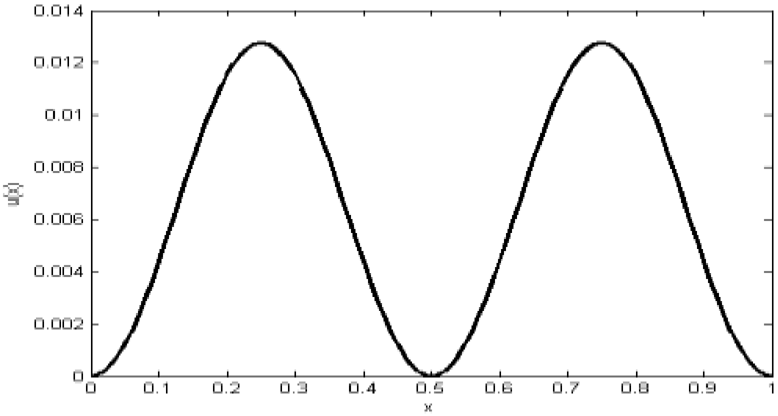

Figure 1: Behavior of solution (22) with $\lambda = 1,$ and $n = 1.$

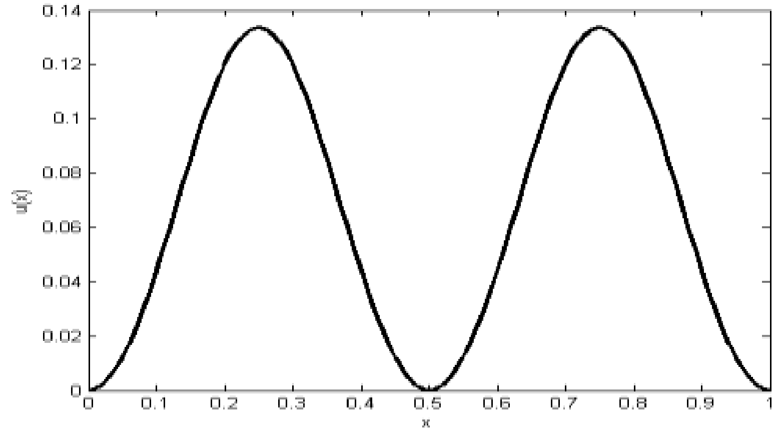

Figure 2: Behavior of solution (23) with $n = 1.$

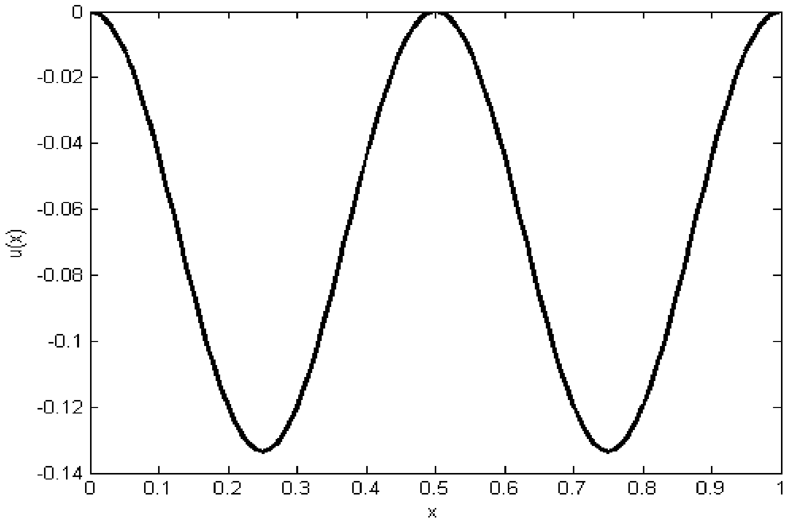

Figure 3: Behavior of solution (24) with $n = 1.$